

Drop Formation in a Liquid Jet

Abstract: A one-dimensional analysis of drop formation in a liquid jet is developed under the assumption that the axial velocity of the axisymmetric, nonviscous liquid jet remains independent of the radial coordinate. The resulting equations are used for both linear and nonlinear analyses. In the linear form, this model provides a stream stability relation comparable to that of Rayleigh; transient solutions are obtained for given initial conditions of an infinite stream. For the nonlinear equations, numerical simulation was done to study the satellite drop formation; with the present model, the satellite drop is always formed.

Introduction

The phenomenon of drop formation in a liquid jet has been studied by many authors. The earliest analysis appears to be that of Lord Rayleigh [1], who made a linearized stability analysis of a nonviscous liquid jet. This work was later extended by Rayleigh himself [2] and, more recently, by others [3-7] who included various effects of viscosity, nonlinearity and ambient air pressure in their studies. Most of these analyses followed the pattern of the classical treatment set by Rayleigh. For linear analysis, solutions were sought through an indirect quantity such as potential functions. Because of the complexities involved, transient initial value problems and steady state boundary value problems, which describe a more realistic drop forming process, have not been solved in closed-form solutions. For a nonlinear analysis, higher order expansion methods or complex numerical analyses have to be used.

In this paper, these complexities have been circumvented through simplification of the hydrodynamic equations, and the original, physically meaningful variables are retained through analysis. For this simplification, the axial velocity of the nonviscous fluid is assumed to be dependent upon only the axial coordinate. This assumption reduces the equations of stream motion to two simple equations—one for momentum balance and the other for continuity. For the quasi-stationary stream analysis, these nonlinear equations are then transformed to a moving coordinate system.

Both linear and nonlinear studies have been performed. The linearization makes closed-form solutions possible. These solutions include the stability of the stream, which shows the optimal condition for drop formation, and solutions for drop breakup time under a given initial per-

turbation and under a continuous external excitation. Due to their simple form, the nonlinear equations can be readily adapted to numerical methods. Typical nonlinear solutions are presented in tabular form.

The closed-form solutions obtained in this analysis are more easily adapted to the design analysis of drop generating devices than the complex results from the classical treatment. Such devices have been described by Sweet [8] and Kamphoefner [9]; the latter is a review paper on the status of the subject through about 1971. Sweet's report includes an analysis of drop formation in which the disturbance growth is approximated by splicing a linear function and an exponential function. In our analysis, the disturbance growth is derived to be a single hyperbolic function.

Basic equations

If the axial velocity and the pressure in the jet are constant over the cross section of the axisymmetric stream and dependent only on axial coordinate z and time t , the Navier-Stokes equations of momentum [10] in cylindrical coordinates become

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} = \nu \left\{ \frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} (r v_r) \right] + \frac{\partial^2 v_r}{\partial z^2} \right\} + \frac{f_r}{\rho}, \quad (1)$$

$$v_\theta = 0, \text{ and} \quad (2)$$

$$\frac{\partial v_z}{\partial t} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \frac{\partial^2 v_z}{\partial z^2} + \frac{f_z}{\rho}, \quad (3)$$

where ρ is density; ν , kinematic viscosity; p , pressure; v , velocity; and f , body force in directions of respective coordinates.

The continuity equation [10] for incompressible flow of an infinitesimal element is

$$\frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{\partial v_z}{\partial z} = 0 \quad (4)$$

or, as applied in a control volume of variable cross section (Fig. 1),

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial z} (Av_z) = 0, \quad (5)$$

where $A = \pi r_0^2$ and $r_0(z, t)$ is the time-dependent stream radius.

The pressure term p in Eq. (3) includes the surface tension, which plays a key role in the drop formation. If the internal pressure balances the surface tension for an infinitesimally thin cross section, then

$$p_s = T \left(\frac{1}{r_N} + \frac{1}{r_T} \right) \quad (6)$$

where T is the surface tension and r_N and r_T are mutually orthogonal principal radii of curvature of the curved surface. Referring to Fig. 1, one gets

$$r_N = r_0 \sec \alpha = r_0 [1 + (\partial r_0 / \partial z)^2]^{\frac{1}{2}}, \text{ and} \quad (7)$$

$$r_T = \frac{[1 + (\partial r_0 / \partial z)^2]^{\frac{3}{2}}}{-\partial^2 r_0 / \partial z^2}. \quad (8)$$

Thus the total pressure p is

$$p = p_s + f$$

$$= \frac{T}{[1 + (\partial r_0 / \partial z)^2]^{\frac{1}{2}}} \left\{ \frac{1}{r_0} - \frac{\partial^2 r_0 / \partial z^2}{1 + (\partial r_0 / \partial z)^2} \right\} + f(z, t), \quad (9)$$

where f is the pressure contributed by any source other than the surface tension. If viscosity is ignored, Eq. (1) is decoupled from the rest and it suffices to solve the following equations (denote $v_z = v$ from here on):

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z}, \text{ and} \quad (10)$$

$$\frac{\partial r_0^2}{\partial t} + \frac{\partial}{\partial z} (r_0^2 v) = 0. \quad (11)$$

Quasi-stationary stream analysis

If an infinitely long stream has a velocity v_0 , a change of coordinates,

$$\eta = z - v_0 t, \quad (12)$$

alters the observation reference to one in which the observer follows the stream and results in quasi-stationary hydrodynamic equations:

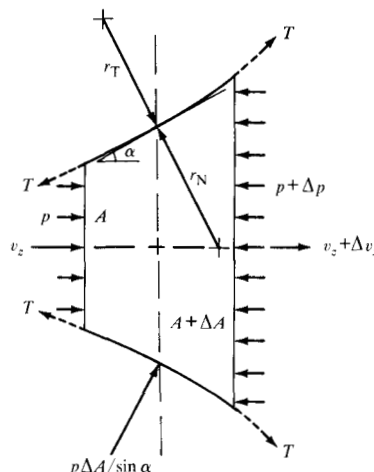


Figure 1 Free body diagram of an infinitesimal length of jet showing variations of pressure, velocity, cross-sectional area and the stream contour. The radius of curvature r_T lies in the plane of the paper and r_N is perpendicular to r_T .

$$\frac{\partial v}{\partial t} + (v - v_0) \frac{\partial v}{\partial \eta} = -\frac{1}{\rho} \frac{\partial p}{\partial \eta}; \quad (13)$$

$$\frac{\partial r_0^2}{\partial t} + \frac{\partial}{\partial \eta} (v - v_0) r_0^2 = 0. \quad (14)$$

Further, if we introduce a velocity variation such that

$$v(\eta, t) = v_0 + u(\eta, t) \text{ or } u(\eta, t) = v(\eta, t) - v_0, \quad (15)$$

then the equations of motion become

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial \eta} = -\frac{1}{\rho} \frac{\partial p}{\partial \eta}, \text{ and} \quad (16)$$

$$\frac{\partial r_0}{\partial t} + u \frac{\partial r_0}{\partial \eta} + \frac{r_0}{2} \frac{\partial u}{\partial \eta} = 0; \quad (17)$$

p is shown by Eq. (9).

Linear analysis

For the linearization, the velocity variation u and the derivative $\partial r_0 / \partial \eta$ are regarded as small quantities and the $u \partial u / \partial \eta$, $(\partial r_0 / \partial \eta)^2$ and $u \partial r_0 / \partial \eta$ terms are ignored. If we set the radius $r_0 = a(1 + \delta)$, then the linearized equations of motion become

$$\frac{\partial u}{\partial t} = \frac{T}{\rho a} \left(\frac{\partial \delta}{\partial \eta} + a^2 \frac{\partial^3 \delta}{\partial \eta^3} \right) + \frac{1}{\rho} \frac{\partial f}{\partial \eta} \text{ and} \quad (18)$$

$$\frac{\partial u}{\partial \eta} = -2 \frac{\partial \delta}{\partial t}. \quad (19)$$

Elimination of u results in

$$\frac{\partial^2 \delta}{\partial t^2} + \frac{T}{2\rho a} \frac{\partial^2 \delta}{\partial \eta^2} + \frac{aT}{2\rho} \frac{\partial^4 \delta}{\partial \eta^4} = -\frac{1}{\rho} \frac{\partial^2 f}{\partial \eta^2} \equiv f_0(\eta, t). \quad (20)$$

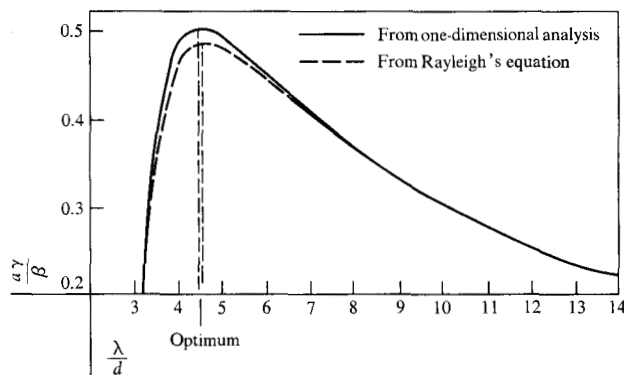


Figure 2 Stability plots of Eq. (23) and Rayleigh's equation [1] as functions of wavelength-jet diameter ratio.

Stability of an infinite stream Since stream breakup requires a wave form that grows in exponential fashion, the solution of Eq. (20) with $f_0 = 0$ takes the form

$$\delta(\eta, t) = \delta_0 e^{\gamma t + ik\eta}, \quad (21)$$

where γ is a growth factor, $k = 2\pi/\lambda$ and λ is the wave length. Substitution into Eq. (20) results in

$$k^4 - \frac{k^2}{a^2} + \frac{\gamma^2}{a^2\beta^2} = 0 \quad \text{where } \beta^2 \equiv \frac{T}{2\rho a}; \text{ or} \quad (22)$$

$$\frac{a^2\gamma^2}{\beta^2} = a^2k^2(1 - a^2k^2) = \frac{4\pi^2a^2}{\lambda^2} \left(1 - \frac{4\pi^2a^2}{\lambda^2}\right). \quad (23)$$

This is the stability equation similar to Rayleigh's equation [1]. Equation (23) shows that $a^2k^2 < 1$ is required for the stream to break up. In terms of wave length, the requirement is $\lambda > 2\pi a = \pi d$, i.e., that the stream will be broken into drops separated by distances greater than the jet circumference. Comparison between Eq. (23) and Rayleigh's results showed little difference. Stability plots are shown in Fig. 2.

The optimum condition for breakup means maximum γ , which is obtained from

$$\frac{\partial}{\partial(a^2k^2)} [a^2k^2(1 - a^2k^2)] = 1 - 2a^2k^2 = 0:$$

that is,

$$k_{\text{opt}}^2 = \frac{1}{2a^2} \text{ or } \lambda_{\text{opt}} = \sqrt{2} \pi d = 4.443d.$$

This compares well with the result obtained from Rayleigh's equation [1], which yields $\lambda_{\text{opt}} = 4.508d$.

Initial value problems If a uniform jet is given an initial disturbance in the form of a velocity distribution or radius variation, the stream will break up under the influence of

surface tension. Suppose the initial disturbance is sinusoidal in nature; the form of the solution is then

$$\delta(\eta, t) = A(t) \cos(2\pi\eta/\lambda).$$

Substituting δ into Eq. (20), with $f_0 = 0$, we obtain

$$\frac{d^2A}{dt^2} - \beta^2 \frac{4\pi^2}{\lambda^2} \left(1 - \frac{4\pi^2}{\lambda^2} a^2\right) A = 0, \quad \frac{4\pi^2}{\lambda^2} a^2 < 1,$$

with solution

$$A(t) = C_1 \cosh \gamma_0 t + C_2 \sinh \gamma_0 t,$$

where

$$\gamma_0^2 = \beta^2 \frac{4\pi^2}{\lambda^2} \left(1 - \frac{4\pi^2}{\lambda^2} a^2\right) = \frac{T}{2\rho a} \frac{4\pi^2}{\lambda^2} \left(1 - \frac{4\pi^2}{\lambda^2} a^2\right) \quad (24)$$

and

$$\delta(\eta, t) = (C_1 \cosh \gamma_0 t + C_2 \sinh \gamma_0 t) \cos(2\pi\eta/\lambda); \quad (25)$$

C_1 and C_2 are determined from initial conditions.

Consider the case of initial disturbance on the velocity.

At $t = 0$,

$$\delta(\eta, 0) = 0, \quad v(\eta, 0) = v_0 + \Delta v \sin(2\pi\eta/\lambda) \text{ or } u(\eta, 0) = v(\eta, 0) - v_0 = \Delta v \sin(2\pi\eta/\lambda).$$

From Eq. (19),

$$\frac{\partial \delta}{\partial t}(\eta, 0) = -\frac{1}{2} \frac{\partial U}{\partial \eta}(\eta, 0) = -\Delta v \frac{\pi}{\lambda} \cos(2\pi\eta/\lambda).$$

These conditions yield $C_1 = 0$ and $\gamma_0 C_2 = \pi\Delta v/\lambda$. Therefore

$$\delta(\eta, t) = \frac{\pi\Delta v}{\lambda\gamma_0} \cos(2\pi\eta/\lambda) \sinh \gamma_0 t. \quad (26)$$

The stream breaks into drops when the amplitude of δ becomes 1. Hence, the linear theory predicts the breakup time to be

$$t_b = \frac{1}{\gamma_0} \sinh^{-1}(\lambda\gamma_0/\pi\Delta v). \quad (27)$$

When this is normalized to jet velocity v_0 , the so-called "break off distance" is obtained as

$$Z_b = v_0 t_b = v_0/\gamma_0 \sinh^{-1}(\lambda\gamma_0/\pi\Delta v). \quad (28)$$

Note that in Eq. (26) when $\gamma_0 t$ is large, the function $\sinh \gamma_0 t$ may be approximated by $\frac{1}{2}e^{\gamma_0 t}$, which was used by Sweet in his analysis [8], resulting in a logarithmic expression for t_b (Eq. 27).

Radially disturbed jet Drop formation can be accomplished by placing around a jet one or a series of equally spaced rings which exert pulsating radial pressure or axial forces on the stream. The forces may be electrostatic or magnetic in character. For the simplified analy-

sis, this effect can be studied by assuming an infinite number of rings. This makes the external pulsation travel with the stream. Then the pressure term becomes

$$p = -\frac{T}{a} \left(\frac{\partial \delta}{\partial \eta} + a^2 \frac{\partial^2 \delta}{\partial \eta^2} \right) + p_0 + p_e \cos(2\pi\eta/\lambda), \quad (29)$$

where p_e is the pressure amplitude resulting from the external disturbance. The linearized equation of motion, Eq. (20), is rewritten as

$$\frac{\partial^2 \delta}{\partial t^2} + \beta^2 \frac{\partial^2 \delta}{\partial \eta^2} + a^2 \beta^2 \frac{\partial^4 \delta}{\partial \eta^4} = \frac{2\pi^2}{\lambda^2} \frac{p_e}{\rho} \cos(2\pi\eta/\lambda) = \bar{p}_e \cos(2\pi\eta/\lambda), \quad (30)$$

where

$$\bar{p}_e = \frac{2\pi^2}{\lambda^2} \frac{p_e}{\rho}.$$

The solution is again in the form

$$\delta = A(t) \cos(2\pi\eta/\lambda), \text{ and} \\ A(t) = -(\bar{p}_e/\gamma_0^2) + C_1 \cosh \gamma_0 t + C_2 \sinh \gamma_0 t, \quad (31)$$

where

$$\gamma_0^2 = \frac{2T\pi^2}{\rho a \lambda^2} \left(1 - \frac{4\pi^2 a^2}{\lambda^2} \right).$$

If the initial conditions are $\delta = 0$ and $\partial \delta / \partial t = 0$, then we obtain

$$\delta(t) = -\frac{p_e a (1 - \cosh \gamma_0 t) \cos(2\pi\eta/\lambda)}{T(1 - 4\pi^2 a^2/\lambda^2)}. \quad (32)$$

The breakup time according to this linear theory is

$$t_b = \frac{1}{\gamma_0} \cosh^{-1} \left(1 + \frac{\lambda^2 \rho \gamma_0^2}{2\pi^2 p_e} \right). \quad (33)$$

If

$$\lambda^2 = \lambda_{\text{opt}}^2 = 8\pi^2 a^2 \text{ and } \gamma_{\text{opt}}^2 = T/8\rho a^3, \text{ then}$$

$$\delta_{\text{opt}} = -\frac{2ap_e}{T} (1 - \cosh \gamma_{\text{opt}} t) \cos \frac{\eta}{(2a)^{1/2}}, \text{ and}$$

$$t_{b-\text{opt}} = \left(\frac{8\rho a^3}{T} \right)^{1/2} \cosh^{-1} \left(1 + \frac{T}{2ap_e} \right). \quad (34)$$

Equations (27) and (33) allow comparison of breakup efficiency for drop formation by different disturbances when values of Δv and p_e can be estimated.

• Nonlinear analysis

The solution of the initial value problem has been described above for the linearized one-dimensional hydro-

dynamic equations. That solution, however, describes the stream contour as a sinusoidal form, which is true only during the early stages of the stream dynamics. In the later stages, because the slope of the contour becomes large, the linear analysis is no longer valid. Hence, more realistic stream dynamics can only be studied through the nonlinear equations such as Eqs. (16) and (17). Because of the simplicity of those equations, their numerical difficulty is minimal and they are easily adapted to a computer program as we have done to obtain the following results.

We introduce the following dimensionless parameters: $\delta = r_0/a$, $w = u/u_0$, $\bar{z} = \eta/\lambda$, $\bar{t} = u_0 t/\lambda$, $H = a/\lambda$ and $Q = T/(pau_0^2)$. Then the equations of stream dynamics (9), (16) and (17) become

$$\frac{\partial w}{\partial \bar{t}} = -\frac{\partial}{\partial \bar{z}} \left(\frac{w^2}{2} + \bar{p} \right) \text{ and} \quad (35)$$

$$\frac{\partial \delta^2}{\partial \bar{t}} = -\frac{\partial}{\partial \bar{z}} (\delta^2 w); \text{ where} \quad (36)$$

$$\bar{p} = \frac{Q}{[1 + H^2(\partial \delta / \partial \bar{z})^2]^{1/2}} \left[\frac{1}{\delta} - \frac{H^2(\partial^2 \delta / \partial \bar{z}^2)}{1 + H^2(\partial \delta / \partial \bar{z})^2} \right]. \quad (37)$$

Note that in this formulation there are only two physical parameters: H , the wave length parameter and Q , the surface tension-disturbance parameter. The quantity u_0 in the parameter Q is any quantity with the dimensions of velocity. In the following analysis, u_0 will denote the velocity-disturbance modulation amplitude.

Numerical analysis An exact solution to Eqs. (35)-(37) is not known; however, this set of equations belongs to the class of hyperbolic partial differential equations for which numerical methods have been extensively studied. Here, the Lax-Wendroff method [11] for finite difference equations will be adopted. The variables chosen are $s = \delta^2$ and $m = \delta^2 w$. The quantity s represents the cross-sectional area of the stream and m is proportional to the momentum. Equations (35)-(37) become

$$\frac{\partial m}{\partial \bar{t}} + \frac{\partial}{\partial \bar{z}} \left(\frac{m^2}{s} \right) = -\frac{s \partial \bar{p}}{\partial \bar{z}}, \quad (38)$$

$$\frac{\partial s}{\partial \bar{t}} + \frac{\partial m}{\partial \bar{z}} = 0, \text{ and} \quad (39)$$

$$\bar{p} = \frac{2Q}{[4s + H^2(\partial s / \partial \bar{z})^2]^{1/2}} \left\{ 1 - \frac{H^2[2s(\partial^2 s / \partial \bar{z}^2) - (\partial s / \partial \bar{z})^2]}{4s + H^2(\partial s / \partial \bar{z})^2} \right\}. \quad (40)$$

Then the following relationships are obtained (from here on, $\bar{t}$, $\bar{z}$, and $\bar{p}$ will be denoted by t , z , and p , respectively, for typographical convenience):

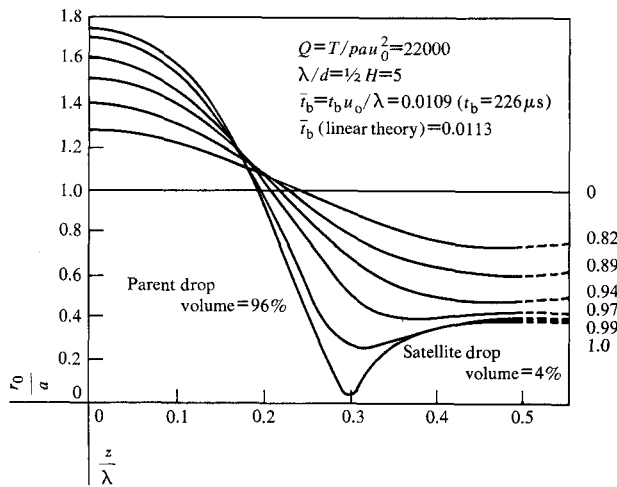


Figure 3 Typical nonlinear variation of stream contour as time elapses from $t=0$ to breakup time, $t=t_b$. The result shows that the satellite drop is 4% of the total drop volume when $Q = 22000$ and $H = 0.1$.

$$s(z, t + \Delta t) = s(z, t) + \Delta t \left(\frac{\partial s}{\partial t} \right)_{z,t} + \frac{\Delta t^2}{2} \left(\frac{\partial^2 s}{\partial t^2} \right)_{z,t} + O(\Delta t^3).$$

Substituting Eq. (39) for the time derivatives, we get

$$s(z, t + \Delta t) \approx s(z, t) - \Delta t \left(\frac{\partial m}{\partial z} \right)_{z,t} - \frac{\Delta t^2}{2} \left(\frac{\partial}{\partial z} \frac{\partial m}{\partial t} \right)_{z,t} \\ = s(z, t) - \Delta t \left(\frac{\partial m}{\partial z} \right)_{z,t} + \frac{\Delta t^2}{2} \left\{ \frac{\partial}{\partial z} \left[\frac{\partial}{\partial z} \left(\frac{m^2}{s} \right) + s \frac{\partial p}{\partial z} \right] \right\}_{z,t} \quad (41)$$

Similarly,

$$m(z, t + \Delta t) = m(z, t) - \Delta t \left[\frac{\partial}{\partial z} \left(\frac{m^2}{a} \right) + a \frac{\partial p}{\partial z} \right] \\ - \frac{\Delta t^2}{2} \left\{ \frac{\partial}{\partial z} \left[- \frac{2m}{s} \left(\frac{\partial}{\partial z} \left(\frac{m^2}{s} \right) + s \frac{\partial p}{\partial z} \right) \right. \right. \\ \left. \left. + \frac{m^2}{s^2} \frac{\partial m}{\partial z} \right] - \frac{\partial p}{\partial z} \frac{\partial m}{\partial z} + s \frac{\partial}{\partial z} \frac{\partial p}{\partial t} \right\}_{z,t} \quad (42)$$

From Eq. (40),

$$\frac{\partial p}{\partial t} = \frac{4Q(A_p + H^2 B_p - 3H^2 C_p)}{[4s + H^2 (\partial s / \partial z)^2]^{5/2}}, \quad (43)$$

where

$$A_p = 4s \left(\frac{\partial m}{\partial z} \right),$$

$$B_p = 2s \left[2s \frac{\partial^3 m}{\partial z^3} - \frac{\partial s}{\partial z} \frac{\partial^2 m}{\partial z^2} - \frac{\partial^2 s}{\partial z^2} \frac{\partial m}{\partial z} \right] + 4 \left(\frac{\partial s}{\partial z} \right)^2 \frac{\partial m}{\partial z}, \text{ and}$$

$$C_p = \left(\frac{\partial s}{\partial z} \right)^2 \left[\frac{\partial^2 s}{\partial z^2} \frac{\partial m}{\partial z} + \frac{\partial s}{\partial z} \frac{\partial^2 m}{\partial z^2} + s \frac{\partial^3 m}{\partial z^3} \right] - 3s \frac{\partial^2 s}{\partial z^2} \frac{\partial s}{\partial z} \frac{\partial^2 m}{\partial z^2}.$$

Similarly we can calculate $\frac{\partial p}{\partial z}$ by replacing $\frac{\partial m}{\partial z}$ with $-\frac{\partial a}{\partial z}$ in Eq. (43).

Equations (41)-(43) enable computation of the state variables, m and s , at $t + \Delta t$ in terms of their respective values at t . A successful computation depends on the ability to control the numerical stability. In particular, the grid size of Δt and Δz is such that the predicted states must be within the boundaries of characteristic lines [12]. Due to the nonlinear characteristics, this range varies with time. Thus, a variation of the time interval Δt is required. One scheme used here to insure a proper Δt is to compare, at each step, two predicted state values, e.g., $m_1(z, t + 2\Delta t)$ based on $m(z, t)$ and $m_2(z, t + 2\Delta t)$, which was calculated from $m(z, t + \Delta t)$, which in turn had been obtained from $m(z, t)$. If the difference between m_1 and m_2 falls within a specified limit, the size Δt is acceptable. If the difference is too large, Δt is reduced until the required accuracy is obtained. The computation of stream contour continues until a singularity occurs, signifying a negative cross-sectional area. The final contour line is taken to be that obtained just before the singularity.

Numerical results One simple problem to study is the stream dynamics of a finite, quasi-stationary jet caused by an initial disturbance with a periodic distribution. For such a problem the solution is periodic; therefore, it is necessary to examine only one wavelength. The bound-

Table 1 Satellite volumes and breakup time.*

Q	$4(H = 0.125)$			$5(0.1)$			$7(0.0714)$		
	v_s/v_t	$\bar{t}_b$	$(\bar{t}_{bl})$	v_s/v_t	$\bar{t}_b$	$(\bar{t}_{bl})$	v_s/v_t	$\bar{t}_b$	$(\bar{t}_{bl})$
300	0.0114	0.0626	(0.0717)	0.0432	0.057	(0.061)	0.1618	0.052	(0.052)
2000	0.0107	0.0319	(0.0355)	0.0405	0.0283	(0.0297)	0.1473	0.0256	(0.0266)
7000	0.0107	0.0198	(0.0217)	0.0400	0.0174	(0.0180)	0.142	0.0157	(0.0161)
30000	0.0107	0.0111	(0.012)	0.0405	0.0096	(0.00993)	0.1385	0.00865	(0.0884)
200000	0.0107	0.00509	(0.00542)	0.0405	0.00433	(0.00446)	0.136	0.0039	(0.00396)
700000	0.0107	0.00299	(0.00317)	0.0406	0.00254	(0.0026)	0.136	0.00227	(0.00231)

* v_s/v_t is the ratio of satellite volume to total drop volume; $\bar{t}_b$, breakup time; $(\bar{t}_{bl})$, breakup time from linear theory.

ary conditions were obtained from the conditions of periodicity. For the solutions presented here, the wavelength is divided into 20 segments and in the first set $s(z, 0) = 1$ and $m(z, 0) = -\sin 2\pi z$.

A typical solution is plotted in Fig. 3. The curves map the contours of a stream segment at various times. Only half the wavelength is shown.

Since there are only two lumped parameters Q and H involved in the model, it appeared to be of interest to study their effects on drop formation, especially on the secondary drop called the "satellite" drop. For this purpose the volume ratios of satellite and parental drops have been computed for various combinations of Q and H . In the range covered, satellites were always formed. As seen from the table, satellite volumes are strongly influenced by λ/d , but only weakly by Q , which is the parameter involving surface tension and disturbance amplitude. The volume ratios and the breakup times are tabulated in Table 1. For comparison, the times predicted by linear theory, Eq. (27), are given in parentheses. The breakup time from the linear theory agreed well with that from the nonlinear solution.

Nonharmonic initial distributions, such as sawtooth and rectangular functions, have been tried and have resulted in little difference in satellite formation.

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