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Optimal Pricing for an Unbounded Queue

Abstract: The maximization of expected reward is considered for an $M_p/M/s$ queuing system with unlimited queue capacity. The system is controlled by dynamically changing the price charged for the facility's service in order to discourage or encourage the arrival of customers. For the finite queue capacity problem, it has been shown that all optimal policies possess a certain monotonicity property, namely, that the optimal price to advertise is a non-decreasing function of the number of customers in the system. The main result presented here is that for the unlimited capacity problem, there exist optimal stationary policies at least one of which is monotone. Also, an algorithm is presented, with numerical results, which will produce an ε-optimal policy for any $\varepsilon > 0$, and an optimal policy if a simple condition is satisfied.

Introduction

The control of the operations of service facilities in order to maximize some economic gain function has been the subject of a considerable number of papers in the recent literature. The arrival process [1-8] and the service mechanism [7, 9-15], are the two general areas of control which are usually considered, and since the latter is often more amenable to control, it has been the subject of most of the published work.

It appears that the optimal control of a service facility in an open market has received somewhat less attention. By open market, we mean to describe a situation in which potential customers are free to take their business to any one of a number of competitors. One approach to modeling such a system is to allow an arriving customer to change his mind and leave if too many customers are already waiting for service [1, 2, 4, 6, 8, 10]. An alternate approach, more closely related to classical supply-demand relationships, is to allow the potential customer to choose whether or not to patronize the facility based on the current advertised price. It is not unreasonable to assume that the higher the price, the less likely it is that a given individual will buy the facility's service. This assumption can be implemented by requiring the mean customer arrival rate (a reflection of demand) to be a decreasing function of price. The model is completed by the addition of a (possibly non-linear) holding cost which penalizes the facility for keeping its customers waiting.

Motivation for this approach is supplied by Leeman [16], who discusses the concept of controlling queues through the use of price and cites several examples:

"When an analyst in operations research encounters a queue, he seldom, if ever, looks into the alternative of introducing or changing a price in order to shorten or eliminate the queue. In practice, of course, prices often are used to reduce queues; examples are peak-load charges for electricity, higher daytime prices for parking, and higher Saturday prices for haircuts. But casual observation suggests that there are many unexplored, yet promising, possibilities of queue reduction through the use of price."

Two of the "possibilities" given in [16] are the use of price to ". . . reduce queues on congested highways and urban streets," and the introduction of take-off and landing charges to reduce queues of aircraft waiting to take off or waiting (stacked) to land. One of the advantages Leeman cites for the price approach is an improvement in the allocation of existing service facilities:

"Those who value services at particular points in space and time bid them away from others who value them less, so that scarce spatial-temporal bottlenecks are allocated to those who value them highly rather than on a first-come, first-served basis or on the basis of centrally established priorities."

One of the important applications of queuing theory today is in modeling and controlling the behavior of virtual memory computing systems. Here, an arriving customer represents a request for system resources (CPU time or space, I/O facilities, etc.) and a service completion represents the partial or complete satisfaction of that request. A major problem area in the operation of a virtual system is the fact that user loads are often grossly unbalanced over a day's time. When such a situation arises, the portion of time the system spends managing the queue (as opposed to doing useful work) increases drastically. This phenomenon is commonly known as "thrashing."

A potentially valuable approach to this load balancing problem is to set prices for services which are based, at least in part, on the level of congestion, i.e., the length of the request queue. Such a policy would charge higher prices during peak load intervals, thus encouraging users to take advantage of the lower rates charged during low usage periods.

Our objective is not simply the control of queues, but rather the maximization of reward through queue controls. The manager, in our model, must carefully balance the consequences of a price change. For example, if he increases the price at some point, the arrival rate of new customers is reduced but, on the positive side, the holding costs tend to decrease and, of course, each arriving customer pays more.

Situations where the present model has some relevance are those in which the customer's primary motivation for selecting the given facility is price, not queue size (although price does give the informed customer a limited amount of queue size information). This preference for the price criterion over that of expected waiting time may come about in various ways: (a) the customer may be ignorant of the queue size; (b) he may be indifferent to the length of his expected wait; (c) he may have made significant personal or economic commitments before he discovers the length of the queue; or (d) he may be persuaded to remain in spite of the queue length.

Examples might include cases where (a) a surrogate (an employee, a written purchase order, an application for a bank loan, a request for time or space in a virtual computer system, etc.) is sent to the facility instead of the actual customer; (b) the value of the customer's time is small compared to that of the service being purchased; (c) the customer hired a babysitter, paid a parking fee, walked several blocks and up three flights of stairs in order to get to the facility; and (d) the facility's holding cost is disbursed (in whole or in part) to the arriving customer as compensation for the inconvenience or expense of his expected wait.

The purpose of this paper is two-fold: first, to quantify the trade-offs mentioned above by means of a mathematical model, in order to obtain dynamic pricing policies which are optimal (or near optimal); and second, to ascertain the form of optimal policies.

Underlying queuing optimization is the more general theory of Markov and Semi-Markov Decision Processes. Our work leans heavily on results of Fox [17], Lippman [18], and Ross [19] for the method used to prove the existence of optimal stationary policies. In the development of our algorithm, the method of Ross [19] and Derman [20] is invaluable.

The physical system can be described as an unbounded $M_p/M/s$ queue with variable arrival rate. The arrival process is Poisson with rate λ_p a strictly decreasing function of the currently advertised price p. The service times for the s servers are independent, exponentially distributed random variables with mean $1/\mu$. Control of the system is effected by increasing or decreasing the price p in order to discourage or encourage the arrival of customers. At each transition (customer arrival or service completion), the manager of the facility must choose one of a finite number of prices to advertise until the next transition.

Definitions and system operation

The queuing reward system described above can be modeled as a Semi-Markov Decision Process (SMDP) with action space P given by $P = \{p_1, \cdots, p_K\}$, where $0 \le p_1 < p_2 < \cdots < p_K < \infty$, $K < \infty$, and state space $I = \{0, 1, \cdots\}$. Here, K is the number of prices available to the manager of the facility. The number of servers in the system is denoted by s. If the manager choses action $p \in P$ when the system is in state i, the transition probabilities are given by

$$q_{i,i+1}(p) = \lambda_p/[(i \wedge s)\mu + \lambda_p], \tag{1}$$

$$q_{i+1,i}(p) = 1 - q_{i+1,i+2}(p), \qquad i = 0, 1, \cdots, \tag{2}$$

where $0 < \lambda_{p_K} < \cdots < \lambda_{p_1} < s\mu$, and where $(i \wedge s)$ denotes the minimum of i and s. The net reward received immediately following a customer arrival when there are i customers in the system is $p - c_i$; p is the currently advertised price and c_i represents a holding cost. The assumption that $\lambda_{p_1} < s\mu$ ensures that all states are positive recurrent.

The assumption that the holding cost c_i is a lump sum (associated with an arriving customer) is made for computational convenience. It is innocuous since the long range average rate of return per unit time will not be influenced by when (during the customer's stay in the system) this cost is assessed. The holding cost function $c : I \to \mathscr{R}$ is assumed to satisfy

$$0 \le c_0 = c_1 = \cdots = c_{s-1} \le c_s \le \cdots < p_K. \tag{3}$$

The equalities above are reasonable since all customers who arrive when there is at least one server free have the same expected time in the system, namely, $1/\mu$. We require the costs to be bounded by p_K in order to ensure the possibility of a positive reward for each arriving customer. There is no cost or reward associated with a departure (service completion). If, when in state i, action

p is chosen, then the time until the next transition is an exponentially distributed random variable with mean $1/[(i \wedge s)\mu + \lambda_p]$.

A policy A is a sequence $A_1, A_2, \cdots$ of decision rules, where the nth decision rule A_n tells how to select an action in P after completion of the $(n-1)$th transition (note that transition zero occurs at time zero). A stationary policy A is a map from the state space $I = \{0, 1, \cdots\}$ to the action space P; i.e., a stationary policy $A = (a_0, a_1, \cdots)$ always chooses action a_i whenever the system is in state i.

For convenience, we summarize the system operation: When in state i, the facility advertises a price $p \in P$ for its services. The arrival process is then Poisson with rate λ_p. When a potential customer arrives at the facility, he pays the facility p units. He is then served by any nonbusy server. If all servers are busy $(s \leq i)$, he joins the queue. When the facility accepts a customer, it is assessed a cost c_i. The queue is emptied on a first-come, first-served basis with no priorities, and the system is reviewed when a new customer is accepted and when a service is completed. At the time of each review, a new price can be selected.

Criteria for optimality

We choose the expected average rate of return over an infinite time horizon as our measure of performance. Let A be any pricing policy (not necessarily stationary), and let $R(t)$ denote the total reward earned by time t. Also, let d_n be the reward earned during the nth transition interval. Define

$$\phi^1(A)(i) = \liminf_{t\to\infty} \mathrm{E}_A\left[\frac{R(t)}{t} \,\middle|\, X_0 = i\right], \text{ and} \tag{4}$$

$$\phi^2(A)(i) = \liminf_{n\to\infty} \frac{\mathrm{E}_A\left[\sum_{j=1}^{n} d_j \mid X_0 = i\right]}{\mathrm{E}_A\left[\sum_{j=1}^{n} t_j \mid X_0 = i\right]}, \tag{5}$$

where X_{j-1} is a random variable representing the state of the system just before the jth transition, and t_j is the time interval between the $(j-1)$th and jth transition. We will demonstrate the existence of a stationary policy which is optimal (maximizes ϕ^1 and ϕ^2 over the set of all policies).

We obtain two main results. The first of these shows that the model with infinite queue capacity is essentially a limiting case of the model with finite queue capacity. In particular, we show that

$$g^* = \lim_{M\to\infty} g_M^*, \tag{6}$$

where g^* and g_M^* denote the optimal expected rates of return for the problems of size $M = \infty$ and $M < \infty$ and, consequently, that there is an optimal stationary policy A^* which satisfies the monotonicity property $a_0^* \leq a_1^* \leq \cdots$.

The second result is an algorithm which for any $\varepsilon > 0$ determines an ε-optimal policy in a finite number of steps. Also, a finite algorithm is developed for finding optimal solutions under the added condition that $c_i = c_{i+1}$ for i sufficiently large. Finally, we demonstrate that every optimal stationary policy is monotone if the above condition does not hold, whereas if this condition does hold, we give an example of an optimal stationary policy which is not monotone.

Existence of an optimal stationary policy

For the case $M < \infty$ [23] we were able to employ the results of Fox [17], Lippman [18], and Ross [21] to establish the existence of a stationary policy A^* which simultaneously maximized $\phi^1(A)$ and $\phi^2(A)$. Our principal objective in this section is to establish the same result when $M = \infty$.

We begin now by showing that the optimal rate of return for the model with $M < \infty$ converges to the optimal rate of return for the model with $M = \infty$.

Let $R_0(A)$ be the expected reward earned under the stationary policy A, starting from state 0, until the first return to state 0, and let $T_0(A)$ be the expected elapsed time for this event. This yields

$$R_0(A) = (a_0 - c_0) + \sum_{i=1}^{\infty} (a_i - c_i) \prod_{j=1}^{i} \frac{\rho_j}{(j \wedge s)} \tag{7}$$

and

$$T_0(A) = \frac{1}{\mu\rho_0}\left[1 + \sum_{i=0}^{\infty} \prod_{j=0}^{i} \frac{\rho_j}{(s \wedge (j+1))}\right], \tag{8}$$

and so, using a result of John's and Miller [22] and Lipman [18], we have

$$\phi^1(A) = \frac{R_0(A)}{T_0(A)}, \tag{9}$$

and

$$g^* = \sup_{A} \phi^1(A) = \sup_{A\in\Delta} \phi^1(A), \tag{10}$$

where Δ is the set of all stationary policies.

Theorem 1. Let g_M^* be the maximum expected rate of return for a model of size $M < \infty$, then $g^* = \lim_{M\to\infty} g_M^*$.

Proof. We assume (without loss of generality) that $M > s$. Let $A^* = (a_0, a_1, \cdots)$ be a policy such that $g^* = \phi^1(A^*)$. Such a policy exists since $g^* = \sup_A \phi^1(A)$ and Lippman [18] proves that under conditions satisfied by our model, there exists a ϕ^1-optimal policy which is stationary. Thus,

$$g^* - g_M{}^* \le \mu\rho_0\left[\frac{(a_0 - c_0) + \sum_{i=1}^{\infty} (a_i - c_i) \prod_{j=1}^{i} \frac{\rho_j}{(j \wedge s)}}{1 + \sum_{i=0} \prod_{j=0}^{i} \frac{\rho_j}{(s \wedge (j+1))}}\right]$$

$$- \mu\rho_0\left[\frac{(a_0 - c_0) + \sum_{i=1}^{M-1} (a_i - c_i) \prod_{j=1}^{i} \frac{\rho_j}{(j \wedge s)}}{1 + \sum_{i=0}^{M-1} \prod_{j=0}^{i} \frac{\rho_j}{(s \wedge (j+1))}}\right]$$

$$\le \frac{\mu\rho_0 \sum_{i=M}^{\infty} (a_i - c_i) \prod_{j=1}^{i} \frac{\rho_j}{(j \wedge s)}}{1 + \sum_{i=0}^{\infty} \prod_{j=0}^{i} \frac{\rho_j}{(s \wedge (j+1))}}$$

$$\le \frac{\mu\rho_0 (p_k - c_0) \sum_{i=M}^{\infty} \frac{1}{s! s^{i-s}} \prod_{j=1}^{i} \rho_j}{1 + \sum_{i=0}^{\infty} \prod_{j=0}^{i} \frac{\rho_j}{s}}$$

$$= \frac{\mu(p_k - c_0)\left(\frac{s^{s+1}}{s!}\right) \sum_{i=M}^{\infty} \prod_{j=0}^{i} \left(\frac{\rho_j}{s}\right)}{1 + \sum_{i=0}^{\infty} \prod_{j=0}^{i} \left(\frac{\rho_j}{s}\right)}$$

$$\le \left[\frac{\mu(p_k - c_0)\left(\frac{s^{s+1}}{s!}\right)\left(1 - \frac{\rho'}{s}\right)}{1 - \frac{\rho''}{s}}\right]\left(\frac{\rho''}{s}\right)^{M+1}, \tag{11}$$

where $\rho' = \min_{a \in P} \lambda_a/\mu$, and $\rho'' = \max_{a \in P} \lambda_a/\mu$. Thus, $g^* - g_M{}^* \le bx^{M+1}$, with $0 < b < \infty$ and $0 < x < 1$.

Now let $A_M{}^* = (a_0, a_1, \cdots, a_M)$ be an optimal policy for the finite model of size M. The existence of $A_M{}^*$ is guaranteed by the results of Low [23]. Using a similar analysis, we then have

$$g_n{}^* - g^* \le \left[\frac{\mu(p_k - c_0)\left(\frac{s^{s+1}}{s!}\right)\left(1 - \frac{\rho'}{s}\right)}{1 - \frac{\rho''}{s}}\right]\left(\frac{\rho''}{s}\right)^{M+2}. \tag{12}$$

Thus, $g_M{}^* - g^* \le bx^{M+2}$, which implies $|g^* - g_M{}^*| \le bx^{M+1}$, with $0 < b < \infty$, $0 < x < 1$, and both b and x independent of M. This completes the proof. Q.E.D.

Define the functions $F : \{-1, 0, 1, \cdots\} \to \mathscr{R}$ and $h : \{-1, 0, 1, \cdots\} \to \mathscr{R}$ by

$$F(-1) = 0, \tag{13}$$

$$F(i) = \max_{a \in P} \left\{a - c_i - \frac{g^*}{\lambda_a} + \frac{(i \wedge s)\mu}{\lambda_a} F(i-1)\right\}, \quad i = 0, 1, \cdots, \tag{14}$$

$$h(-1) = 0, \text{ and} \tag{15}$$

$$h(i+1) = h(i) - F(i), \qquad i = -1, 0, 1, \cdots. \tag{16}$$

We now show that F is bounded and that h is linearly bounded.

Lemma 1. There exists a constant $B < \infty$ such that $0 \le F(i) \le B$, $i = 0, 1, \cdots$.

Proof. We first show that F is non-negative. Let $i \in I$ be fixed. Since $F(i, g)$ is continuous in g, $g^* = \lim_{M \to \infty} g_M{}^*$ by Theorem 1 and since in [23] it was shown that $F(i, g_M{}^*) > 0$ for all $M > i$, we can conclude that

$$F(i) = F(i, g^*) = \lim_{M \to \infty} F(i, g_M{}^*) \ge 0. \tag{17}$$

We now show that F is bounded above. Denote by a_i the maximizing price in (14) and let $\rho_i = \lambda_{a_i}/\mu$ for $i = 0, 1, \cdots$. If we solve for $a_0, a_1, \cdots$, we obtain

$$F(i) = \left\{\mu\rho_0\left[\frac{(a_0 - c_0) + \sum_{j=1}^{i} (a_j - c_j) \prod_{k=1}^{j} \frac{\rho_k}{(k \wedge s)}}{1 + \sum_{j=0}^{i-1} \prod_{k=0}^{j} \frac{\rho_k}{(s \wedge (k+1))}}\right] - g^*\right\}$$

$$\times \left[\frac{1 + \sum_{j=0}^{i-1} \prod_{k=0}^{j} \frac{\rho_k}{(s \wedge (k+1))}}{\mu\rho_i \prod_{k=0}^{i-1} \frac{\rho_k}{(s \wedge (k+1))}}\right]. \tag{18}$$

Using $g^* \ge \phi^1(a_0, a_1, \cdots)$ and considerable algebraic manipulation, we obtain

$$F(i) \le \left\{\left[(a_0 - c_0)\rho_0 + \rho_0 \sum_{j=1}^{i} (a_j - c_j) \prod_{k=1}^{j} \frac{\rho_k}{(k \wedge s)}\right]\right.$$

$$\times \left(\frac{1}{s}\right)\left[1 + \sum_{j=i+1}^{\infty} \prod_{k=i+1}^{j} \left(\frac{\rho_k}{s}\right)\right]$$

$$- \left[\sum_{j=i+1}^{\infty} (a_j - c_j) \prod_{k=i+1}^{j} \left(\frac{\rho_k}{s}\right)\right]$$

$$\left.\times \left[1 + \sum_{j=0}^{i-1} \prod_{k=0}^{j} \frac{\rho_k}{(s \wedge (k+1))}\right]\right\}$$

$$\times \left[\frac{1}{1 + \sum_{j=0}^{\infty} \prod_{k=0}^{j} \frac{\rho_k}{(s \wedge (k+1))}}\right], \tag{19}$$

where we have assumed (without loss of generality) that $i > s$. Now, $a_j \ge c_j$ for all $j \in$ I, for if not, we would have

$$0 \le F(j) = a_j - c_j - \frac{g^*}{\lambda_{a_j}} + \frac{(j \wedge s)\mu}{\lambda_{a_j}} F(i-1)$$

$$< \frac{(j \wedge s)\mu F(j-1) - g^*}{\lambda_{a_j}}$$

$$\le \frac{(j \wedge s)\mu F(j-1) - g^*}{\lambda_{p_k}}$$

$$< (p_k - c_j) - \frac{g^*}{\lambda_{p_k}} + \frac{(j \wedge s)\mu}{\lambda_{p_k}} F(j-1) \le F(j). \tag{20}$$

Hence,

$$F(i) \le \frac{\rho_0}{s} \left[\frac{(a_0 - c_0) + \sum_{j=1}^{i} (a_j - c_j) \prod_{k=1}^{j} \frac{\rho_k}{(k \wedge s)}}{1 + \sum_{j=0}^{\infty} \prod_{k=0}^{j} \frac{\rho_k}{(s \wedge (k+1))}} \right] \times \left[1 + \sum_{j=1}^{\infty} \left(\frac{\rho''}{s}\right)^j \right]$$

$$\le \frac{g^*}{s\mu \left(1 - \frac{\rho''}{s}\right)} < \infty, \tag{21}$$

where we have used ρ'' to indicate λ_{p_1}/μ. This completes the proof.

Lemma 2. For any $i \in I$, h satisfies

$$h(i) = \max_{a \in P} \left\{ \frac{\lambda_a}{(i \wedge s)\mu + \lambda_a} [a - c_i + h(i+1)] + \frac{(i \wedge s)\mu}{(i \wedge s)\mu + \lambda_a} h(i-1) - \frac{g^*}{(i \wedge s)\mu + \lambda_a} \right\}, \tag{22}$$

and

$$|h(i)| \le B\, i. \tag{23}$$

Proof. By definition, we have

$$h(i+1) = h(i) - \max_{a \in P} \left\{ a - c_i - \frac{g^*}{\lambda_a} + \frac{(i \wedge s)\mu}{\lambda_a} [h(i-1) - h(i)] \right\}, \tag{24}$$

which implies

$$h(i) \left[1 + \frac{(i \wedge s)\mu}{\lambda_a} \right] \ge a - c_i - \frac{g^*}{\lambda_a} + \frac{(i \wedge s)\mu}{\lambda_a} h(i-1) + h(i+1) \tag{25}$$

for all a $\in P$, with equality for $a = a_i$. Therefore,

$$h(i) \ge \frac{\lambda_a}{(i \wedge s)\mu + \lambda_a} [a - c_i + h(i+1)] + \frac{(i \wedge s)\mu}{(i \wedge s)\mu + \lambda_a} h(i-1) - \frac{g^*}{(i \wedge s)\mu + \lambda_a} \tag{26}$$

for all $a \in P$, with equality for $a = a_i$, which implies the first result. Equation (23) follows immediately from Lemma 1 and equations (18) and (22).

Lemma 3. If X_n is the state of the system after n transitions, then for each $i \in I$ there is a constant m_i such that

$$\sup_n \sup_A E_A[|h(X_n)| : X_0 = i] \le m_i < \infty. \tag{27}$$

Proof. In view of (23), it is clearly sufficient to show that for each $i \in I$ there exists a positive number m_i such that

$$\sup_n \sup_A E_A[X_n : X_0 = i] \le m_i < \infty. \tag{28}$$

It is apparent that the stochastic process $\{X_n : A, X_0 = i\}$ is a (possibly non-stationary) random walk over the non-negative integers with a reflecting barrier at the origin. If $a_{in} \in P$ is the price selected by A when in state i, just after the nth transition, then the probability $f_{jn} = Pr\{X_{n+1} = j+1 : X_n = j\}$ is given by $(f_{0n} \equiv 1)$

$$f_{jn} = \frac{\lambda_{a_{jn}}}{(j \wedge s)\mu + \lambda_{a_{jn}}} \le \frac{\lambda_{p_1}}{\mu + \lambda_{p_1}}, \text{ for } \begin{cases} j = 0, 1, \cdots, \\ n = 0, 1, \cdots \end{cases} \tag{29}$$

Consider a sequence $U_0, U_1, \cdots$ of random variables drawn from the uniform distribution over (0, 1). Let $X_0' = i$, and

$$X_{n+1}' = X_n' + Y_n', \qquad n = 0, 1, \cdots \tag{30}$$

where

$$Y_n' = \begin{cases} 1 \text{ if } U_n > 1 - r_{X_n} \\ -1 \text{ otherwise.} \end{cases} \tag{31}$$

It is clear that $\{X_n' : A, X_0' = \mathrm{i}\}$ has the same finite dimensional distributions as $\{X_n : A, X_0 = \mathrm{i}\}$. Now define $X_0 = i$ and

$$X_{n+1} = X_n + Y_n, \qquad n = 0, 1, \cdots, \tag{32}$$

where

$$Y_n = \begin{cases} 1 \text{ if } U_n > \dfrac{\mu}{\mu + \lambda_{p_1}} \\ -1 \text{ otherwise.} \end{cases} \tag{33}$$

By construction, $Y_n \ge Y_n'$, $n = 0, 1, \cdots$, so that $X_n \ge X_n'$. Consequently, we need only consider the process $\{X_n : X_0 = i\}$, where

$$f_{jn} = \Pr\{X_{n+1} = j+1 : X_n = j\} = \begin{cases} 1 \text{ if } j = 0 \\ r \equiv \dfrac{\lambda_{p_1}}{\mu + \lambda_{p_1}}, \end{cases} \quad j = 1, 2, \cdots, \tag{34}$$

and

$$q_{jn} = Pr\{X_{n+1} = j-1 : X_n = j\} = \begin{cases} 0 \text{ if } j = 0 \\ q \equiv 1 - r, \end{cases} \quad j = 1, 2, \cdots, \tag{35}$$

because

$$E_A[X_n : A, X_0 = i] \le E[X_n : X_0 = i].$$

Kac [24; pp. 378-391] has derived the probability mass function for X_n given $X_0 = i$. It is not difficult to show that this distribution has a finite mean for each i, which completes the proof. Q.E.D.

We can now assert the existence of an optimal stationary policy which is determined by (22).

Theorem 2. For each $i \in I$,

$$g^* = \phi^2(A^*)(i) = \sup_A \phi^2(A)(i) = \sup_A \phi^1(A)(i) = \phi^1(A^*)(i), \qquad (36)$$

where A^* is any policy which, for each $i \in I$, maximizes the right hand side of (22), or, equivalently, (14).

Proof. Combining Lemmas 2 and 3 with Ross [21 (Theorem 7.7, p 163)] obtain the second equality. The other equalities follow from (10), Ross [21] (Theorem 7.6 p 162), and the existence of a stationary ϕ^1-optimal policy. Q.E.D.

We complete this section with a theorem which, in conjunction with Theorem 2, demonstrates that a stationary policy A is optimal if and only if it solves (14).

Theorem 3. Suppose $\phi^1(\mathrm{A}) = g^*$ for some stationary policy $A = (a_0, a_1, \cdots)$. Then a_i must maximize the right hand side of (14).

Proof. Suppose for some $i_0 \in I$ that

$$F(i_0) = a_{i_0} - c_{i_0} - \frac{g^*}{\lambda_{i_0}} + \frac{(i_0 \wedge s)\mu}{\lambda_{i_0}} F(i_0 - 1) + \delta$$

for $\delta > 0$. Then, for $i > i_0$, equation (18) gives us

$$F(i) \geq \mu\rho_0 \left[\frac{(a_0 - c_0) + \sum_{j=1}^{i} (a_j - c_j) \prod_{k=1}^{j} \frac{\rho_k}{(k \wedge s)}}{\mu\rho_i \prod_{k=0}^{i-1} \frac{\rho_k}{(s \wedge (k+1))}} \right] - g^* \left[\frac{1 + \sum_{j=0}^{i-1} \prod_{k=0}^{j} \frac{\rho_k}{(s \wedge (k+1)}}{\mu\rho_i \prod_{k=0}^{i-1} \frac{\rho_k}{(s \wedge (k+1)}} \right] + \left[\frac{\prod_{k=1}^{i_0} \frac{\rho_k}{(k \wedge s)}}{\mu\rho_i \prod_{k=0}^{i-1} \frac{\rho_k}{(s \wedge (k+1))}} \right] \delta, \qquad (37)$$

which implies

$$\left[\mu\rho_i \prod_{k=0}^{i-1} \frac{\rho_k}{(s \wedge (k+1))} \right] F(i) \geq \mu\rho_0 \left[(a_0 - c_0) + \sum_{j=1}^{i} (a_j - c_j) \prod_{k=1}^{j} \frac{\rho_k}{(k \wedge s)} \right] - g^* \left[1 + \sum_{j=0}^{i-1} \prod_{k=0}^{j} \frac{\rho_k}{(s \wedge (k+1))} \right] + \left[\prod_{k=1}^{i_0} \frac{\rho_k}{(k \wedge s)} \right] \delta. \qquad (38)$$

Since both sides of this inequality are bounded, we may pass to the limit. This yields

$$0 \geq \mu\rho_0 \left[(a_0 - c_0) + \sum_{j=1}^{\infty} (a_j - c_j) \prod_{k=1}^{j} \frac{\rho_k}{(k \wedge s)} \right] - g^* \left[1 + \sum_{j=0}^{\infty} \prod_{k=0}^{j} \frac{\rho_k}{(s \wedge (k+1))} \right] + \left[\prod_{k=1}^{i_0} \frac{\rho_k}{(k \wedge s)} \right] \delta, \qquad (39)$$

which implies

$$g^* \geq \phi^1(A) + \left[\frac{\prod_{k=1}^{i_0} \frac{\rho_k}{(k \wedge s)}}{1 + \sum_{j=0}^{\infty} \prod_{k=0}^{j} \frac{\rho_k}{(s \wedge (k+1))}} \right] \delta, \qquad (40)$$

which contradicts our hypothesis. Q.E.D.

Monotonicity of A*

We are now in a position to present our main result for the infinite queue capacity model: namely, $a_0^* \leq a_1^* \leq \cdots$ for some optimal stationary policy. For the purpose of this section we need some additional notation. Let $a_i^*(M)$ be an optimal price for state i in the finite problem of size M, and define

$$A_M^*(i) = [a_0^*(M), a_1^*(M), \cdots, a_i^*(M)] \qquad (41)$$

for $i < M$. We formalize the monotonicity result in the following theorem.

Theorem 4. There is an optimal policy A^* which is monotone; moreover, there is an integer r such that

$$a_0^* \leq a_1^* \leq \cdots \leq a_r^* = a_{r+1}^* = \cdots. \qquad (42)$$

Proof. Fix $i \in I$, and consider the sequence $\langle A_M^*(i) \rangle$. Since there are but $K^{i+1} < \infty$ possible values for each vector $A_M^*(i)$ in our sequence, there is a subsequence $\langle M_j \rangle$ for which $A_{M_j}^*(i)$ is constant. Also, Theorem 1 and the continuity of F (k, g) with respect to g imply that

$$F(k) = \lim_{j \to \infty} F(k, g_{M_j}^*), \qquad k = -1, 0, \cdots, i. \qquad (43)$$

Consequently, if follows from these two facts that if price $a_k^*(M_j)$ is chosen at state k, then

$$F(k) = \lim_{j \to \infty} \left[a_k^*(M_j) - c_k - \frac{g^*_{M_j}}{\lambda_{a_k^*(M_j)}} + \frac{(k \wedge s)\mu}{\lambda_{a_k^*(M_j)}} F(k-1, g_{M_j}^*) \right] = a_k^*(M_1) - c_k - \frac{g^*}{\lambda_{a_k^*(M_1)}} + \frac{(k \wedge s)\mu}{\lambda_{a_k^*(M_1)}} F(k-1). \qquad (44)$$

Hence, we can conclude from Theorem 2 that $a_k^*(M_1)$ is an optimal price for state k, $k = 0,1,\cdots,j$. Since it was shown in [23] that $a_k^*(M_1) \leq a_{k+1}^*(M_1)$, $k - 0,\cdots$, i − 1, we have

$$a_0^* \leq a_1^* \leq \cdots \leq a_i^* \tag{45}$$

for each $i \in I$. Q.E.D.

Finding ε-optimal policies (ε > 0)

We now show that an ε-*optimal* policy can be determined for any $\varepsilon > 0$ in a finite number of steps. Define $\tilde{g}_M$ as the expected rate of return for the infinite model when the policy

$$\tilde{A}_M = (a_0, a_1, \cdots, a_{M-1}, p_K, p_K, \cdots) \tag{46}$$

is used, where $(a_0, \cdots, a_{M-1})$ is optimal for the problem of size M. Also define ε_M by

$$\varepsilon_M = \left[\frac{\mu(p_k - c_0)\left(\frac{s^{s+1}}{s!}\right)\left(1 - \frac{\rho'}{s}\right)}{1 - \frac{\rho''}{s}}\right]\left(\frac{\rho''}{s}\right)^{M+1} \tag{47}$$

where $\rho' = \min_{a\in P} \lambda_a/\mu$ and $\rho'' = \max_{a\in P} \lambda_a/\mu$.

Theorem 5. Let $\varepsilon > 0$ be given. Then there is an $M < \infty$ such that $\tilde{A}_M$ is ε-optimal.

Proof. We know $g^* - g_M^* \leq \varepsilon_M$ by Theorem 1. Similarly, it is easy to show that $g_M^* - \tilde{g}_M \leq \varepsilon_M$. Therefore,

$$g^* - \tilde{g}_M = (g^* - g_M^*) + (g_M^* - \tilde{g}_M) \leq \varepsilon_M + \varepsilon_M = 2\varepsilon_M. \tag{48}$$

Q.E.D.

Thus, for any $\varepsilon > 0$, an ε-optimal policy can be found by first using equation (47) to determine a sufficiently large M, and then using the results of [23] to solve a finite problem of size M.

We conclude this section by showing how a sequence of policies may be generated which converges to an optimal policy. The convergence is, of course, not uniform.

Theorem 6. Let $\varepsilon_n = 1/n$, and suppose $\langle \tilde{A}_{M_n} \rangle$ is a sequence of ε_n-optimal policies. Then for each $i \in I$, there exists an $N_i < \infty$ such that for every $n > N_i$, $a_i(M_n) \in Q_i$, where $Q_i \subseteq P$ is the set of optimal prices for state i, i.e., maximizers of (14). In particular, if a_i^* is unique, then $a_i(M_n) \to a_i^*$ as $n \to \infty$.

Proof. Suppose for some $i \in I$, that no such N_i exists. Then there must exist a subsequence $\langle n_j \rangle$ for which $a_i(M_{n_j})$ is not a member of Q_i. Further, since P is finite, there must exist a subsequence $\langle n_{j_k} \rangle$ over which $a_i(M_{n_{j_k}})$ is constant. But by Theorem 1 and the proof of Theorem 4, this implies that this constant maximizes (14), and by Theorem 2, any such price must be optimal. Q.E.D.

Finding optimal policies

We now treat the more difficult computational problem that arises when $\varepsilon = 0$. We will use the following condition:

C1. $c_{k_0} = c_{k_0+1} = \cdots$ for some $k_0 \in I$, with $k_0 \geq s$.

As in the finite queue capacity case [23], we extend the domain of the function F defined in (13) and (14) to $\{-1, 0, 1, \cdots\} \times \mathscr{R}$ as follows:

$$F(-1, \cdot) = 0, \tag{49}$$

and

$$F(i, g) = \max_{a\in P}\left[a - c_i - \frac{g}{\lambda_a} + \frac{(i \wedge s)\mu}{\lambda_a} F(i-1, g)\right] \tag{50}$$

for i $\in I$, and $g \in \mathscr{R}$. Clearly, $F(\cdot, g^*) = F(\cdot)$.

We show that under condition C1, an optimal policy A^* can be obtained in a finite number of steps. Theorem 4 guarantees the existence of an optimal stationary policy of the form

$$a_0^* \leq a_1^* \leq \cdots \leq a_{r-1}^* \leq a_r^* = a_{r+1}^* = \cdots \tag{51}$$

for some $r \in I$. Define a_∞^* by

$$a_\infty^* = a_r^* = a_{r+1}^* = \cdots. \tag{52}$$

Our first goal is to determine this price. Toward this end we prove in Lemma 4 that $\lim_{i\to\infty} F(i)$ exists; in Theorem 7 we characterize the form of this limit and show how it is useful in finding a_∞^*.

Lemma 4. Let r be the smallest index ($r \geq s$) such that $a_r^* = a_{r+1}^* = \cdots$. Then $F(i) \geq F(i-1)$ for all $i \geq r$.

Proof. We define the difference Δ_i by

$$\Delta_i = F(i) - F(i-1), \tag{53}$$

and we denote $\lambda_{a_\infty^*}$ by λ_∞, and λ_∞/μ by ρ_∞. Fix $i \geq r$. Since $a_i^* = a_{i+1}^* = \cdots = a_\infty^*$, we have

$$F(i) = a_\infty^* - c_i - \frac{g^*}{\lambda_\infty} + \frac{s\mu}{\lambda_\infty} F(i-1), \tag{54}$$

and

$$F(i+1) = a_\infty^* - c_{i+1} - \frac{g^*}{\lambda_\infty} + \frac{s\mu}{\lambda_\infty} F(i). \tag{55}$$

Subtracting, we get

$$\Delta_{i+1} = -(c_{i+1} - c_i) + \frac{s\mu}{\lambda_\infty} \Delta_i. \tag{56}$$

By iterating this procedure we obtain

$$\Delta_{i+t} = -\Big[(c_{i+t} - c_{i+t-1}) + (c_{i+t-1} - c_{i+t-2})\left(\frac{s\mu}{\lambda_\infty}\right) + \cdots + (c_{i+1} - c_i)\left(\frac{s\mu}{\lambda_\infty}\right)^{t-1}\Big] + \left(\frac{s\mu}{\lambda_\infty}\right)^t \Delta_i, \text{ for } t = 0, 1, \cdots. \tag{57}$$

Multiplying both sides by $(\rho_\infty/s)^t$ yields

$$\Delta_i = \left[(c_{i+1} - c_i)\left(\frac{\rho_\infty}{s}\right) + \cdots + (c_{i+t} - c_{i+t-1})\left(\frac{\rho_\infty}{s}\right)^t\right] + \left(\frac{\rho_\infty}{s}\right)^t \Delta_{i+t}. \tag{58}$$

By Lemma 1, $|\Delta_{i+t}| \le B < \infty$ for some number B and all $t = 0, 1, \cdots$.

Thus,

$$\lim_{t\to\infty} \left(\frac{\rho_\infty}{s}\right)^t \Delta_{i+t} = 0, \tag{59}$$

so that

$$\Delta_i = \sum_{j=i}^{\infty} (c_{j+1} - c_j)\left(\frac{\rho_\infty}{s}\right)^{j-i+1}, \tag{60}$$

which is non-negative by the montonicity of the cost function c. Q.E.D.

We know by Lemmas 1 and 4 that the sequence $\langle F(i)\rangle$ has a unique limit point which we denote by $F(\infty)$. Also, denote the unique limit of $\langle c_i\rangle$ by c_∞. The next theorem characterizes $F(\infty)$.

Theorem 7. The number $F(\infty)$ satisfies

$$F(\infty) = \min_{a\in P} \left\{\frac{g^* - (a - c_\infty)\lambda_a}{s\mu - \lambda_a}\right\}. \tag{61}$$

Furthermore, $a_\infty{}^*$ can be chosen to be the smallest of the prices solving (61), and one of the following must hold:

a. If $c_\infty > c_i$ for all $i \in I$, then $a_\infty{}^*$ is unique, or
b. If *C1* holds, then $a_\infty{}^*$ can be chosen to be any price minimizing (61).

Proof. From equation (14) and the definition of Δ_i, we obtain (for $i \ge s$)

$$F(i) = \max_{a\in P} \left[a - c_\infty - \frac{g^*}{\lambda_a} + \frac{s\mu}{\lambda_a} F(i) + (c_\infty - c_i) - \frac{s\mu}{\lambda_a}\Delta_i\right]. \tag{62}$$

Thus,

$$F(i) \ge a - c_\infty - \frac{g^*}{\lambda_a} + \frac{s\mu}{\lambda_a} F(i) + (c_\infty - c_i) - \frac{s\mu}{\lambda_a}\Delta_i \tag{63}$$

for all $a \in P$, with equality for $a = a_i{}^*$. This leads to

$$F(i) = \min_{a\in P} \left[\frac{g^* - (a - c_\infty)\lambda_a}{s\mu - \lambda_a} + \frac{\Delta_i - (c_\infty - c_i)\dfrac{\rho_a}{s}}{1 - \dfrac{\rho_a}{s}}\right]. \tag{64}$$

By Theorem 4, we know that there exists $r \in I$ such that $a_\infty{}^*$ is optimal for all $i \ge r \ge s$. Thus,

$$F(i) = \frac{g^* - (a_\infty{}^* - c_\infty)\lambda_\infty}{s\mu - \lambda_\infty} + \frac{\Delta_i - (c_\infty - c_i)\dfrac{\rho_\infty}{s}}{1 - \dfrac{\rho_\infty}{s}} \tag{65}$$

for all i sufficiently large. As i approaches infinity, the first term on the right is independent of i, and the second term approaches zero. By taking the limit, we see that $F(\infty)$ has the required form, and that $a_\infty{}^*$ solves (61). To prove the second assertion, we write

$$F(\infty) = \max_{a\in P} \left\{\left[a - c_i - \frac{g^*}{\lambda_a} + \frac{s\mu}{\lambda_a} F(i-1)\right] - (c_\infty - c_i) + \frac{s\mu}{\lambda_a}[F(\infty) - F(i-1)]\right\}. \tag{66}$$

If $c_\infty > c_i$ for all $i \in I$, then $F(\cdot)$ is strictly increasing for sufficiently large i by (61). Now suppose $a' \in P$ minimizes (61). Then $a' \ge a_i{}^*$ for all $i \ge r$, which implies $a' \ge a_\infty{}^*$. The reason for this is that the maximand above is the sum of the maximand of $F(i)$ and a function strictly increasing in a. By the same line of reasoning, $a_{i+1}{}^* \ge a_i{}^*$ for i sufficiently large, which implies $a_\infty{}^*$ must be unique.

If C1 holds, then $F(i) = F(\infty)$ for all i sufficiently large, and it is clear that any price minimizing (61) will maximize (14). Q.E.D.

We now prove two necessary and sufficient conditions for g to equal g^*. These will be used later in the development of a finite algorithm.

Theorem 8. Let $g \in \mathscr{R}$ be given. Then the following three statements are equivalent:

a. $g = g^*$,
b. $g = \phi^1(A(g))$, and
c. $\lim_{i\to\infty} F(i, g)$ exists.

Moreover, if $\bar{A}$ is a given stationary policy, then $\bar{A}$ is optimal if and only if $\bar{A} = A(\phi^1(\bar{A}))$, where $A(\cdot)$ represents a policy generated by $F(i, \cdot)$ for $i = 0, 1, \cdots$.

Proof.

a $\Rightarrow$ *b*. If $g = g^*$, then

$$\phi^1(A(g)) = \phi^1(A(g^*)) = \phi^1(A^*) = g^* = g. \tag{67}$$

b $\Rightarrow$ *a*. Suppose $g = \phi^1(A(g))$ for some g. Then, using the representation of F given in the proof of Lemma 1, and assuming (without loss of generality) that $i > s$, we can write

$$F(i, g) = \max_{(a_0, \cdots, a_i)} \left\{ \left[(a_0 - c_0)\rho_0 + \rho_0 \sum_{j=1}^{i} (a_j - c_j) \prod_{k=1}^{j} \frac{\rho_k}{(k \wedge s)} \right] \times \left[\frac{1}{s}\right]\left[1 + \sum_{j=i+1}^{\infty} \prod_{k=i+1}^{j} \left(\frac{\rho_k}{s}\right)\right] - \left[\sum_{j=i+1}^{\infty} (a_j - c_j) \prod_{k=i+1}^{j} \left(\frac{\rho_k}{s}\right)\right] \times \left[1 + \sum_{j=0}^{i-1} \prod_{k=0}^{j} \frac{\rho_k}{(s \wedge (k+1))}\right] \times \left[\frac{1}{1 + \sum_{j=0}^{\infty} \prod_{k=0}^{j} \frac{\rho_k}{(s \wedge (k+1))}}\right] \right\}. \tag{68}$$

Again, we have written a_i for $a_i(g)$, and ρ_i for λ_{a_i}/μ.

If $g = g^*$, we are done. If $g > g^*$, then $\phi^1(A(g)) > g^*$, which is impossible. Suppose then that $g < g^*$. Since $g^* = \lim g_M^*$, we must have $g < g_M^*$ for all sufficiently large M. Thus, by the results of the finite queue case [23], $a_0(g) \le a_1(g) \le \cdots$, which implies that for some finite $r > s$, $a_r(g) = a_{r+1}(g) = \cdots$. Call this common value a_∞. Then for i sufficiently large, we have

$$F(i, g) = \max_{(a_0, \cdots, a_i)} \left\{ \frac{\rho_0}{s} \left[\frac{(a_0 - c_0) + \sum_{j=1}^{i} (a_j - c_j) \prod_{k=1}^{j} \frac{\rho_k}{(k \wedge s)}}{1 + \sum_{j=0}^{\infty} \prod_{k=0}^{j} \frac{\rho_k}{(s \wedge (k+1))}} \right] \times \left[\frac{1}{1 - \frac{\rho_\infty}{s}}\right] - \frac{\rho_\infty}{s} \left[\frac{\frac{a_\infty}{1 - \frac{\rho_\infty}{s}} - \sum_{j=i+1}^{\infty} c_j \left(\frac{\rho_\infty}{s}\right)^{j-i+1}}{1 + \sum_{j=0}^{\infty} \prod_{k=0}^{j} \frac{\rho_k}{(s \wedge (k+1))}} \right] \times \left[1 + \sum_{j=0}^{i-1} \prod_{k=0}^{j} \frac{\rho_k}{(s \wedge (k+1))}\right] \right\}. \tag{69}$$

Since the maximand above has a limit as i approaches infinity, and since $(a_0, \cdots, a_i)$ achieves the maximum for each i, it must be true that $A(g) = (a_0, \cdots)$ achieves the maximum in the limit. That is,

$$\frac{\phi^1(A(g))}{s\mu\left(1 - \frac{\rho_\infty}{s}\right)} - \frac{(a_\infty - c_\infty)\frac{\rho_\infty}{s}}{1 - \frac{\rho_\infty}{s}} \ge \frac{\phi^1(A)}{s\mu\left(1 + \frac{\rho_\infty}{s}\right)} - \frac{(a_\infty - c_\infty)\frac{\rho_\infty}{s}}{\left(1 - \frac{\rho_\infty}{s}\right)} \tag{70}$$

for all stationary policies A. But this implies that $\phi^1(A(g) \ge \phi^1(A)$ for all stationary A, which means $\phi^1(A(g)) = g^*$, and thus, by statement b, that $g = g^*$.

$a \Rightarrow c$.

If $g = g^*$, then

$$\lim_{i\to\infty} F(i, g) = \lim_{i\to\infty} F(i) = F(\infty), \tag{71}$$

which is finite by Lemmas 1 and 4.

$c \Rightarrow a$

If $g = g^* + \varepsilon$, then

$$F(0, g) = \max_{\alpha\in P} \left\{a - c_0 - \frac{g^*}{\lambda_a} - \frac{\varepsilon}{\lambda_a}\right\} = a_0 - c_0 - \frac{g^*}{\lambda_0} - \frac{\varepsilon}{\lambda_0} \le F(0) - \frac{\varepsilon}{\lambda_0}. \tag{72}$$

Continuing in this manner, we get

$$F(i, g) \le F(i) - \frac{\varepsilon}{\mu} \left[\frac{1 + \sum_{j=0}^{i-1} \prod_{k=0}^{j} \frac{\rho_k}{(s \wedge (k+1))}}{\rho_0 \prod_{j=1}^{i} \frac{\rho_k}{(k \wedge s)}} \right], \quad i \in I. \tag{73}$$

Clearly, the right hand side is unbounded below, so that $F(i, g) \to -\infty$ as $i \to \infty$.

If $g = g^* - \varepsilon$, then a similar analysis yields

$$F(i, g) \ge F(i) + \frac{\varepsilon}{\mu} \left[\frac{1 + \sum_{j=0}^{i-1} \prod_{k=0}^{j} \frac{\rho_k^*}{(s \wedge (k+1))}}{\rho_0^* \prod_{j=1}^{i} \frac{\rho_k^*}{(k \wedge s)}} \right], \tag{74}$$

where ρ_k^* denotes $\lambda_{a_k^*}/\mu$. Since the right hand side is unbounded above, we see that $F(i, g) \to \infty$ as $i \to \infty$. This completes the proof of the equivalence of a, b, and c.

If $\bar{A} = A^*$, then we have

$$A(\phi^1(\bar{A})) = A(\phi^1(A^*)) = A(g^*) = A^* = \bar{A}. \tag{75}$$

If $\bar{A} = A(\phi^1(\bar{A}))$, then set $g = \phi^1(\bar{A})$. This gives

$$A(g) = A(\phi^1(\bar{A})) = \bar{A}, \tag{76}$$

so

$$\phi^1(A(g)) = \phi^1(\bar{A}) = g. \tag{77}$$

Thus, since $(b) \Rightarrow (a)$, $g = g^*$, so $\bar{A} = A^*$. Q.E.D.

We now extend the definition of $F(\infty)$. Define $F(\infty, \cdot) : \mathscr{R} \to \mathscr{R}$ by

$$F(\infty, g) = \min_{a \in P} \left[\frac{g - (a - c_\infty)\lambda_a}{s\mu - \lambda_a} \right]. \tag{78}$$

Note that $F(\infty) = F(\infty, g^*)$. In Theorem 9, we show how this function can be useful in testing for optimality.

Theorem 9 Suppose $C1$ holds. Then for each $i \geq k_0$ and each $g \in \mathscr{R}$, both of the following statements are true:

a. $F(i, g) > (=)[<] F(i - l, g)$ implies $g < (=)[>] g^*$, and
b. $F(i, g) > (=)[<] F(\infty, g)$ implies $g < (=)[>] g^*$.

Proof. We first prove statement a. If, for $\delta > 0$,

$$F(i, g) = F(i - 1, g) + \delta, \text{ then,} \tag{79}$$

$$\begin{aligned} F(i + 1, g) &= \max_{a \in P} \left[a - c_{k_0} - \frac{g}{\lambda_a} + \frac{s\mu}{\lambda_a} F(i - 1, g) \right. \\ &\quad \left. + \frac{s\mu}{\lambda_a} \delta \right] \\ &\geq \max_{a \in P} \left[a - c_{k_0} - \frac{g}{\lambda_a} + \frac{s\mu}{\lambda_a} F(i - 1, g) \right] \\ &\quad + \frac{s\mu}{\lambda_{p_1}} \delta \\ &= F(i, g) + \frac{s}{\rho''} \delta = F(i - 1, g) \\ &\quad + \left[1 + \frac{s}{\rho''} \right] \delta, \end{aligned} \tag{80}$$

where $\rho'' = \max_{a \in P} \lambda_a / \mu$. Consequently, we have

$$\begin{aligned} F(i + t, g) &\geq F(i - 1, g) + \left[1 + \left(\frac{s}{\rho''} \right) + \left(\frac{s}{\rho''} \right)^2 \right. \\ &\quad \left. + \cdots + \left(\frac{s}{\rho''} \right)^t \right] \delta \end{aligned} \tag{81}$$

for $t = 0, 1, \cdots$. Thus, by Theorem 8 and its proof, $g < g^*$. If

$$F(i, g) = F(i - 1, g), \text{ then} \tag{82}$$

$$F(i + t, g) = F(i - 1, g) \text{ for } t = 0, 1, \cdots, \tag{83}$$

which implies that $\lim F(j, g)$ exists, and thus, by Theorem 8, that $g = g^*$.

The proof for $F(i, g) < F(i - 1, g)$ is essentially the same as for $>$. This completes the proof of a.

Now suppose $F(i, g) = F(\infty, g) + \delta$ for some $\delta > 0$. Then,

$$\begin{aligned} F(i + 1, g) &= \max_{a \in P} \left\{ a - c_{k_0} - \frac{g}{\lambda_a} + \frac{s\mu}{\lambda_a} F(\infty, g) + \frac{s\mu}{\lambda_a} \delta \right\} \\ &\geq F(\infty, g) + \left(\frac{s}{\rho''} \right) \delta, \end{aligned} \tag{84}$$

and, consequently,

$$F(i + t, g) \geq F(\infty, g) + \left(\frac{s}{\rho''} \right)^t \delta, \qquad t = 0, 1, \cdots. \tag{85}$$

Thus, by Theorem 8, $g < g^*$.

The proofs for (=) and [<] are essentially the same as the proofs above. Q.E.D.

We now summarize in Lemma 5 some properties of $F(\infty, g)$ which we require for the algorithm.

Lemma 5. The function $F(\infty, \cdot)$ is strictly increasing, piecewise linear, concave, and unbounded above and below. Moreover, the interior of each linear segment corresponds to one and only one price.

Proof. Let $\delta > 0$. That $F(\infty, g)$ is strictly increasing can be seen immediately from

$$\begin{aligned} F(\infty, g + \delta) &= \min_{a \in P} \left[\frac{g - (a - c_\infty)\lambda_a}{s\mu - \lambda_a} + \frac{\delta}{s\mu - \lambda_a} \right] \\ &\geq F(\infty, g) + \frac{\delta}{s\mu - \lambda_{p_k}} > F(\infty, g). \end{aligned} \tag{86}$$

Also, the equation above shows that the minimizing price is a nondecreasing function of g, since the minimand for $g + \delta$ is the sum of the minimand for g and a function decreasing in a. Thus, since P is finite, and $F(\infty, g)$ is continuous in g, $F(\infty, g)$ is piecewise linear. Let (g_1, g_2) be the interior of a linear segment of $F(\infty, g)$. In (g_1, g_2) the slope of the segment is $1/[s\mu - \lambda_{a_\infty}(g)]$, which corresponds to exactly one price. It is clear that $F(\infty, g)$ is concave since the slopes of the linear segments decrease as $a_\infty(g)$ increases. It is obvious that $F(\infty, g)$ is unbounded above and below. Q.E.D.

A finite algorithm

Define the function $\mathscr{F}$ by

$$\mathscr{F}(g) = F(k_0, g) - F(\infty, g) \tag{87}$$

for $g \in \mathscr{R}$, where k_0 is defined in C1. It is not difficult to show that $\mathscr{F}$ is piecewise linear, strictly decreasing, and convex. Using Theorem 9, we can show that $\mathscr{F}$ has the unique root g^*.

For any $g \in \mathscr{R}$, we define $G(g)$ to be the g-intercept of that line segment of $\mathscr{F}$ lying directly above g; if there are two such segments, pick the rightmost one. The algorithm is outlined below.

I Let $g \in \mathscr{R}$ be given.
II Compute $\mathscr{F}(g)$ and $A(g)$ by performing the $(k_0 + 2)$ sequential maximizations indicated by equations (49, 50 and 78).
III If $\mathscr{F}(g) = 0$, then $g = g^*$, $A(g) = A^*$, and the process terminates. If not, then replace g with $G(g)$, and return to step II. The algorithm is finite by the properties of $\mathscr{F}$ outlined above.

It is easy to show that $a_i(g)$ is nonincreasing in g for each $i = 1, 2, \cdots, k_0$, and that $a_\infty(g)$ is nondecreasing. Also, the algorithm produces an increasing sequence $g_1 < g_2 < \cdots < g_L = g_{L+1} = g^*$. We wish now to find an upper bound $\hat{L}$ on the number of iterations required to reach optimality.

Define $\underline{a}_i$ to be the largest maximizer of $(a - c_i)\lambda_a$. It can be shown (see [23]) that $c_i < \underline{a}_i \le a_i^*$ for all $i \in I$, and also that $\underline{a}_0 \le \underline{a}_1 \le \cdots \le \underline{a}_{k_0} = \cdots$. Since the algorithm produces a decreasing sequence $[a_0(g_1), \cdots, a_{k_0}(g_1)] > [a_0(g_2), \cdots, a_{k_0}(g_2)] > \cdots$ and an increasing sequence $a_\infty(g_1) \le a_\infty(g_2) \le \cdots$, we can compute a value for $\hat{L}$.

$$\hat{L} = K_{k_0} + 1 + \sum_{i=0}^{k_0} (K - K_i), \tag{88}$$

where K_i is the index in P of $\underline{a}_i$.

We prove one interesting result which may not be obvious from the geometric interpretation given above. Define the policy $\tilde{A}(g)$ by

$$\tilde{a}_i(g) = \begin{cases} a_i(g) \text{ for } i = 0, \cdots, k_0, \text{ and} \\ a_\infty(g) \text{ for } i > k_0, \end{cases} \tag{89}$$

where $a_i(g)$ is any maximizer of $F(i, g)$, and $a_\infty(g)$ is any minimizer of $F(\infty, g)$. For fixed g, we denote $a_i(g)$ by a_i, $a_\infty(g)$ by a_∞, λ_{a_i}/μ by ρ_i, and λ_{a_∞}/μ by ρ_∞.

Theorem 10. Suppose C1 holds. Then, $G(g) = \phi^1(\tilde{A}(g))$.

Proof. Using equations (18) and (61), we obtain

$$\mathscr{F}(g) = \mu\rho_0 \left[\frac{(a_0 - c_0) + \sum_{i=1}^{k_0} (a_i - c_i) \prod_{j=1}^{i} \frac{\rho_j}{(j \wedge s)}}{\mu\rho_{k_0} \prod_{j=0}^{k_0-1} \frac{\rho_j}{(s \wedge (j+1))}} \right] - g \left[\frac{1 + \sum_{i=0}^{k_0-1} \prod_{j=0}^{i} \frac{\rho_j}{(s \wedge (j+1))}}{\mu\rho_{k_0} \prod_{j=0}^{k_0-1} \frac{\rho_j}{(s \wedge (j+1))}} \right] - \left[\frac{\frac{g}{s\mu} - (a_\infty - c_\infty)\frac{\rho_\infty}{s}}{1 - \frac{\rho_\infty}{s}} \right], \tag{90}$$

which, upon solving for the g-intercept, yields

$$G(g) = \mu\rho_0 \times \left[\frac{(a_0 - c_0) + \sum_{i=1}^{k_0} (a_i - c_i) \prod_{j=1}^{i} \frac{\rho_j}{(j \wedge s)} + \sum_{i=k_0+1}^{\infty} (a_\infty - c_\infty) \prod_{j=1}^{i} \frac{\rho_j}{(j \wedge s)}}{1 + \sum_{i=0}^{k_0} \prod_{j=0}^{i} \frac{\rho_j}{(s \wedge (j+1))} + \sum_{i=k_0+1}^{\infty} \prod_{j=0}^{i} \frac{\rho_j}{(s \wedge (j+1))}} \right], \tag{91}$$

which is just $\phi^1(\tilde{A}(g))$. Q.E.D.

Numerical example

In this section, we examine a sample problem. The algorithm defined above was coded by the author in APL/360 [26]. A problem is completely defined by specifying s, μ, c, P, and λ. Let $s = 2$, $\mu = 12$, $c = (10, 10, 15, 25, 40, 40, \cdots)$, $P = (1, 2, \cdots, 99)$, and $\lambda_p = 10.1 - 0.1p$.

We begin by setting $g_0 = \phi^1(\underline{A})$, which yields

$$\underline{A} = (55, 55, 58, 63, 70, 70, \cdots)$$

and

$$g_0 = 205.1521962.$$

The results are summarized below.

j	g_j	$A(g_j)$	$\mathscr{F}(g_j)$
[0]	205.1521962	56 56 60 67 94 73 73 73 ···	40.76173377
[1]	205.1854732	56 56 60 67 83 73 73 73 ···	18.74884642
[2]	205.2248295	56 56 60 66 75 73 73 73 ···	3.92947009
[3]	205.2370933	56 56 60 66 73 73 73 73 ···	0.1198211141
[4]	205.2374961	56 56 60 66 73 73 73 73 ···	1.498956514E¯11

Thus, $g^* = 205.2374961$ and

$$A^* = (56, 56, 60, 66, 73, 73, \cdots).$$

A sufficient condition for monotonicity

In Theorem 4, we showed that there is always an optimal stationary policy which is monotone. In this section we show that if condition C1 is not satisfied, i.e., if $c_\infty > c_i$ for all $i \in I$, then every optimal stationary policy is montone. We begin with an example of an optimal stationary policy which is not monotone. Let

$$s = 1, \mu = 6, c_0 = c_1 = \cdots = 2,$$

$$P = (3, 5), \lambda_3 = 3, \text{ and } \lambda_5 = 1.$$

It is easy to show that $g^* = 3$, and that $F(i) = 0$ for all $i \in I$. Thus

$$F(i) = \max [3 - 2 - 3/3, 5 - 2 - 3/1]$$
$$= \max [0, 0] = 0,$$

and by Theorem 2, a_i^* may be chosen to be either 3 or 5 for $i = 0, 1, \cdots$. Therefore, every stationary policy is optimal, including the non-montone policy $(5, 3, 5, \cdots)$.

Lemma 6. The following four statements are equivalent:

a. $F(0) = 0$,
b. $F(i) = 0$ for all $i \in I$,
c. $g^* = \max_{a \in P} [(a - c_0)\lambda_a]$, and
d. $c_\infty = c_0$.

Proof.

$c \Rightarrow a$.

Suppose c holds. Then,

$$0 \le F(0) = \max_{a \in P} \left\{ \frac{(a - c_0)\lambda_a - \max_{a' \in P} [(a' - c_0)\lambda_{a'}]}{\lambda_a} \right\} \le 0 \tag{92}$$

$a \Rightarrow c$. If a holds, we have

$$\max_{a \in P} \left[\frac{(a - c_0)\lambda_a - g^*}{\lambda_a} \right] = 0, \tag{93}$$

which implies $g^* \ge (a - c_0)\lambda_a$ for all $a \in P$, with equality for at least one $a \in P$.

$c \Rightarrow d$.
Suppose c holds, but that $c_\infty > c_0$. Let $i \ge 1$ be the smallest index such that $c_i > c_0$. Then,

$$\begin{aligned} F(i) &= -(c_i - c_0) + \max_{a \in P} \left[\frac{(a - c_0)\lambda_a - g^*}{\lambda_a} \right. \\ &\quad \left. + \frac{(i \wedge s)\mu}{\lambda_a} F(i-1) \right] \\ &< \max_{a \in P} \left[\frac{(i \wedge s)\mu}{\lambda_a} F(i-1) \right] \\ &\le \frac{(i \wedge s)\mu}{\lambda_{p_K}} F(i-1). \end{aligned} \tag{94}$$

By the first part of this proof, we know that $F(0) = 0$, and since, by assumption, $c_0 = \cdots = c_{i-1} < c_i$, we have

$$0 = F(0) = \cdots = F(i-1) > \frac{\lambda_{p_K}}{(i \wedge s)\mu} F(i), \tag{95}$$

which is impossible by Lemma 1.

$d \Rightarrow c$.
Let $g = \max\{(a - c_0)\lambda_a\}$ and suppose $c_\infty = c_0$. Then, $F(i, g) = 0$ for all $i \in I$ since

$$F(0, g) = \max_{a \in P} \left[\frac{(a - c_0)\lambda_a - g}{\lambda_a} \right] = 0, \tag{96}$$

and if $F(i-1, g) = 0$, we have

$$F(i, g) = \max_{a \in P} \left[\frac{(a - c_0)\lambda_a - g}{\lambda_a} \right] = 0. \tag{97}$$

Thus, $\lim_{i \to \infty} F(i, g) = 0$, and by Theorem 8, this implies $g = g^*$.

$a \Rightarrow b$.
Since a and d are equivalent, and since b is implied by a and d, b is implied by a.

$b \Rightarrow a$. Trivial. Q.E.D.

Lemma 7. If $a_i^* > a_{i+1}^*$, then

$$F(i) < \frac{g^*}{(s \wedge (i+1))\mu} \tag{98}$$

and

$$F(i) \le F(i-1). \tag{99}$$

Proof. The proof is identical to the proof of Lemma 5 of [23] with $g = g^*$. Q.E.D.

Theorem 11. Suppose $c_\infty > c_i$ for all $i \in I$, and let $A^* = (a_0^*, a_1^*, \cdots)$ be any optimal stationary policy. Then $a_0^* \le a_1^* \le \cdots$.

Proof. In view of Theorem 3, we need only consider policies which maximize the right hand side of (14). Suppose $a_1^* < a_0^*$. Then by Lemma 7, $F(0) \le F(-1) = 0$, which implies $F(0) = 0$ by Lemma 1, which, in turn, implies $c_\infty = c_0$ by Lemma 6. Thus, $a_0^* \le a_1^*$.

Now suppose $a_{i+1}^* < a_i^*$ for $1 \le i \le s-1$. It follows from Lemma 7 that

$$a - c_0 - \frac{g^*}{\lambda_a} + \frac{i\mu}{\lambda_a} F(i-1) \le F(i-1) \text{ for all } a \in P. \tag{100}$$

In particular, for $a = a_{i-1}^*$, we have

$$\begin{aligned} F(i-1) &\ge \left[a_{i-1}^* - c_0 - \frac{g^*}{\lambda_{a_{i-1}}} + \frac{(i-1)\mu}{\lambda_{a_{i-1}}} F(i-2) \right] \\ &\quad + \frac{i\mu}{\lambda_{a_{i-1}^*}} F(i-1) - \frac{(i-1)\mu}{\lambda_{a_{i-1}^*}} F(i-2) \\ &= F(i-1) + \frac{i\mu}{\lambda_{a_{i-1}^*}} \left[F(i-1) \right. \\ &\quad \left. - \frac{(i-1)}{i} F(i-2) \right], \end{aligned} \tag{101}$$

or, equivalently,

$$F(i-1) \le \frac{(i-1)}{i} F(i-2) \le F(i-2). \tag{102}$$

By iterating this inequality, we obtain

$$F(i) \le F(i-1) \le \cdots \le F(0) \le F(-1) = 0, \tag{103}$$

which, by Lemmas 1 and 6, implies $c_\infty = c_0$. This completes the proof for $0 \le i \le s-1$.

If $a_{i+1}^* < a_i^*$ for $i \ge s$, we have $F(i) \le F(i-1)$ by Lemma 7. Thus,

$$\begin{aligned} F(i+1) &= -\left\{ (c_{i+1} - c_i) + \frac{s\mu}{\lambda_{a_{i+1}^*}} [F(i-1) - F(i)] \right\} \\ &\quad + \left[a_{i+1}^* - c_i - \frac{g^*}{\lambda_{a_{i+1}^*}} + \frac{s\mu}{\lambda_{a_{i+1}^*}} F(i-1) \right] \\ &\le F(i). \end{aligned} \tag{104}$$

Iterating this inequality, we obtain

$$F(i) \geq F(i+1) \geq \cdots,$$

but this is impossible by Lemma 4 and Eq. (60). This completes the proof. Q.E.D.

Special case: constant cost

Consider the situation where the cost function is constant over all states. This would be the case, for example, if the number of servers was infinite. Theorem 12 shows that, in this case, an optimal policy can be found analytically.

Theorem 12. If $c_\infty = c_0$, then the policy $\underline{A} = (\underline{a}_0, \underline{a}_0, \underline{a}_0, \cdots)$ is optimal, where $\underline{a}_0$ is the largest maximizer of $(a - c_0)\lambda_a$.

Proof. By Lemma 6, $g^* = (\underline{a}_0 - c_0)\lambda_{\underline{a}_0}$ and $F(i) = 0$ for all $i \in I$. Thus,

$$F(i) = \max_{a \in P} \left[\frac{(a - c_0)\lambda_a - g^*}{\lambda_a} \right] = 0, \tag{105}$$

which implies $\underline{a}_0$ is optimal for each $i \in I$ by Theorem 2. Q.E.D.

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