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Solution of Queuing Problems by a Recursive Technique

Abstract: A recursive method for efficient computational analysis of a wide class of queuing problems is presented. Interarrival and service times are described by multidimensional Markovian processes while arrival and service rates are allowed to be state dependent.

Introduction

Traditional solution techniques [1-5] for queuing problems typically deal with a limited class of queuing models: Poisson, or general input with infinite waiting room, or finite source input as in the classical machine repairman model (e.g., [6]). However, queuing problems of practical importance are often more complex: arrival and service times may be state dependent, the number of sources and/or waiting room may be limited, etc. Moreover, the traditional solution is usually presented by means of Laplace transforms and generating functions. In evaluating mean value or standard deviation of a random variable from such expressions by computer, a great deal of computing time and/or memory might be necessary.

We demonstrate below how an efficient solution to many queuing problems is possible via a new recursive technique. Interarrival and service times are assumed to be general provided they possess rational Laplace transforms (e.g., hyperexponential, hypoexponential, Erlangian, general Erlangian). Arrival and service rates are allowed to be state dependent, i.e., dependent on the number of customers in the queue. Therefore, the repairman model is included as a special case.

The equilibrium state probabilities are obtained by a recursive technique. This allows us to determine other important performance characteristics such as queue length distribution, mean waiting time, waiting time distribution, etc.

Principle of the recursive technique

• *General remarks*

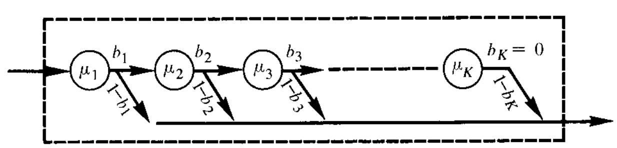
Many distribution functions (d.f.) of practical interest (e.g., hyperexponential, Erlangian, general Erlangian) can be interpreted as a combination of several fictitious stages of service, each having exponentially distributed duration. In other words, processes with such a d.f. can be described exactly by means of a multidimensional random variable. This "phase" concept was introduced by Erlang [2] and generalized by Cox [7]. Cox showed that any distribution with a rational Laplace transform can be represented by a single sequence of fictitious phases of exponential service. The phases may have different mean service rates and are passed through with a certain branching probability b_γ as shown in Fig. 1. Furthermore, any given distribution function of general type (e.g., constant d.f., step functions, . . .) can be approximated with arbitrary accuracy by means of such a "phase" process. Thus many queuing problems can be, either exactly or approximately, modeled by means of a multidimensional Markovian process (i.e., a Markovian process described by a multidimensional random variable [5]). The main problem is the solution of the system of equations for such a process.

The technique presented in the following sections takes advantage of the special recursive structure of the systems of equations. Compared with other known methods (such as matrix inversion, iterative procedures, etc.) the recursive method is easy to program and is superior with respect to computing time and/or memory.

• *Principles of solution*

As mentioned earlier, all queuing problems can be represented by means of a one- or multidimensional Marko-

Figure 1 Representation of distributions with a rational Laplace transform by means of fictitious stages.



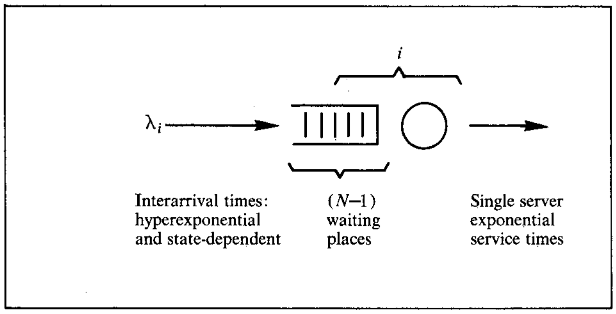


Figure 2 The investigated system $H_2/M/1/\lambda_i$, μ_i (i is the number of customers in the system).

vian process. Markovian processes can be described by means of the so-called Chapman-Kolmogoroff system of equations [3-5].

All these systems of equations possess the following typical feature: There exists a subset of the state probabilities which we define as boundaries, and if the values of the boundaries are known, the recursive solution of the total system of equations can be carried out efficiently.

Therefore, the main ideas of our method are

- To determine the boundaries and to derive expressions for all remaining state probabilities as functions of the boundary values.
- To solve a reduced system of equations for these (unknown) boundaries.
- To determine all interesting state probabilities, and other performance values as well, by means of the (now known) boundaries.

Accordingly, the algorithm is characterized by the following three steps: 1) reduction step, 2) solution step, 3) evaluation step.

Reduction step Consider, as an example, a two-dimensional state space with the states (i, j) where $i = 1, 2, \cdots, N$ and $j = 1, 2, \cdots, K$. Each state occurs with the (yet unknown) state probabilities $P_{i,j}$ ($i = 1, 2, \cdots, N$; $j = 1, 2, \cdots, K$). Furthermore, assume that the states (N, γ) where $\gamma = 1, 2, \cdots, K$ are the boundary states. We introduce the following substitution

$$P_{i,j} = \sum_{\gamma=1}^{K} C_{i,j}^{\gamma} P_{N,\gamma}. \tag{1}$$

The number of steady state equilibrium equations is $N \cdot K$. However, for every fixed γ ($\gamma = 1, 2, \cdots, K$), the coefficients $C_{i,j}^{\gamma}$ can be determined by solving recursively a subsystem of $(N-1) \cdot K$ linear equations by assuming $P_{N,\gamma} = 1$ and $P_{N,\xi} = 0$ for $\xi \neq \gamma$. After obtaining all coefficients $C_{i,j}^{\gamma}$ it is a simple matter to make the substitution (1) to the remaining $(K-1)$ independent equations as well as to the normalizing condition

$$\sum_i \sum_j P_{i,j} = 1. \tag{2}$$

These K equations represent the *reduced system* of equations with K unknown state probabilities $P_{N,\gamma}$ rather than $N \cdot K$ unknowns.

Solution step When solving the reduced system of K equations, one of the common techniques (such as matrix inversion) can be applied and a remarkable saving of computer time and/or memory is obtained.

In many cases, however, a more elegant and still more efficient solution is possible by a second (third, ...) reduction step, i.e., by a second (third, ...) substitution. This multistep reduction technique is a powerful tool for the straightforward solution of queuing problems (for an example, see the subsection on substitutions, below).

Evaluation step Having solved the reduced system of equations, one can find from the original system of equations all state probabilities of interest. In addition, all performance values of interest such as probability of waiting, moments, and distribution of the queue length and waiting time, etc. can be determined.

Remark If only special performance values rather than all state probabilities are of interest, these values can be determined in parallel with the reduction and solution steps, i.e., without determining explicitly all state probabilities.

Application of the recursive technique

In order to demonstrate the procedure of the new technique, two examples of basic queuing systems have been chosen.

As a first example, $H_2/M/1/\lambda_i$, μ_i (for definition see below) is presented in detail. In addition, a numerical example for this problem is given in the Appendix. Then the approach to the problem with general distribution is outlined by dealing with the $M/G/1/\lambda_i$ queuing system.

• *The system* $H_2/M/1/\lambda_i$, μ_i

Figure 2 shows the main characteristics of the investigated system: hyperexponential interarrival times of second order with a state-dependent arrival rate $\lambda_i(\lambda_{i1}, \lambda_{i2})$, finite waiting room with $(N-1)$ waiting places, and one server. The service times are exponentially distributed with the state-dependent service rate μ_i. The system is assumed to be in statistical equilibrium, i.e., it is assumed that it is stationary.

State space and transition coefficients Figure 3 illustrates all states as well as all possible transitions for the considered system. State (i, j) denotes that there are i demands in the system ($i = 0, \cdots, N$) and the arrival process is in phase j, corresponding to a hyperexponential distribution of second order ($j = 1, 2$).

Remark In describing the arrival process, the common representation of hyperexponential distributions by parallel phases has been used rather than the general method of Cox (cf. the subsection on general remarks, above).

Equations of state For stationarity the Chapman-Kolmogoroff equations give the following recurrence relations:

$$P_{i+1,j} = \frac{\mu_i + \lambda_{i,j}}{\mu_{i+1}} P_{i,j} - b_j \frac{\lambda_{i-1,1}}{\mu_{i+1}} P_{i-1,1} - b_j \frac{\lambda_{i-1,2}}{\mu_{i+1}} P_{i-1,2}, \tag{3}$$

where $i = 0, 1, \cdots, N$; $j = 1, 2$ if we define $P_{i,j} = 0$ for $i < 0$ or $i > N$, $\lambda_{i,j} = 0$ for $i = N$, and $\mu_i = 0$ for $i = 0$.

Method of solution Consider the state diagram and the recurrence formula: Let the boundaries be the states $(0, 1)$ and $(0, 2)$; if the values of $P_{0,1}$ and $P_{0,2}$ are known, all remaining state probabilities $P_{i,j}$ can be determined recursively from (3).

Remark Notice that in the recurrence formula the variables $P_{i,j}$ are expressed as linear combinations of $P_{s,t}$ where $s < i$. This makes the substitution straightforward. In the more general case (when $s \leq i$) the solution is still possible. However, one must then solve a subsystem to obtain the recurrence relations. Here we introduce the substitution

$$P_{i,j} = C^1_{i,j} P_{0,1} + C^2_{i,j} P_{0,2}. \tag{4}$$

The main procedure of the algorithm is to determine all coefficients $C^\gamma_{i,j}$ ($\gamma = 1, 2$) and to solve the reduced system of equations of two unknowns. More precisely, all coefficients $C^\gamma_{i,j}$ can be determined by using $2N$ of the $2(N+1)$ linear equations (3) and substituting all state probabilities according to (4). Hence,

$$C^\gamma_{i+1,j} = \frac{\mu_i + \lambda_{i,j}}{\mu_{i+1}} C^\gamma_{i,j} - b_j \frac{\lambda_{i-1,1}}{\mu_{i+1}} C^\gamma_{i-1,1} - b_j \frac{\lambda_{i-1,2}}{\mu_{i+1}} C^\gamma_{i+1,2} \tag{5}$$

with $C^1_{0,1} = C^2_{0,2} = 1$; $C^2_{0,1} = C^1_{0,2} = 0$; $C^\gamma_{i,j} = 0$ for $i < 0$. Now the *reduced system* of equations is obtained by substituting (4) into one of the two remaining equations, say,

$$0 = \mu_N \cdot P_{N,1} - b_1 \cdot \lambda_{N-1,1} \cdot P_{N-1,1} - b_1 \cdot \lambda_{N-1,2} \cdot P_{N-1,2}, \tag{6}$$

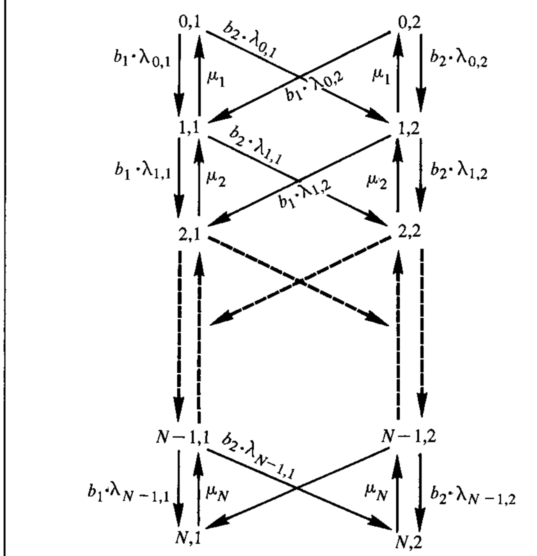


Figure 3 State space and transition coefficients for the system $H_2/M/1/\lambda_i, \mu_i$ (λ_i is the arrival rate; μ_i is the service rate; i is the number of customers in the system; branching probabilities b_1, b_2 are assumed to be the same for all i; $b_1 + b_2 = 1$).

as well as into the normalizing condition

$$\sum_i \sum_j P_{i,j} = 1. \tag{7}$$

This leads directly to the following two relations for the boundary values:

$$P_{0,1}\left[\mu_N \cdot C^1_{N,1} - b_1 \cdot \lambda_{N-1,1} \cdot C^1_{N-1,1} - b_i \cdot \lambda_{N-1,2} \cdot C^1_{N-1,2}\right] + P_{0,2}\left[\mu_N C^2_{N,1} - b_1 \cdot \lambda_{N-1,1} \cdot C^2_{N-1,1} - b_1 \cdot \lambda_{N-1,2} \cdot C^2_{N-1,2}\right] = 0, \tag{8}$$

$$P_{0,1} \sum_i \sum_j C^1_{i,j} + P_{0,2} \sum_i \sum_j C^2_{i,j} = 1. \tag{9}$$

Equation (8) can be solved for $P_{0,2}$ in terms of $P_{0,1}$. If we represent the solution as

$$P_{0,2} = f_2 P_{0,1}, \tag{10}$$

the solution of the state probability $P_{0,1}$ is given by the normalizing condition (9),

$$P_{0,1} = \left(\sum_i \sum_j C^1_{i,j} + f_2 \sum_i \sum_j C^2_{i,j}\right)^{-1}. \tag{11}$$

Algorithm

Reduction step Solve the system of equations (3) with the boundary values $P_{0,1} = 1$ and $P_{0,2} = 0$. This procedure yields all coefficients $C^1_{i,j}$. Sum up all values $C^1_{i,j}$; store

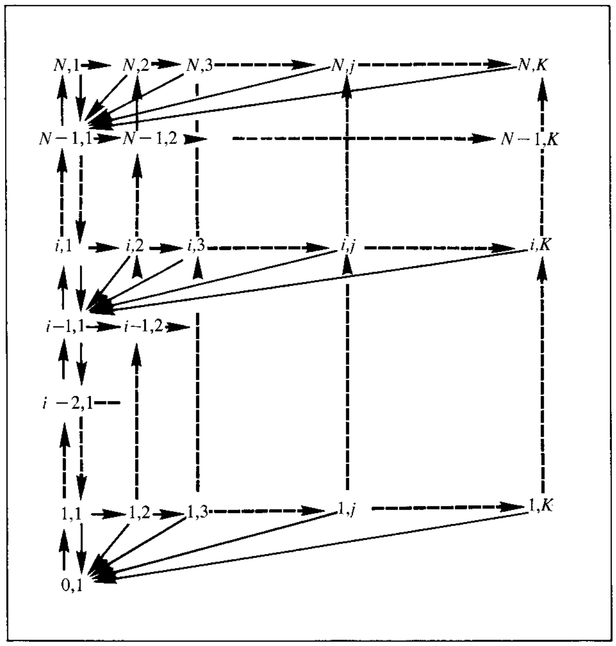


Figure 4 State space and possible transitions for the system $M/G/1/\lambda_i$.

only this sum S_1 as well as the coefficients $C^1_{N,j}$ and $C^1_{N-1,j}$ ($j = 1, 2$). Correspondingly, solve the system of equations (3) with the boundary values $P_{0,1} = 0$ and $P_{0,2} = 1$. Store sum S_2 and the coefficients $C^2_{N,j}$ and $C^2_{N-1,j}$, as well.

Solution step Determine the boundary value $P_{0,1}$ according to (11). Notice that the summations already have been performed and all necessary coefficients are stored. Determine the second boundary value $P_{0,2}$ using (10).

Evaluation step Determine all state probabilities $P_{i,j}$ of interest according to (3). Determine all performance values of interest by means of the known state probabilities.

• *The system* $M/G/1/\lambda_i$

Consider the queuing system $M/G/1/\lambda_i$, i.e., Markovian input with the state-dependent rate λ_i, general service time with service rate μ, and a single server. If the concept of Cox is followed (cf. the subsection on general remarks, above), the general service time distribution can be approximated with arbitrary accuracy by means of a phase process described by the state rates μ_γ and the branching probabilities b_γ for each phase γ ($\gamma = 1, 2, \cdots, K$).

State space and transition probabilities Using the phase representation of the service time, the queuing process can be described exactly by a two-dimensional Markovian process. The state space and possible transitions are outlined in Fig. 4.

Equations of state The Chapman-Kolmogoroff equations give the following set of recurrence relations. For $0 < i \le N$, $1 < j \le K$,

$$P_{i,j} = P_{i,j-1} \frac{\mu_{j-1} \cdot b_{j-1}}{\lambda_i + \mu_j} + P_{i-1,j} \cdot \frac{\lambda_{i-1}}{\lambda_i + \mu_j} \tag{12}$$

if we define $\lambda_i = 0$ for $i = N$ and $P_{0,j} = 0$ for $j = 2, \cdots, K$. For $0 \le i \le N$,

$$\sum_{j=1}^{K} P_{i,j} \cdot \mu_j(1 - b_j) + P_{i-2,1} \cdot \lambda_{i-2} = P_{i-1,1} \cdot (\lambda_{i-1} + \mu_1) \tag{13}$$

if we define $P_{-1,1} = 0$.

Substitution The boundary states are the states $(i, 1)$, $i = 0, \cdots, N$ as shown in Fig. 4. (It is also possible to choose other boundaries. Then, however, a second substitution leading to a more efficient solution step is not possible.) In order to reduce the system of equations, the following substitution can be made:

$$P_{i,j} = C^0_{i,j} \cdot P_{0,1} + C^1_{i,j} \cdot P_{1,1} + \cdots + C^i_{i,j} \cdot P_{i,1} = \sum_{y=0}^{i} C^y_{i,j} \cdot P_{y,1}. \tag{14}$$

Note that all coefficients $C^y_{x,j} = 0$ for $y > x$, all j. Substituting (14) into (13),

$$\sum_{y=0}^{i} P_{y,1} \sum_{j=1}^{K} \mu_j(1 - b_j) \cdot C^y_{i,j} + \sum_{y=0}^{i-2} P_{y,1} \cdot \lambda_{i-2} \cdot C^y_{i-2,1} = \sum_{y=0}^{i-1} P_{y,1} \cdot C^y_{i-1,1} \cdot (\lambda_{i-1} + \mu_1).$$

Note that $P_{i,1}$ occurs only in the first term; all other terms are linear combinations of $P_{s,1}$ where $s < i$. Note also that the second term can be extended because $C^i_{i-2,1} = 0$. Therefore,

$$P_{i,1} = \sum_{y=0}^{i-1} P_{y,1} \times \frac{\left\{ C^y_{i-1,1} \cdot (\lambda_{i-1} + \mu_1) - C^y_{i-2,1} \cdot \lambda_{i-2} - \sum_{j=1}^{K} \mu_j \cdot (1 - b_j) \cdot C^y_{i,j} \right\}}{\sum_{j=1}^{K} \mu_j(1 - b_j) \cdot C^i_{i,j}},$$

and, if we represent the denominator by g_i and the numerator by f^y_i,

$$P_{i,1} = \sum_{y=0}^{i-1} P_{y,1} \cdot f^y_i / g_i. \tag{15}$$

Second substitution From the recurrence relation (15), one can determine all $P_{i,1}$ in terms of $P_{0,1}$, recursively; i.e.,

$$P_{i,1} = d_i \cdot P_{0,1}. \tag{16}$$

Therefore, starting with $d_0 = 1$, one has

$$d_i = \sum_{y=0}^{i-1} d_y \cdot (f_i^y / g_i). \tag{17}$$

Finally, the normalizing condition

$$\sum_{i=0}^{N} \sum_{j=1}^{K} P_{i,j} = 1 = \sum_{i=0}^{N} \sum_{j=1}^{K} \sum_{y=0}^{i} C_{i,j}^{y} P_{y,1}$$

$$= \sum_{i=0}^{N} \sum_{j=1}^{K} \sum_{y=0}^{N} C_{i,j}^{y} \cdot P_{y,1}$$

$$= \sum_{i=0}^{N} \sum_{j=1}^{K} \sum_{y=0}^{N} C_{i,j}^{y} \cdot d_y \cdot P_{0,1}$$

leads to

$$P_{0,1} = \left(\sum_{y=0}^{N} d_y \sum_{i=0}^{N} \sum_{j=1}^{K} C_{i,j}^{y} \right)^{-1}. \tag{18}$$

Algorithm

Reduction step Determine all coefficients $C_{i,j}^y$, i.e., solve the system of equations (12) recursively by putting $P_{y,1}$ equal to one and all other boundary values $P_{i,1}$ $(i \neq y)$ equal to zero $(y = 0, 1, 2, \cdots, N)$. Remember that $C_{i,j}^y = 0$ for $y > i$. Store only f_i^y, g_i and $\Sigma_i \Sigma_j C_{i,j}^y$

Solution step Determine all coefficients d_i $(i = 1, 2, \cdots, N)$ according to Eqs. (15) and (16). Then determine the actual value of $P_{0,1}$ by means of Eq. (18).

Evaluation step Determine all state probabilities $P_{i,j}$ of interest according to Eqs. (16) and (14), respectively. Determine other performance values of interest using the known state probabilities.

Summary

A new and efficient recursive technique that is suitable for solving a wide class of queuing problems is presented above. The procedure of the recursive algorithm is described and illustrated via two basic queuing problems. Furthermore, a detailed numerical example is given in the Appendix.

The general applicability and efficiency of the recursive technique has been found to be valid for a wide class of queuing problems, including

- $M/H_k/1/\lambda_i, \mu_i$, $k = 1, 2, 3$.
- $H_2/E_k/1/\lambda_i, \mu_i$, $k = 1, 2, \cdots$.
- $GI/M/n/\lambda_i, \mu_i$.
- $GI/G/1/\lambda_i, \mu_i$.
- $M/M/n/\lambda_i, \mu_i$ with preemptive priorities.
- Two- and multi-queue models.

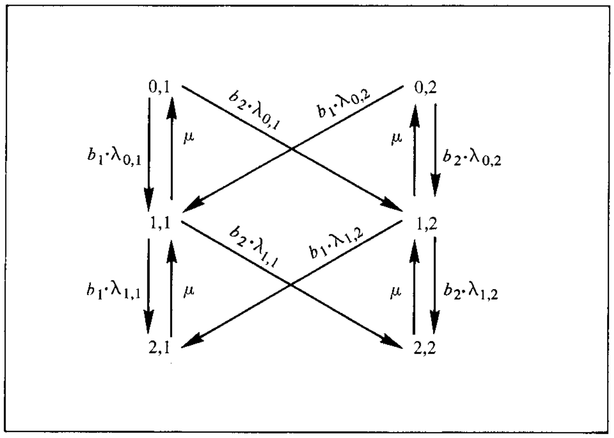


Figure 5 State space and transition coefficients for the example in the Appendix.

Obviously the first three cases are included in the GI/G/1 system. However, they have been studied separately because their special structures allow more efficient computation in comparison with the general case.

Appendix: Numerical example for the queuing system $H_2/M/1/\lambda_i, \mu_i$

- *General remarks and parameters*

The following sections demonstrate the application of the algorithm presented in the subsection on the system $H_2/M/1/\lambda_i, \mu_i$. The state space as well as the corresponding equations of the state are first described. The algorithm is then performed in a step-by-step manner.

As a numerical example, the following parameter values are chosen:

- Maximum number of customers in the system: $N = 2$.
- State-dependent arrival rates: $\lambda_{0,1} = 0.8\ s^{-1}$, $\lambda_{0,2} = 0.4\ s^{-1}$, $\lambda_{1,1} = 0.4\ s^{-1}$, $\lambda_{1,2} = 0.2\ s^{-1}$.
- Probabilities of branching: $b_1 = b_2 = 0.5$.
- Service rates: $\mu_i = \mu = 1\ s^{-1}$.

State space See Fig. 5.

Equations of state The equations of state, generally described by (3), are, in this case,

(a) $P_{1,1} = \dfrac{\lambda_{0,1}}{\mu} P_{0,1}$.

(b) $P_{1,2} = \dfrac{\lambda_{0,2}}{\mu} P_{0,2}$.

(c) $P_{2,1} = \frac{\mu + \lambda_{1,1}}{\mu} P_{1,1} - b_1 \frac{\lambda_{0,1}}{\mu} \cdot P_{0,1} - b_1 \cdot \frac{\lambda_{0,2}}{\mu} \cdot P_{0,2}.$

(d) $P_{2,2} = \frac{\mu + \lambda_{1,2}}{\mu} P_{1,2} - b_2 \cdot \frac{\lambda_{0,1}}{\mu} \cdot P_{0,1} - b_2 \cdot \frac{\lambda_{0,2}}{\mu} \cdot P_{0,2}.$

(e) $0 = P_{2,1} - b_1 \cdot \frac{\lambda_{1,1}}{\mu} \cdot P_{1,1} - b_1 \cdot \frac{\lambda_{1,2}}{\mu} \cdot P_{1,2}.$

(f) $0 = P_{2,2} - b_2 \cdot \frac{\lambda_{1,1}}{\mu} \cdot P_{1,1} - b_2 \cdot \frac{\lambda_{1,2}}{\mu} \cdot P_{1,2}.$

The normalizing condition is

(g) $1 = P_{0,1} + P_{0,2} + P_{1,1} + P_{1,2} + P_{2,1} + P_{2,2}.$

Therefore, by introducing the substitution (4), the coefficient f_2 is given by the following expression:

$$f_2 = -\frac{\mu \cdot C^1_{2,1} - b_1 \cdot \lambda_{1,1} \cdot C^1_{1,1} - b_1 \cdot \lambda_{1,2} \cdot C^1_{1,2}}{\mu \cdot C^2_{2,1} - b_1 \cdot \lambda_{1,1} \cdot C^2_{1,1} - b_1 \cdot \lambda_{1,2} \cdot C^2_{1,2}}.$$

Algorithm

Reduction step Solve Eqs. (a) – (d) for the boundary values $P_{0,1} = 1$ and $P_{0,2} = 0$. The calculated values correspond to the coefficients $C^1_{i,j}$. The numerical values are $C^1_{0,1} = 1$, $C^1_{0,2} = 0$, $C^1_{1,1} = 0.8$, $C^1_{1,2} = 0$, $C^1_{2,1} = 0.72$, $C^1_{2,2} = -0.4$.

Therefore, the sum S_1 of all coefficients $C^1_{i,j}$ is $S_1 = 2.12$. Solve Eqs. (a) – (d) for the boundary values $P_{0,1} = 0$ and $P_{0,2} = 0$. Correspondingly, one obtains the coefficients $C^2_{0,1} = 0$, $C^2_{0,2} = 1$, $C^2_{1,1} = 0$, $C^2_{1,2} = 0.4$, $C^2_{2,1} = -0.2$, $C^2_{2,2} = 0.28$, $S_2 = 1.48$.

Solution step The coefficient f_2 is given by

$$f_2 = -\frac{0.72 - 0.2 \times 0.8 - 0.1 \times 0}{-0.2 - 0.2 \times 0 - 0.1 \times 0.4} = 2.3333.$$

Therefore, the actual boundary values are

$$P_{0,1} = (S_1 + f_2 \cdot S_2) = 0.1794$$

and

$$P_{0,2} = f_2 \cdot P_{0,1} = 0.4187.$$

Evaluation step Suppose that all state probabilities as well as the mean queue length are of interest. Then, the numerical values of state probabilities are obtained by evaluating once more Eqs. (a) – (d) with the actual boundary values $P_{0,1} = 0.1794$, $P_{0,2} = 0.4187$. This leads directly to $P_{1,1} = 0.1435$, $P_{1,2} = 0.1675$, $P_{2,1} = 0.0455$, $P_{2,2} = 0.0455$. Finally, the mean value of jobs in the system is given by

$$L = \sum_i \sum_j i \cdot P_{i,j},$$

which is, numerically, $L = 0.4930$.

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