M. Karnaugh

Loss of Point-to-Point Traffic in Three-Stage Circuit Switches

Abstract: A theoretical study is made of simple analytical models for the point-to-point loss of telecommunication traffic caused by blocking in three-stage circuit switches. Two new models are compared with Jacobaeus' frequently used model and with some simulation results to determine regions of acceptable accuracy. The effects of random hunting and sequential hunting for routes are compared by simulation.

The results apply to space-division link systems and also to some time-division switches of current interest. In the case of random hunting, the new models give improved agreement with simulation results. The overestimate of loss inherent in the Jacobaeus method, however, is found to be acceptably low when the numbers of primary and tertiary matrix switches are not small, e.g. > 10. We lack a good analytical model for the sequential hunting method, which is found to result in lower traffic loss for the switches being studied.

1. Introduction

Recent progress in the development of miniaturized electronic circuits and, particularly, large-scale integrated digital electronics has made time-division digital (PCM) switching techniques very attractive for telephone applications [1, 2]. This has resulted in increased interest in three-stage switches, of which there are some especially convenient time-division forms.

Although it has long been known that three-stage switches can be made strictly nonblocking [3], considerable savings in components can be realized by permitting a very small but positive probability that a call will be lost because of link congestion (i.e., blocking) in the switch. It is therefore important to be able to estimate the dependence of the loss on the switch parameters, on the switch control (i.e., route hunting) algorithm, and on the offered traffic.

Probably the best known and most widely used method for the estimation of loss in link systems is that of C. Jacobaeus [4], which is described also in a book by R. Syski [5] and in a survey paper by K. Kümmerle [6]. This approximate method is distinguished by its simplicity and ease of computation. Its sources of error were known to Jacobaeus, who correctly pointed out that they cause an overestimate of the loss. The overestimate, which leads to safe designs, was readily accepted at a time (1950) when digital computers were not generally available.

In this paper we apply the method of Jacobeaus specifically to loss in three-stage switches. Alternative loss formulas are derived from two other models that remove some of the sources of error. The formulas are then compared with one another and with a few simulation results in order to determine their regions of acceptable accuracy. It is important to bear in mind that the method of Jacobaeus has been applied to a much broader range of problems than are considered in this paper, which is not intended to be a critique of the more general usefulness of his method.

As preliminaries, the structures of the three-stage space-division switch and its time-division analogs are described, and a mathematical definition of the loss is given.

The approach taken in this paper is to offer a self-contained treatment of a few related heuristic models. Accordingly, only the simplest cases will be considered and no attempt will be made to delineate the various possible extensions of the models to more complex cases. Switches with Bernoullian offered traffic will receive the most attention because this is the only type of offered traffic for which the models are exactly comparable.

This paper is concerned only with point-to-point loss, also called "point loss"; i.e., a call is considered to be lost when the connection from a particular inlet of the switch to a particular outlet is blocked. The "group loss" of calls from a particular inlet to any one of a group of outlets will not be considered.

2. Notation

Each elementary switching element, or crosspoint, is functionally a single-pole or multi-pole, single-throw switch. Crosspoints may be made of metallic contacts, e.g., reed relays or crossbar switches, or semiconductor gates. Figure 1(a) represents a switching matrix that connects a set of five horizontal conductors to a set of five vertical conductors by means of 25 crosspoints. Any switching matrix of this type will be called a *matrix* for brevity.

Constraints are frequently placed on the operation of the crosspoints in a matrix. We shall assume that no more than one crosspoint in any row or in any column may at any time be closed, i.e., conducting. In this fashion, any one-to-one connection pattern of horizontals to verticals may be realized.

In order to simplify block diagrams of switches, the representation in Fig. 1(b) will be substituted for that in Fig. 1(a). More generally, Fig. 1(c) represents a matrix with L inlets and K outlets, the convention of flow being from left to right.

In a space-division switch, each inlet or outlet of a matrix can provide no more than one communication channel. In a time-division switch, the fundamental time period, called a frame, is subdivided into S equal, periodic subintervals, called time-slots, each of which can independently provide a communication channel. The S channels are said to be in time-division multiplex (TDM). A time-division matrix can assume an independent connecting state in each of its S time-slots. Therefore, it is functionally equivalent to S separate space-division matrices. Figure 2(a) represents a time-division matrix with K time-slots.

A time-division matrix cannot shift any communication channel from one time-slot to any other time-slot. This function, called time-slot interchange, is performed by a buffer, which stores data from the incoming channels and permits it to be read out again in any order and in any subset of the outgoing time-slots having the necessary cardinality. This function requires that the buffer have a random access capability. Although more limited time-slot interchange may be achieved by partial-frame memories or shift registers [7], we assume that the buffers have full-frame storage and random access capability.

Figure 2(b) shows the representation we use for a buffer having S input time-slots and T output time-slots. This buffer can provide no more than the minimum of S and T channels of communication. It is the time-switching analog of a matrix with S inlets and T outlets.

A very small space-division switch is shown in Fig. 3(a). This switch can connect any of the four inlets to any of the four outlets, but not all sets of one-to-one connections are possible. For example, if a connection from inlet 1 to outlet 1 is already established, it is impossible to add a connection from inlet 2 to outlet 2 because the needed link (1, 1) is already in use. In this case, the desired connection is said to be *blocked*. Attempts to establish new calls in the presence of blocking result in traffic *loss*.

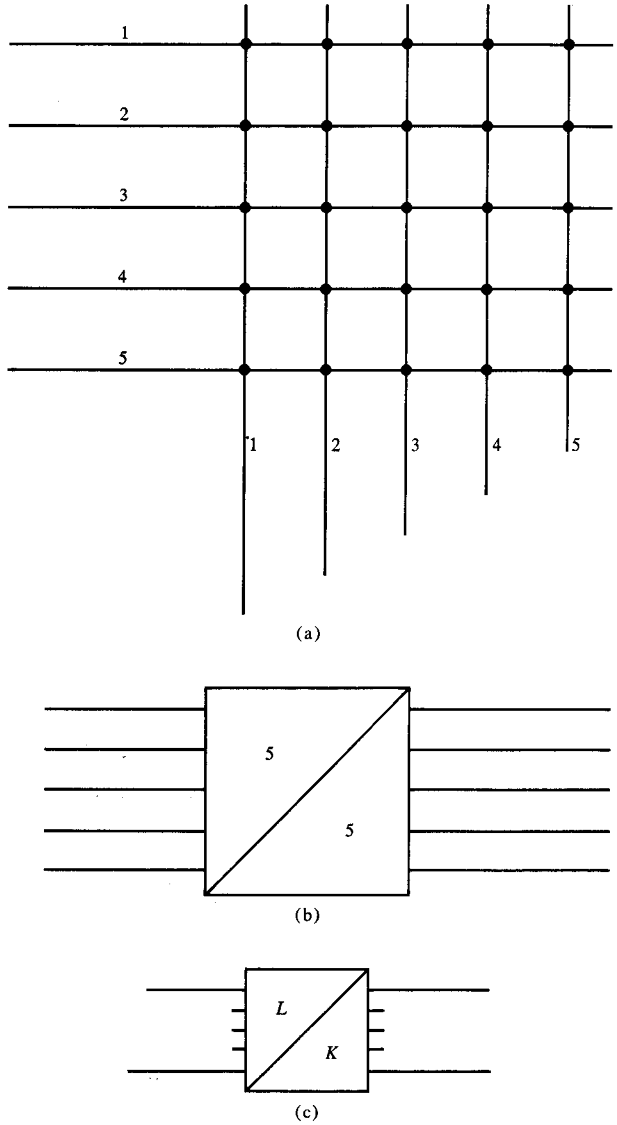

Figure 1 Simplified representation of crosspoint matrix diagrams. (a) A five-by-five crosspoint matrix. (b) Simplified representation of the five-by-five matrix. (c) Representation of an L-by-K matrix.

Not all of the matrices will be shown explicitly in diagrams of larger switches. Figure 3(b) represents a two-stage network having N primary matrices and M secondary matrices. Notice that each connection must pass through two crosspoints and one link.

Folded switches, in which each line appears as both an inlet and an outlet, have some interesting properties. When transmission through the switch is bidirectional, either of two inlet-to-outlet connections will suffice to establish a call [8]; but when the switch is unidirectional, both connections must be made [1]. For simplicity, we

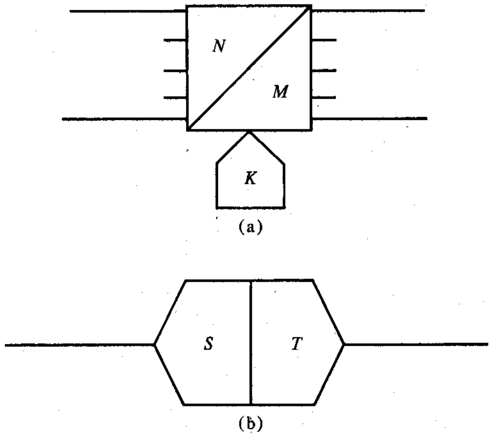

Figure 2 The time-division matrix and the buffer. (a) Representation of a time-division matrix with N inlets, M outlets, and K time-slots. (b) Representation of a buffer with S input time-slots and T output time-slots. The buffer is capable of full-frame storage and random access for time-division switching.

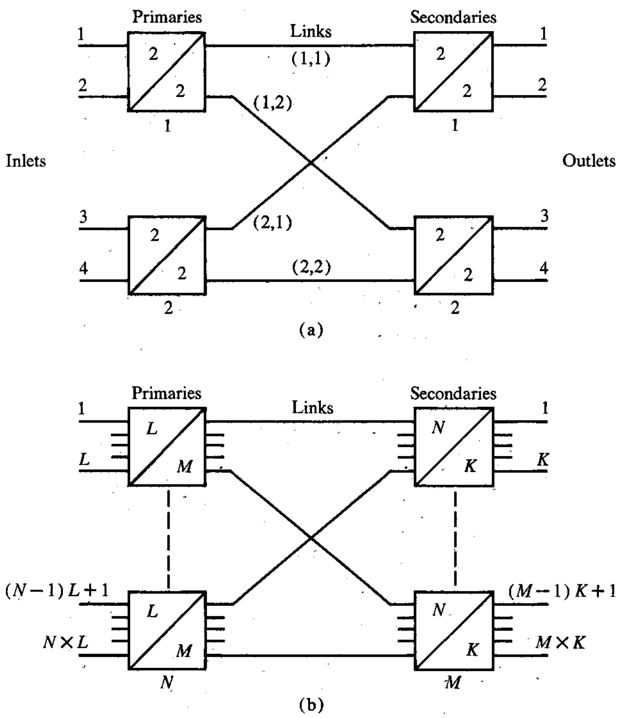

Figure 3 Two-stage networks. (a) A very small two-stage space division switch. (b) A two-stage space division switch with N primary matrices and M secondary matrices.

assume that the inlets and the outlets of our switches form disjoint sets, and we require a single, unique connection for each new call.

The customary algebraic notation is used in the analysis that follows. Square brackets, however, are reserved for the arguments of functions.

3. Three-stage switches

A three-stage space-division switch is illustrated in Fig. 4(a). There are exactly K possible routes for a desired connection, each one traversing three matrices and two links. To establish a new call through the switch via a particular route, both the A-link and the B-link of that route must be idle. Blocking occurs when none of the K routes has a pair of idle links.

C. Clos [3] has demonstrated that this switch is non-blocking when $K = L + T - 1$. A considerable saving in crosspoints is possible, however, when occasional blocking is tolerated. We are concerned with estimating the loss when

$$K < L + T - 1. \tag{1}$$

In addition, we assume that

$$K \geq \max\,[L, T] \tag{2}$$

so that the primary and the tertiary between which a new connection is desired have at least one idle link on each. Blocking then results from a failure to match idle links on any route. Condition (2) is typical of a central switch for trunks and preconcentrated lines.

There are analogies between the three-stage space-division switch in Fig. 4(a) and at least two types of time-division switches having buffers. These analogies permit the same estimates of loss to be used for all three types of switch.

Consider the time-space-time switch in Fig. 4(b). Here, the inlet channels are time-division multiplexed on each of the N inlet lines and the outlet channels are time-division multiplexed on each of the M outlet lines. There are L time-slots per frame at the inlets and T time-slots per frame at the outlets, the frames all having equal periods. The primaries and tertiaries are buffers, and the secondary is a time-division matrix having K time-slots.

As in the space-division switch, there are K routes for any desired connection. These routes are not, however, spatially disjoint. Instead, each one occupies a different time-slot in the same physical pair of buses. Each bus provides K links in time-division multiplex. As before, blocking occurs when there is no matching idle pair of links; but the matching must occur in time. The secondary matrix cannot permute time-slots.

A space-time-space switch is shown in Fig. 4(c). The inlets are in TDM with L time-slots per frame and the outlets are in TDM with T time-slots per frame. The primary and tertiary are time-division matrices, and the K secondaries are buffers. To find a route, the calling time-slot must be idle in an A bus and the called time-slot must be idle in the B bus of the same buffer. The K routes are now spatially disjoint.

The primary in Fig. 4(c) is functionally equivalent to L space-division matrices having N inputs and K outputs each, Similarly, the tertiary is functionally equivalent to T space division matrices having K inputs and M outputs each. Each secondary buffer is the time analog of a space-division matrix. Therefore, the analogy with the space-division switch of Fig. 4(a) requires transposition of the parameters (N, L) and also of (M, T) in formulas or simulations.

Nonblocking three-stage time-division switches resembling these were disclosed in a patent of H. Inose and T. Saito [9]. M. J. Marcus [10] has published an article on space-time equivalents in connecting networks containing a broader range of analogies than we require for the present purposes. M. Huber [11] has published an early paper on congestion in time-division switches, emphasizing group loss. Inasmuch as estimates for the loss in the space-division switch can also be applied to the time-division analogs, we refer specifically to Fig. 4(a) in the analysis that follows.

Description of the switch is not complete until one has defined the route-hunting algorithm. Whenever more than one of the K routes is available to a new call, a rule will be needed for making the selection. For example, we might specify one of the following rules.

1. Random hunting: Test the routes in random order, selecting the first one available.
2. Sequential hunting: Test the routes in a fixed sequence, selecting the first one available.
3. Sequential hunting with random start: Test the routes in a fixed cyclic sequence but with a random starting point, selecting the first one available.
4. High-occupancy hunting: Test the routes in order of decreasing occupancy of the secondary traversed, with a fixed sequence when occupancies are equal, selecting the first one available.

In the case of sequential hunting, there is also a choice to be made as to whether the sequence followed is the same for all primary-tertiary connections. The influence of some of these hunting methods on the loss has been studied by D. Bazlen, G. Kampe, and A. Lotze [12].

Of the four hunting algorithms defined above, only random hunting is accurately modeled in our analysis. Sequential hunting (with a primary-tertiary invariant sequence) gives the three-stage switch a lower loss, however, and we shall see a few simulations of this.

It is a reasonable conjecture that sequential hunting with random start is very similar to random hunting. It is also a reasonable conjecture that high-occupancy hunting is very similar to sequential hunting. We have not simulated these hunting algorithms. V. E. Beneš has published a study of optimal routing [13] which suggests the superiority of high-occupancy hunting over random hunting; but he gives numerical results only for a three-stage switch with four inlets and four outlets, made from 2×2 matrices.

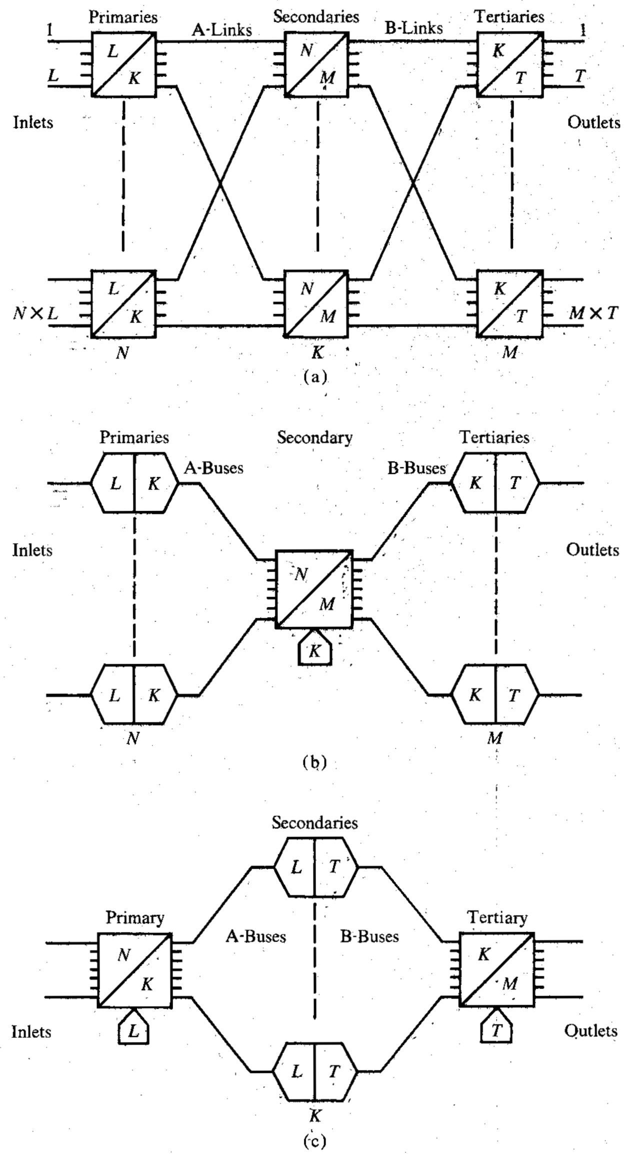

Figure 4 Three-stage networks. (a) A three-stage space-division switch with N primaries, K secondaries, and M tertiaries. (b) A three-stage, time-space-time switch. (c) A three-stage, space-time-space switch.

4. Models for the offered traffic

It is traditional to idealize the offered traffic model in order to arrive at simple estimates of the loss. The "offered traffic" is the traffic that would be carried by the switch if it were strictly nonblocking.

The offered traffic is assumed to arise from one or more independent Markovian birth and death processes. It is further assumed that each primary receives equal traffic and that each tertiary receives equal traffic. A number of different models may be employed, and one should be selected that reasonably approximates the case to be analyzed.

Consider a single process that generates all of the traffic in the switch. Let C represent the number of calls in the switch. The range of this random variable is

$$0 \leq C \leq W, \tag{3}$$

where W is a constant such that

$$0 < W \leq \min\,[NL, MT]. \tag{4}$$

We assume that the rate of requests for new calls and the rate at which existing calls are terminated both depend only on C. In particular, the rate of requests for new calls is $\lambda[C]$, where

$$\lambda[C] > 0 \text{ when } C < W,$$
$$= 0 \text{ when } C = W. \tag{5}$$

The rate of termination of calls is assumed to be μC, with $\mu > 0$. New calls occur at random between the idle inlets and the idle outlets. Terminations occur at random among the existing calls.

The definition of the offered traffic process may now be completed by specifying the function $\lambda[C]$. The three most common choices, and the resulting distributions, $P[C]$, of C are given below. More detail is to be found in Syski's book [14].

1. *Bernoulli*

$$\lambda[C] = \lambda_0(W - C), \tag{6}$$

with $\lambda_0 > 0$, a constant.

The resulting Bernoulli distribution is

$$P[C] = \binom{W}{C} a^C (1-a)^{W-C};. \tag{7}$$

$$a = \lambda_0/(\lambda_0 + \mu). \tag{8}$$

2. *Engset*, with $V > W$, (V being the number of independent sources of calls)

$$\lambda[C] = \lambda_0(V - C) \text{ when } C < W, \tag{9}$$
$$= 0 \qquad \text{when } C = W.$$

The Engset (truncated Bernoulli) distribution is

$$P[C] = \binom{V}{C} (\lambda_0/\mu)^C / D_1, \tag{10}$$

where the denominator is

$$D_1 = \sum_{C=0}^{W} \binom{V}{C} (\lambda_0/\mu)^C. \tag{11}$$

3. *Erlang* (infinite number of sources)

$$\lambda[C] = \lambda_0 \text{ when } C < W,$$
$$= 0 \text{ when } C = W. \tag{12}$$

The Erlang (truncated Poisson) distribution is

$$P[C] = (\lambda_0/\mu)^C / C!\, D_2, \tag{13}$$

where

$$D_2 = \sum_{C=0}^{W} (\lambda_0/\mu)^C / C!. \tag{14}$$

It is also possible to apply one of the above processes to model the traffic offered at a single primary or a single tertiary. It is easy to see, however, that the traffic in the entire switch and the traffic in each of its several primaries cannot simultaneously be of either the Engset type or the Erlang type.

If the traffic offered to the entire switch is Bernoullian, with $W = NL = MT$, then it can be shown that the traffic offered to any subset of the inlets is also Bernoullian, having the same values of λ_0 and μ. It is uniquely for this case that the three loss formulas to be compared are exactly comparable in offered traffic. Therefore, this case will be used as the basis for numerical comparisons of the formulas and for simulation.

5. Definition and estimation of the loss

Loosely speaking, the loss due to blocking in a switch is the expected fraction of the offered calls that will be blocked during some period of statistical equilibrium. We employ a more precise definition that is consistent with that of V. E. Beneš [15], who calls this quantity "the probability of blocking."

Beneš makes the simplifying assumption that blocked calls are cleared, that is, they do not change the state of the system. This assumption is not entirely realistic because blocked calls may result in repeated trials. This is a matter of little importance in the region of interest, where the loss is small.

Assuming a stationary, Markov-type offered traffic process, the blocked calls cleared hypothesis, a switch structure, and a route-hunting algorithm, Beneš shows that the model is a stationary, finite-state Markov process.

Suppose we observe such a system for a time interval $(0, t]$, keeping count of the number of times, $\alpha[t]$, that a new call is attempted and the number of times, $\beta[t]$, that an attempt is blocked. Beneš shows that the limit,

$$b = \lim_{t\to\infty} \beta[t]/\alpha[t], \tag{15}$$

exists and is constant with probability one. We call this quantity the *loss*.

The proof given by Beneš also implies that the limits

$$r_\alpha = \lim_{t\to\infty} \alpha[t]/t, \tag{16}$$

$$r_\beta = \lim_{t\to\infty} \beta[t]/t \tag{17}$$

exist and are constant with probability one.

Since $r_\alpha > 0$, it follows that

$$b = r_\beta / r_\alpha. \tag{18}$$

From this point on, the analysis diverges from that of Beneš.

It is clear that r_α is the time average of the rate at which new calls are attempted. Similarly, r_β is the mean rate at which attempts are blocked. The corresponding probability averages are

$$r_\alpha = \sum_\sigma P[\sigma]\lambda[\sigma], \tag{19}$$

$$r_\beta = \sum_\sigma P[\sigma]\lambda[\sigma]P[B|\sigma], \tag{20}$$

where σ indexes the states of the system, $P[\sigma]$ is the equilibrium probability of being in state σ, $\lambda[\sigma]$ is the calling rate in state σ, and $P[B|\sigma]$ is the conditional probability that an attempted call will be blocked, given state σ. This leads to the loss formula

$$b = \frac{\sum_\sigma P[\sigma]\lambda[\sigma]P[B|\sigma]}{\sum_\sigma P[\sigma]\lambda[\sigma]}. \tag{21}$$

Equation (15) is a suitable starting point for the estimation of loss by simulation, and Eq. (21) is a suitable starting point for analytical approximation. Because of the intractably large number of states, however, considerable simplification of this formula will be necessary in order to make the computations practical.

Simplification is achieved by partitioning the universe of all states, indexed by σ, into n subsets, named $\sigma_1, \sigma_2, \cdots \sigma_n$. The equilibrium probability that the system is in the ith subset is

$$P[\sigma_i] = \sum_{\sigma\epsilon\sigma_i} P[\sigma]. \tag{22}$$

When in subset σ_i, the system attempts new calls at the mean rate,

$$\lambda\,[\sigma_i] = \sum_{\sigma\epsilon\sigma_i} P[\sigma]\lambda[\sigma]/P[\sigma_i], \tag{23}$$

and the conditional probability of blocking these calls is

$$P[B|\sigma_i] = \frac{\sum_{\sigma\epsilon\sigma_i} P[\sigma]\lambda[\sigma]P[B|\sigma]}{\sum_{\sigma\epsilon\sigma_i} P[\sigma]\lambda[\sigma]}. \tag{24}$$

It is now easy to verify that

$$b = \frac{\sum_i P[\sigma_i]\lambda[\sigma_i]P[B|\sigma_i]}{\sum_i P[\sigma_i]\lambda[\sigma_i]}. \tag{25}$$

The problem of estimating the loss is thus reduced to the problem of finding a suitable partition of the universe of states of the system and then estimating $P[\cdot]$, $\lambda[\cdot]$, and $P[B|\cdot]$ for the subsets of the partition. In so doing, we accept some inaccuracies in the estimates as the price for avoiding consideration of each state of the system. The computations then become feasible.

Equations (21) and (25) depend upon the existence of equilibrium state probabilities; but they are valid for any specific switch structure, routing method, or traffic.

Alternative calculations of the loss are made possible by the already assumed symmetries of the switch structure, the route-hunting algorithm, and the offered traffic process with respect to permutations of the primary matrices and permutations of the tertiary matrices.

Suppose that we relabel the primaries and the tertiaries. With the assumed symmetries, this causes a permutation of the state indices that leaves the transition matrix of the Markov process invariant. The measure over the state-valued function space is therefore also invariant. From this, it is clear that the mean calling rate between any one of the N primaries and any one of the M tertiaries is precisely r_α/NM; and the mean rate at which these calls are blocked is r_β/NM. Because of this, it does not matter whether the blocking rate and the calling rate are computed for the entire switch, a single primary, or a particular primary-tertiary pair; their ratio still gives the loss. Each of these three choices will be used in one of the three models that follow. More generally, we are free to use the blocking rate and the calling rate between any subset of the primaries and any subset of the tertiaries.

6. Application of Jacobaeus' method

We are now ready to estimate the point-to-point loss in the switch of Fig. 4(a), using approximations that characterize the method of Jacobaeus [4-6].

The blocking rate and the calling rate will be computed for the traffic from a particular primary, called "the primary", to a particular tertiary, called "the tertiary". The subsets of states will be indexed by (x, y), where x is the number of calls in the primary and y is the number of calls in the tertiary.

A simplifying approximation is made that x and y are independently distributed,

$$P[x, y] = P[x]P[y]. \tag{26}$$

The separate distributions are selected, a priori, so as to reasonably approximate the offered traffic. The most commonly used forms are Bernoulli, Erlang, and Engset.

It is also necessary to specify the form of $\lambda[x, y]$. K. Kümmerle [16] points out that a common control system, when seeking one of a group of outlets, will frequently select an idle one at random. This suggests that the probability that the tertiary is the target for a new call is approximately proportional to the number of its idle outlets. Such a model is quite consistent with the choice of a Bernoulli distribution of calls in the tertiary, although other models may be employed.

With this model, Eq. (25) takes the form

$$b = \frac{\sum_{x,y} P[x]P[y]\lambda[x, y]P[B|x, y]}{\sum_{x,y} P[x]P[y]\lambda[x, y]}. \tag{27}$$

The approximation of $P[B|x, y]$ is quite independent of the choice of distributions for x and y and of the calling rate. The primary has $(K - x)$ idle A-links, and the tertiary has $(K - y)$ idle B-links. Jacobaeus assumes that the idle A-links and the idle B-links are distributed at random, without bias, and with mutal independence over the K routes through the switch. As a consequence of these assumptions the number of matching idle link pairs is hypergeometrically distributed [see Appendix A]. The probability of blocking is the probability that this number is zero.

$$\begin{aligned} P[B|x, y] &= Q_0[K, K - x, K - y] \\ &= x!y!/K!(x + y - K)!. \end{aligned} \tag{28}$$

This model is now applied to the specific case in which x and y have Bernoulli distributions with L and T sources, respectively.

$$P[x] = \binom{L}{x} a_1^{\,x}(1 - a_1)^{L-x}, \tag{29}$$

$$P[y] = \binom{T}{y} a_2^{\,y}(1 - a_2)^{T-y}, \tag{30}$$

and, in order to be consistent about the total traffic in the switch,

$$LNa_1 = TMa_2. \tag{31}$$

The calling rate between the primary and the tertiary takes the form

$$\lambda[x, y] = \gamma(L - x)(T - y), \tag{32}$$

where γ is a constant.

The denominator of Eq. (27) is easily seen to be $\gamma L(1 - a_1)T(1 - a_2)$ and Eq. (27) becomes

$$b = \frac{(L-1)!(T-1)!}{K!} \sum_{x=0}^{L-1} \frac{a_1^{\,x}(1-a_1)^{L-1-x}}{(L-1-x)!} \times \sum_{y=0}^{T-1} \frac{a_2^{\,y}(1-a_2)^{T-1-y}}{(T-1-y)!(y+x-K)!}.$$

The summation over y is performed using the identity

$$1 = \sum_{y+x-K=0}^{T-1+x-K} \binom{T-1+x-K}{y+x-K} a_2^{\,y+x-K}(1-a_2)^{T-1-y}.$$

With a subsequent change of variable,

$$i = x + T - 1 - K, \tag{33}$$

we obtain the simple formula

$$b = \frac{(L-1)!(T-1)!}{K!} \left(\frac{LN}{TM}\right)^{T-1} a_1^{\,k} \sum_{i=0}^{S} \left(\frac{TM}{LN}\right)^{i} \frac{(1-a_1)^{S-i}}{i!(S-i)!}, \tag{34}$$

where

$$S = L + T - 2 - K. \tag{35}$$

The most important source of error in Jacobaeus' model is in the hypothesis of mutually independent occupancies of the A-links and B-links. In any attempt to correct this source of error, at least two distinct factors must be considered. First, there may be some existing calls between the primary and the tertiary. The A-link and the B-link on the corresponding number of routes are surely both occupied. Second, we must, in some fashion, take into account the fact that the switch is controlled so as to occupy only complete routes. Therefore, there are always equal numbers of busy A-links and busy B-links incident on each secondary.

Other sources of error include the assumption of independent distributions, $P[x]$ and $P[y]$, and the failure to consider the deformation of these distributions caused by blocking. The latter would require iterative computations for its correction [16], but we find it possible to obtain good agreement with simulation results without that refinement.

7. Global model

Correction of the major sources of error in Jacobaeus' model is facilitated by modeling the traffic through the entire switch. We begin by assuming a stationary, Markov, birth-and-death process for the number of calls, C, in the switch having the equilibrium distribution, $P[C]$. This is determined by the calling rate $\lambda[C]$, as discussed in Section 4.

The subsets of states are indexed by C (i.e., the states are partitioned into $W + 1$ subsets; and the Cth subset consists of all the states having exactly C calls in the switch). The calling rate and blocking rate are measured on the entire switch. All that now remains to be done in order to apply Eq. (25) is to find an approximation to the conditional probability of blocking, $P[B|C]$, given C calls in the switch and a new call being attempted from a randomly selected idle inlet to a randomly selected idle outlet.

The matrix on which the calling inlet terminates is called "the primary", and the matrix on which the called outlet terminates is called "the tertiary". The following additional variables are used in approximating $P[B|C]$:

x = the number of calls in the primary, with distribution, $P[x|C]$,
y = the number of calls in the tertiary, with distribution, $P[y|C]$,
z = the number of calls between the primary and the tertiary, with distribution $P[z|C, x, y]$.

It is approximated by means of the formula,

$$P[B|C] = \sum_{x,y,z} P[x|C]P[y|C]P[z|C,x,y]P[B|C,x,y,z], \tag{36}$$

where $P[B|C, x, y, z]$ is the conditional probability of blocking, given C, x, y, and z.

Formula (36), by explicitly admitting positive values of z, has partially corrected the hypothesis of statistically independent A-link and B-link occupancies. In addition, the distributions of x and y both have explicit functional dependence upon C.

Although this formula requires a triple summation, we shall see that the summations over x, y, and z can be performed algebraically. This leads to a simple result.

The distributions of x, y, and z are hypergeometric, as follows [see Appendix A]:

$$P[x|C] = Q_x[NL-1, C, L-1], \tag{37}$$

$$P[y|C] = Q_y[MT-1, C, T-1], \tag{38}$$

$$P[z|C, x, y] = Q_z[C, x, y]. \tag{39}$$

It is now necessary to estimate $P[B|C, x, y, z]$. There are $(x-z)$ calls in the primary that do *not* go to the tertiary. We assume that these calls use the remaining $(K-z)$ A-links at random, without bias, and independently of the use of the $(K-z)$ B-links of the tertiary that do *not* carry calls from the primary. Exactly $(y-z)$ of the latter are busy, and these are also selected at random without bias. Under these assumptions, which only partially correct the hypothesis of statistical independence, the distribution of the number of idle link pairs (i.e., available routes for the attempted new call) is hypergeometric. The probability of no available routes is

$$P[B|C, x, y, z] = Q_0[K-z, K-x, K-y]. \tag{40}$$

It is easy to see that this function is maximized in the range

$$0 \leq z \leq \min[x, y]$$

when $z = 0$. Therefore, one overestimates the probability of blocking by assuming that $z = 0$ with probability one, an assumption that reduces formula (40) to Jacobaeus' formula, (28).

Using formulas (36) through (40) and performing the summations as explained in Appendix B, we obtain the result [17],

$$P_{\mathrm{H}}[B|C] = \left\{\binom{NL+MT-2-C}{L+\quad T-2-K}\binom{C}{K}\right\} \Big/ \left\{\binom{NL-1}{L-1}\binom{MT-1}{T-1}\right\}. \tag{41}$$

The subscript, H, in Eq. (41) indicates that this is the higher of two estimates to be used here.

$P_{\mathrm{H}}[B|C]$ still overestimates the probability of blocking when the switch is nearly full. To see this, let us first assume that $N \geq M$ (when $N < M$, a similar argument applies, but with the roles of primaries and tertiaries interchanged). The argument will be based on the fact that the number of busy A-links on any secondary must be equal to the number of busy B-links on that secondary.

Each secondary has N A-links and M B-links. Therefore, an idle B-link has access to *at least* $(N-M+1)$ idle A-links. The tertiary has exactly $(K-y)$ idle B-links, giving it access to *at least* $(N-M+1)$ $(K-y)$ idle A-links. The total number of idle A-links *not* incident on the primary is exactly $(NK-C-K+x)$. Therefore, blocking is impossible when $(N-M+1)$ $(K-y) > NK - C - K + x$ or when

$$C \geq (N-2)K + x + y - (N-M)(K-y) + 1. \tag{42}$$

There are values of C that meet this condition for zero blocking, for which our previous estimate of $P[B|C]$, Eq. (41), gives positive values. For example, let $N = M$, $L = T = K$, and let x and y take their maximum values, $(L-1)$ and $(T-1)$, respectively. Then (42) becomes $C \geq NL-1$, but $C = NL-1$ is included in the summation over C from which the loss is to be computed.

This suggests the possibility of a multiplicative correction to the approximation (40), which exhibits a lamentable independence of C in its present form. The correction factor should go to zero when C equals the right-hand member of (42) and increase smoothly as C decreases. It should be as simple as possible and give good agreement with simulation. A form that meets these conditions is the global correction factor,

$$F = \{(N-2)K + x + y - (N-M)(K-y) + 1 - C\} \div (NK - C). \tag{43}$$

The denominator represents the number of idle A-links.

When x and y are carried through the summations over x and y, as performed in Appendix B, their mean values are

$$\bar{x} = K - T + 1 + \{S(MT - 1)/(NL + MT - 2 - C)\}, \quad (44)$$

$$\bar{y} = K - L + 1 + \{S(NL - 1)/(NL + MT - 2 - C)\}, \quad (45)$$

where, as before, S is defined by Eq. (35). As a consequence, the weighted mean of F with respect to x and y is

$$\hat{F} = \{(N - L)K - (N - M)(L - 1) + 1 + G - C\} \div (NK - C), \quad (46)$$

where

$$G = S\{C + (N - M)(NL - 1)\}/(NL + MT - 2 - C). \quad (47)$$

Averaging F has the minor disadvantage that there may be a few small values for x and y that give negative values for F when C is very large, and these are included in the average. The total contribution of these cases will usually be negligible, because small values of x and y give very little blocking, and values of C near its maximum are very unlikely, except in cases of artificially high offered traffic. While it would have been more correct to define F to be zero when its numerator is negative, this would have complicated the summation excessively.

We now define

$$P_L[B|C] = \hat{F}\, P_H[B|C], \quad \hat{F} \geq 0$$
$$= 0, \quad \hat{F} < 0. \quad (48)$$

Using either estimate of $P[B|C]$, and the three global traffic models defined in Section 4, we have the means for evaluating the loss, b, from Eq. (25), which now has the form

$$b = \sum_{C=K}^{W-1} P[C]\lambda[C]P[B|C] / \sum_{C=0}^{W-1} P[C]\lambda[C]. \quad (49)$$

The upper limit on the summations is $(W - 1)$ because $\lambda[W] = 0$. In the numerator, we observe that $P[B|C] = 0$ when $C < K$. Results follow for the three traffic models, which were defined in Section 4.

1. *Bernoulli*

$$b = \sum_{C=K}^{W-1} \binom{W-1}{C} a^C (1 - a)^{W-1-C} P[B|C] \quad (50)$$

2. *Engset*

$$b = \frac{\lambda_0 V}{\mu \overline{C} D_1} \sum_{C=K}^{W-1} \binom{V-1}{C} \left(\frac{\lambda_0}{\mu}\right)^C P[B|C], \quad (51)$$

where

$$\overline{C} = \frac{\lambda_0 V}{\lambda_0 + \mu} \left\{1 - \frac{1}{D_1} \binom{V-1}{W} \left(\frac{\lambda_0}{\mu}\right)^W \right\} \quad (52)$$

is the mean of the offered traffic process and D_1 is defined in Eq. (11).

3. *Erlang*

$$b = \sum_{C=K}^{W-1} \left(\frac{\lambda_0}{\mu}\right)^C \frac{P[B|C]}{C!} \Big/ \left\{D_2 - \left(\frac{\lambda_0}{\mu}\right)^W \frac{1}{W!}\right\}, \quad (53)$$

where D_2 is defined by Eq. (14). In the case of Erlang traffic, the mean of the offered traffic process is

$$\overline{C} = \left(\frac{\lambda_0}{\mu}\right)\left\{1 - \left(\frac{\lambda_0}{\mu}\right)^W \frac{1}{D_2 W!}\right\}. \quad (54)$$

8. Quasi-global model

It will frequently be desirable to use an Engset or an Erlang model for the traffic in a single primary matrix. These models are incompatible with the global model of Section 7, although they can be used in the method of Jacobaeus. In this section, we provide a quasi-global model that is also applicable to such cases.

The calling rate and the blocking rate are measured on a single primary, called "the primary". The subsets of states are indexed by the number of calls, x, in the primary. The formula for the loss is

$$b = \sum_{x=0}^{L-1} P[x]\lambda[x]P[B|x] / \sum_{x=0}^{L-1} P[x]\lambda[x]. \quad (55)$$

In estimating $P[B|x]$, we use the additional variables, C, y, z, as in Section 7, and also u = the number of calls in all other primaries, with distribution $P[u]$.

It follows that

$$C = u + x. \quad (56)$$

The conditional blocking probability will be estimated from

$$P[B|x] = \sum_{u} P[u]P[B|u, x], \quad (57)$$

where

$$P[B|u, x] = \sum_{y,z} P[y|C]P[z|C, x, y]P[B|C, x, y, z]. \quad (58)$$

The last three functions in Eq. (58) are defined by Eqs. (38), (39) and (40); and the summations over y, z are performed as in Appendix B. The result is

$$P[B|u, x] = \frac{(MT - T)!(T - 1)!u!x!}{k!(u - K + x)!(MT - T + K - x)!(T - 1 - K + x)!} \quad (59)$$

$P[u]$ can be given the Bernoulli, Engset, or Erlang distribution, independently of $P[x]$. While some inconsistency may result, no great harm will be done. In any case, the summation over u can be done algebraically. We select the Bernoulli forms with $W = NL \le MT$ as an example, using

$$P[u] = \binom{NL-L}{u} a^u (1-a)^{NL-L-u}, \qquad (60)$$

$$P[x] = \binom{L}{x} a^x (1-a)^{L-x}, \qquad (61)$$

$$\lambda[x] = \lambda_0(L-x). \qquad (62)$$

The summation over u in Eq. (57) gives

$$P[B|x] = \frac{(NL-L)!(MT-T)!(T-1)!x!a^{K-x}}{K!(MT-T+K-x)!(NL-L-K+x)!(T-1-K+x)!}. \qquad (63)$$

From Eqs (55), (61), (62), and (63), the loss is

$$b = \frac{(NL-L)!(MT-T)!(L-1)!(T-1)!a^K}{K!} \times \sum_{i=0}^{S} \frac{(1-a)^{S-i}}{(MT-1-i)!(NL-L-T+1+i)!(S-i)!i!}, \qquad (64)$$

where $S = L + T - 2 - K$ and $i = x + T - 1 - K$, as in Section 6. This formula gives numerical agreement with formula (50) when the high estimate, $P_H[B|C]$, is used for $P[B|C]$ in (50).

The quasi-global model does not facilitate accurate evaluation of the global correction factor, F, of Eq. (43). A useful correction that still overestimates the loss is obtained from Eq. (43) by substituting $-u$ for $(-C+x)$ in the numerator, substituting for y its maximum value, $(T-1)$, and using the mean values $\bar{u}$ for u and $\overline{C}$ for C. Then,

$$\tilde{F} = \{(N-2)K + T - (N-M)(K-T+1) - \bar{u}\} \div (NK - \overline{C}), \qquad (65)$$

where $\bar{u}$ and $\overline{C} = \bar{u} + \bar{x}$ may be taken from the offered traffic distributions $P[u]$ and $P[x]$.

There are no random variables left in Eq. (65), so it may be used as a multiplicative correction for b, regardless of the assumed traffic. We must recall, however, that $N \ge M$ was assumed in its derivation. Otherwise, we must change Eq. (65) to

$$\tilde{F}' = \{(M-2)K + T - (M-N)(K-L+1) - \bar{u}\} \div (MT - \overline{C}).$$

9. Numerical comparisons

It was pointed out in Section 4 that the method of Jacobaeus, the global model, and the quasi-global model are exactly comparable when the traffic is Bernoullian and $W = NL = MT$. This case is used for numerical comparisons of the three methods and for simulations.

The simulations are designed according to a Bernoulian offered traffic process with parameters λ_0, μ and source occupancy,

$$a_0 = \lambda_0/(\lambda_0 + \mu). \qquad (66)$$

The actual source occupancy, a, is reduced somewhat below a_0 by the loss. To compute a, we note that, in equilibrium, the mean rate of new calls equals the mean rate of call terminations.

$$(1-b)(W - Wa)\lambda_0 = \mu W a, \qquad (67)$$

from which we get

$$a = (1-b)\lambda_0/\{(1-b)\lambda_0 + \mu\}, \qquad (68)$$

and, since $\lambda_0/\mu = a_0/(1-a_0)$,

$$a = (1-b)a_0/(1-ba_0). \qquad (69)$$

The reduction in source occupancy from Eq. (69) is very small. With $b = 0.0607$ and $a_0 = 0.90$, $a = 0.894$. This is the greatest reduction observed in the simulations. The actual occupancy per source, a, is used in plotting all simulation results. This is the same as the average line occupancy.

Figures 5(a) through (g) summarize the numerical results of analysis and simulation. Each of these figures has three curves representing the analytical results. The top curve, labeled J, results from the method of Jacobaeus. The middle curve, Q, results from both the global model and the quasi-global model, but without the global correction factor. The lowest curve, G, results from the global model with the global correction factor as defined by Eq. (48).

A single point on each, encircled, shows how the middle curve is lowered when multiplied by the approximate global correction factor, Eq. (65).

All but one of the graphs also show some results of simulation. The vertical bars indicate 95 percent confidence intervals. Each of these is intersected by a horizontal stroke at the mean value of the samples. Simulation results obtained with sequential hunting of routes are labeled "S.H.". The other simulations employ random hunting.

Switch parameters in the captions of Figs. 5(a) through (g) refer to the switch of Fig. 4(a). The traffic is Bernoulli, with $W = NL = MT$ in each case. Figure 5(g), however, shows a case in which $N > M$ and $L < T$.

The lowest curve in each figure is seen to give remarkable agreement with the results of simulation based on

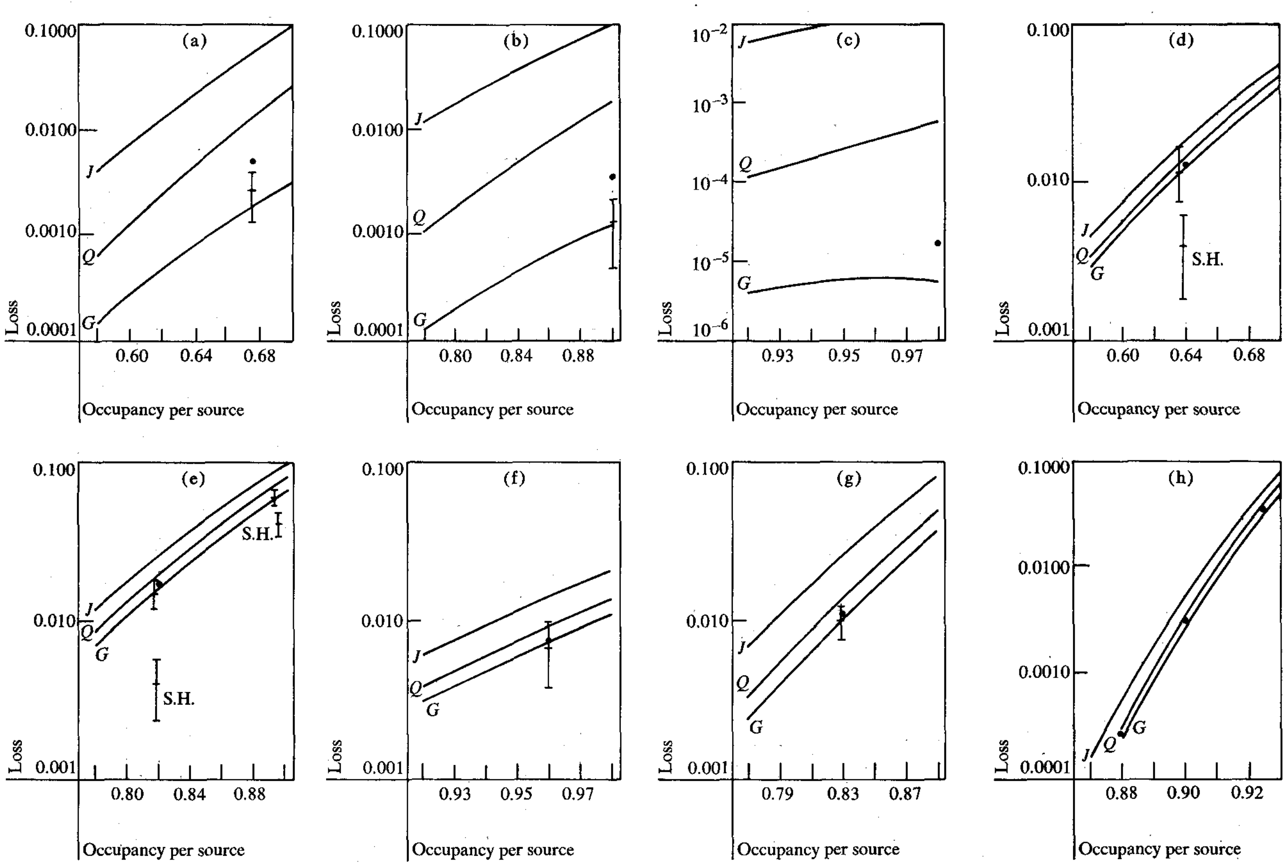

Figure 5 Traffic loss versus the source occupancy plotted for three-stage switches with Bernoulli offered traffic and $W = NL = MT$ sources. Curves J result from Jacobaeus' method; curves Q result from the global and quasi-global models without the global correction factor; curves G result from the global model with the globally corrected conditional probability of blocking, Eq. (48). The isolated points are obtained from the quasi-global model with the approximate global correction factor, Eq. (65). The vertical bars show 95 percent confidence intervals and sample mean values obtained from simulations. Those labeled "S.H." result from sequential hunting, and the others from random hunting. (a) $N = M = 2$, $L = T = K = 24$. (b) $N = M = 2$, $L = T = 24$, and $K = 28$. (c) $N = M = 2$, $L = T = 24$, and $K = 32$. At a source occupancy of 0.98 and with random hunting, no loss was observed in the simulations. (d) $N = M = 10$, $L = T = K = 24$. (e) $N = M = 10$, $L = T = 24$, and $K = 28$. (f) $N = M = 10$, $L = T = 24$, and $K = 32$. At a source occupancy of 0.96, and with *sequential hunting, the simulated mean loss was 0.000407.* (g) $N = 6$, $M = 3$, $L = 12$, $T = 24$, and $K = 24$. (h) $N = M = 10$ and $L = T = K = 500$. This switch was not simulated.

random hunting. The loss obtained from sequential hunting is always less than that obtained from random hunting. The relative decrease in loss provided by sequential hunting becomes more pronounced as the traffic is decreased or as the number of routes, K, is increased. It is clear that sequential hunting is to be recommended and that even the lowest curve significantly overestimates the loss in this case. However, the relative overestimate decreases as the traffic increases toward the overload point.

Two simulation results are not shown on the figures. The first of these was done for the switch of Fig. 5(c) with random hunting and an occupancy of 0.98. No loss at all was observed in six runs of 20,000 trials each. The second was done for the switch of Fig. 5(f) with sequential hunting and an occupancy of 0.96. In this case, 14 runs of 10,000 trials each gave a mean loss of 0.000407 and a standard deviation of 0.000260.

The figures show that the analytical curves are quite far apart when $N = M = 2$, but that they are reasonably close when $N = M = 10$. Both the global correction and the occurrence of calls between the primary and the tertiary become less important as the numbers of primaries and tertiaries increase. As an example, consider the switch of Fig. 5(f), but with $N = M = 20$ instead of 10. At a source occupancy of 0.96, the three curves give predicted losses of 0.0137, 0.0112, and 0.0101. The Jacobaeus curve is independent of N and M, while the other curves rise toward it as N and M increase. Going to $N = M = 100$, the predicted losses are 0.0137, 0.0131, and 0.0129.

We conclude that the method of Jacobaeus can give acceptable accuracy when N and M are sufficiently large, provided that random hunting is assumed. Unfortunately, none of the simple analytical models are accurate for sequential hunting. We must take comfort in the observations that overestimates lead to safe designs and that the relative error is least in the critical overload region. Nevertheless, we might find a very much lower loss for the switch of Fig. 5(h) with sequential hunting, because of the very large number, 500, of routes. This switch was not simulated because of its large size.

10. Additional observations

The preceding section summarizes our evaluation of the analytical models considered here. A few additional comments on the three-stage switches may be of interest.

It is the author's opinion, offered without evidence, that high-occupancy hunting gives the least loss; but it is not likely to be significantly better than sequential hunting, which is a computationally simpler procedure. Although sequential hunting with random start can be carried out with less delay than sequential hunting in some systems, it is likely to give as much loss as random hunting.

With a given hunting method and a fixed traffic load, the loss in a three-stage switch may be decreased by increasing the number of crosspoints in either of two ways. One way is to increase the sizes of the primaries and tertiaries by increasing K, L, and T more or less proportionately. This does not decrease the link occupancies, but it does offer more possible routes, thereby increasing the chance that one will be available. This results in lower loss at a given traffic load, but in a more steeply rising curve, so that the switch may still overload at sufficiently high occupancy. An example of the loss curve in such a switch is shown in Fig. 5(h), where $K = L = T = 500$. Duerdoth and Seymour [1] call such a switch "quasi-non-blocking," presumably because the loss is extremely low at the expected busy-hour traffic load.

A second way to decrease the loss is to increase the ratio of K to L and T, that is, to increase the internal expansion of the switch. This produces lower loss and also reduces the slope of the loss curve. Such a switch is illustrated in Fig. 5(f), where $L = T = 24$ and $K = 32$. At 98 percent occupancy, the loss of this switch is about 0.01 with random hunting and much less with sequential hunting. A switch like this cannot be overloaded; but it requires only 32/47 of the crosspoints that would be needed to make it strictly nonblocking.

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Appendix A: Hypergeometric distribution [18]

The hypergeometric distribution applies to random samples taken without replacement from a finite population containing two types of elements. Suppose there are ν elements, of which α are of type 1 and $(\nu - \alpha)$ are of type 2. If we select β elements at random, without replacement, then the probability that exactly γ of these are of type 1 is

$$Q_\gamma[\nu, \alpha, \beta] = \binom{\alpha}{\gamma}\binom{\nu-\alpha}{\beta-\gamma} \Big/ \binom{\nu}{\beta}$$

$$= \binom{\beta}{\gamma}\binom{\nu-\beta}{\alpha-\gamma} \Big/ \binom{\nu}{\alpha}$$

$$= \frac{(\nu-\alpha)!(\nu-\beta)!\alpha!\beta!}{\nu!\gamma!(\alpha-\gamma)!(\beta-\gamma)!(\nu-\alpha-\beta+\gamma)!}, \tag{A1}$$

when $\gamma \geq 0$, $(\alpha-\gamma) \geq 0$, $(\beta-\gamma) \geq 0$, $(\nu-\alpha-\beta+\gamma) \geq 0$, and is equal to zero otherwise.

The mean of this distribution is

$$\bar{\gamma} = \alpha\beta/\nu. \tag{A2}$$

Because $Q_\gamma[\nu, \alpha, \beta]$ is a discrete probability distribution, we have the important identity

$$\sum_{\gamma=i}^{j} Q_\gamma[\nu, \alpha, \beta] = 1, \tag{A3}$$

whenever $j \geq \min[\alpha, \beta]$

and $\quad i \leq \max[0, \alpha+\beta-\nu]$.

Appendix B: Derivation of Formula (41)

Summation of the right-hand member of Eq. (36) is achieved by the use of the identity (A3). The order of summations will be z, y, and x.

First, defining

$$P[B|C, x, y] = \sum_{z=0}^{x+y-K} P[z|C, x, y]P[B|C, x, y, z] \tag{B1}$$

and substituting from Eqs. (39) and (40), we get

$$P[B|C, x, y] = \sum_{z=0}^{x+y-K} \frac{(C-x)!x!(C-y)!y!}{C!z!(K-z)!(x+y-K-z)!(C-x-y+z)!}. \tag{B2}$$

Use of the identity (A3) in the form

$$1 = \sum_{z=0}^{x+y-K} Q_z[C, K, x+y-K]$$

leads to the formula

$$P[B|C, x, y] = \frac{(C-x)!x!(C-y)!y!}{(C-K)!K!(C-x-y+K)!(x+y-K)!}. \tag{B3}$$

Next, defining

$$P[B|C, x] = \sum_{y=0}^{T-1} P[y|C]P[B|C, x, y] \tag{B4}$$

and substituting from Eqs. (B3) and (38), we get

$$P[B|C, x] = \sum_{y=0}^{T-1} \frac{(MT-T)!(T-1)!(MT-1-C)!C!(C-x)!x!}{(MT-1)!K!(C-K)!(x+y-K)!(T-1-y)!(C+K-x-y)!(MT-T-C+y)!}. \tag{B5}$$

Use of (A3) in the form

$$1 = \sum_{y=0}^{T-1} Q_{x+y-K}[MT-1, T-1-K+x, C]$$

enables us to show that

$$P[B|C, x] = \frac{(MT-T)!(T-1)!(C-x)!x!}{K!(C-K)!(MT-T+K-x)!(T-1-K+x)!}. \tag{B6}$$

Finally, we have

$$P_H[B|C] = \sum_{x=0}^{L-1} P[x|C]P[B|C, x]. \tag{B7}$$

Substituting from Eqs. (B6) and (37), we get

$$P_H[B|C] = \sum_{x=0}^{L-1} \frac{(MT-T)!(T-1)!(NL-L)!(L-1)!(NL-1-C)!C!}{(NL-1)!K!(C-K)!(T-1-K+x)!(L-1-x)!(MT-T+K-x)!(NL-L-C+x)!}. \tag{B8}$$

The use of (A3) in the form

$$1 = \sum_{x=0}^{L-1} Q_{T-1-K+x}[NL+MT-2-C, L+T-2-K, MT-1]$$

enables us to derive formula (41).

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The author is located at the IBM Thomas J. Watson Research Center, Yorktown Heights, New York, 10598.