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Economic Order and Surplus Quantities Model

Abstract: A standard mathematical model for inventory management is known as the Economic Order Quantity (EOQ) model. In this communication the EOQ model is extended to include the possibility of determining how much, if any, excess stock should be sold at the beginning of a decision period. The new model is of practical importance for situations in which a formal inventory management system is to be instituted while substantial inventories exist or when changes in demand, ordering cost, or carrying and interest charges require recomputation of the economic order quantity.

Introduction

The objective of inventory management is to maintain levels of inventory that are optimum with respect to customer demand and cost considerations [1]. The well-known Economic Order Quantity (EOQ) model [2] is predicated upon balancing ordering costs against inventory holding costs. The model presented here is similar to the EOQ model except that at inventory review time an initial inventory level greater than zero is assumed to exist. If that inventory level is considered to be excessive, part of the inventory can be sold and the balance kept in stock until it is depleted. At depletion, the stock is replenished at regular intervals by orders of equal size.

The importance of the new model, which is called the economic order and surplus quantities model, stems from the need to apply the EOQ model to situations in which there is an initial inventory of varying magnitude. These situations can arise when one attempts to apply the EOQ formula for the first time or when the economic order quantities are reviewed and recomputed due to changes in demand, ordering costs, or carrying or interest charges. Development of the model reported here was motivated by the existence of such a situation within IBM; the model is currently being used for managing the inventory of computer components at the Poughkeepsie, New York plant.

Mathematical model

As in the EOQ model, the formulation presented here assumes that items are withdrawn from inventory at a known constant rate r (items per unit time). In contrast to the standard formulation, however, it is assumed that there exists an initial inventory I. The cost of items in the initial stock is c_0 per item. If the initial inventory is considered to be excessive with respect to an estimate of future demand, part of it may be sold or scrapped at a salvage price v per item and the remainder held until depletion. After the remaining stock is depleted, inventory is replenished periodically by equal-sized orders of q items; it is assumed for the model that the orders are filled instantaneously. Ordering costs include a set-up cost s per replenishment, charged at the beginning of a period, and a purchase (or production) cost c per item. An inventory carrying charge (handling, storage, insurance, tax, deterioration and obsolescence costs) h per unit time and an interest charge, or cost of capital, i per unit time are considered separately; both h and i are expressed as fractions of the cost of an item.

The inventory level as a function of time t is illustrated in Fig. 1, where two time phases are indicated. Phase 1 is the period during which the initial inventory is being depleted. If the entire initial inventory is held, it will take $\mathcal{T} = I/r$ time units for the stock to become depleted. However, if Q items ($Q < I$) are held, it will take $T = Q/r$ time units until depletion is reached. In the latter case, the quantity $I - Q$ is sold as surplus. Phase 2 is the periodic replenishment phase during which items are ordered in equal quantities q to last $\tau = q/r$ units of time.

The problem is to determine how much of the initial inventory to hold and how much to sell; how often to order (or make a production run); and what size the order should be to minimize the total discounted cost over an infinite horizon. The total discounted cost is a function of the decision variables Q and q or, equivalently, T and τ. For convenience, the time equivalents of the quantity held in inventory and the economic order quantity, T and τ respectively, are used as the decision variables.

In phase 1 the revenue accrued from selling the surplus is given by

$$v(I-Q) = vr(\mathscr{T}-T), \qquad \mathscr{T} \geq T. \tag{1}$$

The holding cost, calculated using continuous discounting (rather than discrete internal discounting) is given by

$$\int_0^T \exp(-it)\; hc_0 r(T-t)\; dt$$
$$= hc_0 r[Ti^{-1} + \exp(-iT)i^{-2} - i^{-2}]. \tag{2}$$

The notion of discounting implies that interest accrual makes a specified future cost have a lower present value. The higher the interest rate the lower is the present value of a given future cost. Therefore, the total discounted cost in phase 1, for $\mathscr{T} \geq T \geq 0$, is

$$C_1(\mathscr{T}; T) = -vr(\mathscr{T}-T)$$
$$+ hc_0 r\,[Ti^{-1} + \exp(-iT)i^{-2} - i^{-2}]. \tag{3}$$

During phase 2 the cost per order is zero if $q = 0$ and, if $q = r\tau > 0$, is

$$s + cq = s + cr\tau. \tag{4}$$

The holding cost per cycle is

$$\int_0^\tau \exp(-it)\; hcr(\tau - t)\; dt$$
$$= hcr[\tau i^{-1} + \exp(-i\tau)i^{-2} - i^{-2}]. \tag{5}$$

The total discounted cost in phase 2 is

$$\begin{aligned} C_2(\tau) &= \sum_{n=0}^{\infty} \exp(-in\tau)\; \{(s + cr\tau) \\ &\quad + hcr[\tau i^{-1} + \exp(-i\tau)i^{-2} - i^{-2}]\} \\ &= [1 - \exp(-i\tau)]^{-1}\{(s + cr\tau) \\ &\quad + hcr[\tau i^{-1} + \exp(-i\tau)i^{-2} - i^{-2}]\} \\ &= [1 - \exp(-i\tau)]^{-1} \\ &\quad \times [s + cr\tau(1 + hi^{-1})] - hcri^{-2}. \end{aligned} \tag{6}$$

The total discounted cost over the infinite horizon is

$$C(\mathscr{T}; T, \tau) = C_1(\mathscr{T}; T) + \exp(-iT)C_2(\tau),$$
$$\mathscr{T} \geq T \geq 0. \tag{7}$$

The cost function $C(\mathscr{T}; T, \tau)$ is convex in both T and τ; this is a necessary and sufficient condition for the existence of a globally optimal solution of the function. It follows from the convexity properties that the optimal

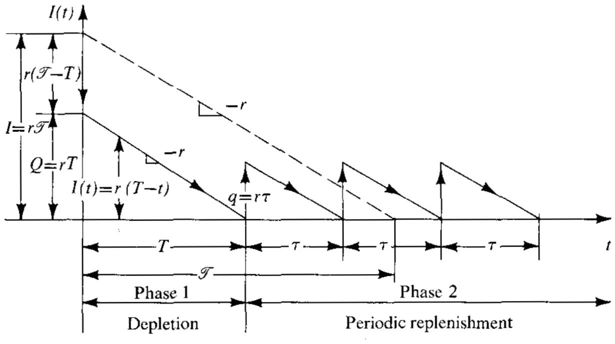


Figure 1 Inventory level as a function of time.

values τ^* and T^* must satisfy the relations $\partial C(\mathscr{T}; T, \tau)/\partial\tau = 0$ and $\partial C(\mathscr{T}; T, \tau)/\partial T = 0$, The first relation gives

$$\exp(-iT)[1 - \exp(-i\tau)]^{-2}\{[1 - \exp(-i\tau)cr(1 + hi^{-1})$$
$$- [s + cr\tau(1 + hi^{-1})]\; i \exp(-i\tau)\} = 0,$$
$$rc(1 + hi^{-1})[1 - \exp(-i\tau) - i\tau \exp(-i\tau)]$$
$$= is \exp(-i\tau),$$
$$rc(1 + hi^{-1})[1 - (1 + i\tau)(1 - i\tau + \tfrac{1}{2}i^2\tau^2 - \cdots)]$$
$$= is(1 - i\tau + \tfrac{1}{2}i^2\tau^2 - \cdots). \tag{8}$$

If we neglect higher-order terms,

$$\tau^* \approx [2s/cr(i+h)]^{\frac{1}{2}} \tag{9}$$

and

$$q^* \approx [2sr/c(i+h)]^{\frac{1}{2}}. \tag{10}$$

The optimal phase 2 cost is given approximately by

$$\begin{aligned} C_2(\tau^*) &\approx [s + cr\tau^*(1 + hi^{-1})] \\ &\quad \times [i\tau^* - \tfrac{1}{2}i^2\tau^{*2}]^{-1} - hcri^{-2} \\ &\approx [s + cr\tau^* + \tfrac{1}{2}hcr\tau^{*2}] \\ &\quad \times [i\tau^* - \tfrac{1}{2}i^2\tau^{*2}]^{-1}. \end{aligned} \tag{11}$$

Insertion of (9) into (11) and algebraic manipulation yields

$$C_2(\tau^*) = cri^{-1} + 2scr(i+h)$$
$$\times \{i[2scr(i+h)]^{\frac{1}{2}} - i^2 s\}^{-1}. \tag{12}$$

When higher-order terms are neglected, the final approximation is

$$C_2(\tau^*) \approx i^{-1}\{cr + [2scr(i+h)]^{\frac{1}{2}}\}$$
$$\approx cri^{-1}[1 + (i+h)\tau^*]. \tag{13}$$

The values of τ^*, q^* and $C_2(\tau^*)$ obtained from Eqs. (9), (10) and (13) are the classical EOQ model results. It is interesting to note that the same results are obtained by

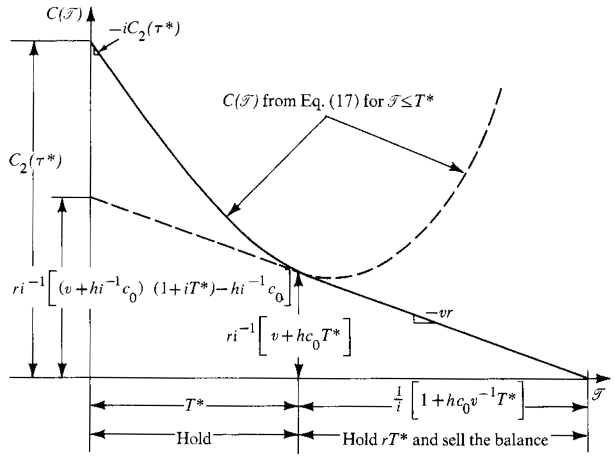
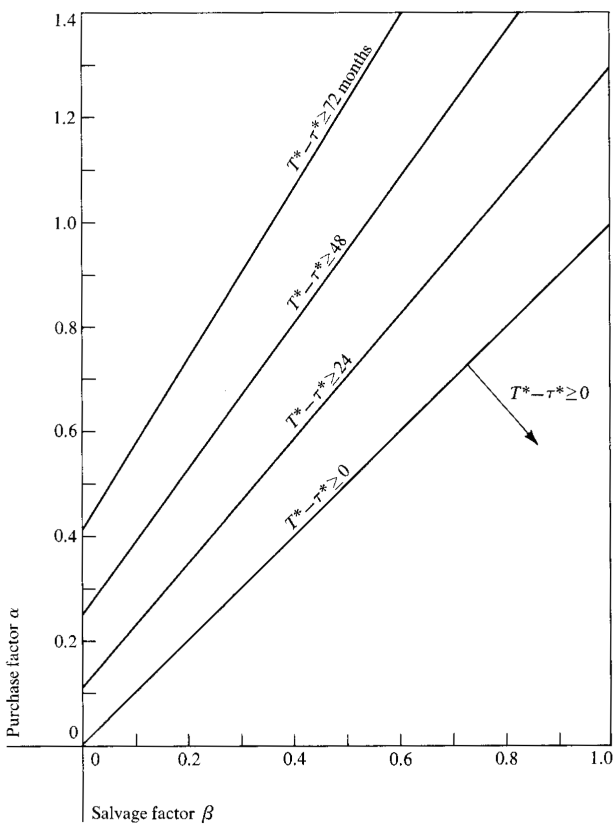


Figure 2 Minimum discounted cost as a function of initial inventory equivalent time.

Figure 3 The difference between optimal values of initial inventory depletion time T^* and periodic ordering time τ^* for selected values of $T^* - \tau^*$. Appropriate values of the independent variables α and β determine the points on a given curve: α is the purchase factor and β is the salvage factor. For this family of curves the interest charge i is eight percent and the carrying charge h is 12 percent. The curves are obtained from solutions of Eq. (23).



minimizing either the total discounted cost or the cost per unit time, when the cost of capital is considered as an integral part of the holding cost [2, 3].

The second relation to be satisfied, namely $\partial C(\mathscr{T}; T, \tau) / \partial T = 0$, yields

$$ri^{-1}[(v + hi^{-1}c_0)\exp(iT^*) - hi^{-1}c_0] = C_2(\tau^*), \text{ or}$$

$$T^* = \begin{cases} i^{-1}\ln\{[ir^{-1}C_2(\tau^*) \\ \quad + hi^{-1}c_0][v + hi^{-1}c_0]^{-1}\}, \\ \quad \text{if } iC_2(\tau^*) > rv; \\ 0, \text{ otherwise.} \end{cases} \tag{14}$$

Substitution of (13) into (14) gives an approximation of T^* in terms of τ^*:

$$T^* \approx \begin{cases} i^{-1}\ln\Big[\{c[1 + (i+h)\tau^*] \\ \quad + hi^{-1}c_0\}(v + hi^{-1}c_0)^{-1}\Big], \\ \quad \text{if } c[1 + (i+h)\tau^*] > v; \\ 0, \text{ otherwise.} \end{cases} \tag{15}$$

The critical value in determining the optimal policy, which is stated as follows, is T^*:

$$\mathscr{T} = \begin{cases} T^* \text{ if } \mathscr{T} > T^*; \\ \mathscr{T} \text{ if } \mathscr{T} \le T^*. \end{cases}$$

If $C(\mathscr{T})$ is defined as the minimum total discounted cost, then

$$C(\mathscr{T}) = \begin{cases} -vr(\mathscr{T} - T^*) + hc_0r[T^*i^{-1} \\ \qquad + \exp(-iT^*)i^{-2} - i^{-2}] \\ \qquad + \exp(-iT^*)\,C_2(\tau^*), \\ \qquad \text{if } \mathscr{T} > T^*; \\ hc_0r[\mathscr{T}i^{-1} + \exp(-i\mathscr{T})i^{-2} - i^{-2}] \\ \qquad + \exp(-i\mathscr{T})\,C_2(\tau^*), \\ \qquad \text{if } \mathscr{T} \le T^*. \end{cases} \tag{16}$$

Substitution of (14) into (16) gives

$$C(\mathscr{T}) = \begin{cases} ri^{-1}[-iv\mathscr{T} + (v + hi^{-1}c_0)(1 + T^*) \\ \qquad - hi^{-1}c_0], \text{ if } \mathscr{T} > T^*; \\ ri^{-1}\{hc_0\mathscr{T} + (v + hi^{-1}c_0) \\ \qquad \times \exp[-i(\mathscr{T} - T^*)] - hi^{-1}c_0\}, \\ \qquad \text{if } \mathscr{T} \le T^*. \end{cases} \tag{17}$$

In words, the optimal inventory policy is to sell $I - Q^* = r(\mathscr{T} - T^*)$ units as surplus if $I = r\mathscr{T} > Q^* = rT^*$ and to hold the initial inventory until depletion if $I \le Q^*$.

As shown in Fig. 2 the optimal discounted cost $C(\mathscr{T})$ is a convex monotonic non-increasing function of $\mathscr{T}$.

When the initial inventory is zero ($\mathscr{T} = 0$), $C(\mathscr{T}) = C_2(\tau^*)$, which is the classical EOQ model cost. It is evident that the higher the initial inventory, the lower the optimal cost. The optimal cost goes to zero as the initial inventory goes to $Q^*(1 + hi^{-1}c_0v^{-1}) + ri^{-1}$. For larger values of initial inventory, the optimal cost becomes negative. In Fig. 2 it is also shown that the slope of $C(\mathscr{T})$ at $\mathscr{T} = 0$ is $-C_2(\tau^*)$ and it follows that if $-iC_2(\tau^*) < -vr$, then T^* is positive; otherwise, T^* is 0 as stated in Eq. (14).

Approximation formulas

From (15) it follows that

$$T^* \approx i^{-1} \ln \left[1 + \frac{ic(i+h)\tau^*}{ic + hc_0}\right] + i^{-1} \ln \left[\frac{ic + hc_0}{iv + hc_0}\right]. \quad (18)$$

Because $\ln(1 + x) \approx x$ for small x, it follows that

$$T^* \approx i^{-1} \left[\frac{ic(i+h)\tau^*}{ic + hc_0}\right] + i^{-1} \ln \left[\frac{ic + hc_0}{iv + hc_0}\right], \quad (19)$$

and, under the assumption that $c \geq c_0$, the following approximations hold:

$$T^* \gtrsim \begin{cases} \tau^* + i^{-1} \ln \left[\dfrac{ic + hc_0}{iv + hc_0}\right], & \text{if } c > v; \\ \tau^*, & \text{otherwise}; \end{cases} \quad (20)$$

$$Q^* \gtrsim \begin{cases} q^* + ri^{-1} \ln \left[\dfrac{ic + hc_0}{iv + hc_0}\right], & \text{if } c > v; \\ q^*, & \text{otherwise}; \end{cases} \quad (21)$$

and

$$C(\mathscr{T}) \lesssim \begin{cases} ri^{-1} \left\{-iv\mathscr{T} + (v + hi^{-1}c_0)\left[1 + i\tau^* + \ln \left(\dfrac{ic + hc_0}{iv + hc_0}\right) - hi^{-1}c_0\right]\right\}, \\ \qquad \text{if } \mathscr{T} > T^*; \\ ri^{-1} \{hc_0\mathscr{T} + (c + hi^{-1}c_0) \times \exp[-i(\mathscr{T} - \tau^*)] - hi^{-1}c_0\}, \\ \qquad \text{if } \mathscr{T} \leq T^*. \end{cases} \quad (22)$$

Approximation (20) can be rewritten as

$$T^* - \tau^* \quad \begin{cases} i^{-1} \ln \left[\dfrac{i\alpha + h}{i\beta + h}\right], \\ \qquad \text{if } \alpha > \beta; \\ 0, \quad \text{otherwise}, \end{cases} \quad (23)$$

where

$\alpha = c/c_0$ (purchase factor) and

$\beta = v/c_0$ (salvage factor).

Although τ^* is a direct function of the demand r, $T^* - \tau^*$ is independent of the demand. Relation (23) can be used as the basis for constructing graphs that specify $T^* - \tau^*$ for different purchase and salvage factors, given the inventory carrying charge and the interest charge. Figure 3 shows a family of curves for $T^* - \tau^*$ as a function of α and β, for $h = 12$ percent and $i = 8$ percent. These curves can be used for inventory control without explicit knowledge of demand.

In general, the purchase factor is greater than or equal to the salvage factor; that is, $\alpha \geq \beta$ or $c \geq v$. However, if $\alpha = \beta$, we conclude from (23) that $T^* \gtrsim \tau^*$, which means that the optimal amount of inventory to hold approaches that calculated from the EOQ model. This situation is also illustrated in Fig. 3.

Summary

In this work we have developed a model and a solution for determining the economic order and surplus quantities of inventory items. The solution is essentially a decision rule which specifies that if the inventory on hand is greater than a critical quantity, that quantity should be held and the balance sold; if the inventory on hand is less than or equal to the critical quantity, the total inventory should be held until depletion. From there on, the conventional EOQ model solution should be followed.

To simplify calculation of the time equivalent of the quantity held in inventory, an approximation formula is used to generate a set of curves that specify the difference between the optimal time equivalents of the quantity held in inventory and the economic order quantity for different purchase and salvage factors, given the inventory carrying and interest charges. Graphs such as those shown in Fig. 3 can easily be used for inventory control.

References

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