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Effects of Dispersion on Steady State Electromagnetic Shock Profiles

Abstract: In nonlinear electromagnetic media the various portions of a wave can travel with different velocities, which can result in the formation of electromagnetic shock waves. The structure of such a steady state shock is determined by an equilibrium between the velocity differences that tend to sharpen the shock and the sources of dispersion that cause a broadening of the shock. Several nonlinear transmission line models are examined for the nature and existence of a single-valued steady state shock. In all cases a nonlinear shunt capacitance is assumed. If the dispersion arises from the relaxation behavior caused by a resistance in series with the nonlinear capacitance, a steady shock always exists, its width decreasing as the extent of the nonlinearity generated by the shock increases. If the series resistance is itself shunted by another capacitance, the relaxation process is not manifested at very high frequencies. This system yields a critical condition for the existence of a continuous single-valued steady state wave profile. If the line has too little dispersion, the steady state profile is multivalued and therefore physically unrealizable.

These dispersion requirements are equivalent to the condition that the velocity of small, high frequency signals ahead of the shock must be greater than the velocity of the shock itself. It is believed that this condition is a broadly applicable criterion for the existence of a stable, single-valued, steady state wave profile. While this hypothesis is not proved analytically, it is supported here by plausibility arguments and by analysis of another system in which the dispersion is included in the linear series inductance rather than in the nonlinear shunt capacitance.

Introduction

The concept that in a nonlinear wave propagation system the various parts of the wave travel with different velocities, and that wave fronts (or tails) can sharpen into shock waves, is deeply imbedded in the classical theory of fluid dynamics [1–3]. The fact that this same concept carries over into the treatment of electromagnetic waves was appreciated by Salinger [4] in a pioneering paper in 1923. However, no further progress seems to have occurred in this field until the 1950s, when practical interest arose in ferroelectrics, in parametric phenomena, and in propagation along magnetically loaded lines in memory arrays. A detailed discussion of electromagnetic shock wave formation and propagation has been provided by one of the authors [5]. A considerable body of experimental and theoretical literature on nonlinear transmission lines has been published since then, and we cite here only a small portion of that literature [6–13]. With the advent of the laser in the 1960s, the concept that different parts of a wave in a nonlinear medium can move with different velocities was used again, first by Chiao et al. [14] in their discussion of the self-focusing of laser beams. Subsequently the concept of self-induced frequency changes in amplitude-modulated laser pulses was also suggested [15], elaborated upon [16], and then observed [17,18]. The behavior of an additional small signal controlled by the propagation velocity profile of a large nonlinear signal has also been treated [19]. While the nonlinear optical phenomena have undoubtedly come to overshadow the earlier baseband propagation problems in their interest, the baseband phenomena are simpler. They are, therefore, a good ground for conceptual exploration, and we return here to this earlier territory, not so much with a specific application or experiment in mind, but simply to understand more carefully what determines the structure of a shock, particularly its width. We are also interested in the condition under which we can in fact expect the resolution of a shock into a continuous, single-valued, invariant, moving profile.

The velocity differences found within an initially spread-out portion of a wave front can cause the front to contract (or expand, depending on the sign of the changes in the nonlinear reactance). If such a contraction continues in an unaltered fashion, it can eventually lead to a multivalued function, in which one part of the wave has overtaken another. As the wave sharpens, and

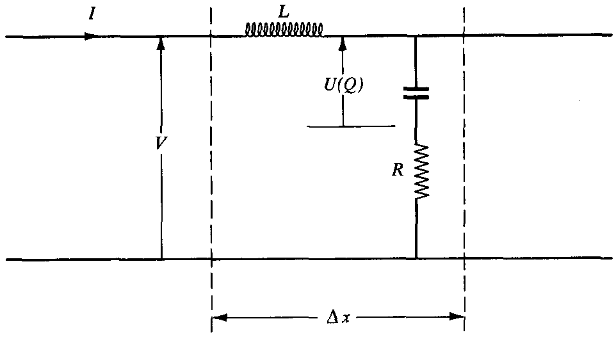

Figure 1 Transmission line having a nonlinear capacitance with a finite relaxation time; L is the inductance, $U(Q)$ characterizes the voltage-charge relationship of the nonlinear capacitance, and R is the resistance providing the damping.

thus involves higher frequencies, dispersive effects that were neglected at first will inevitably manifest themselves and can balance the contraction effects, leading to a profile that propagates without change. An analysis of this balancing process arising from the inertial and relaxation effects of charges in a dielectric was presented some years ago [20]. Since this work has never reached full formal publication a *portion* of it is presented in the second section, which gives a detailed analysis of the effect of relaxation (but without allowing for inertial effects) on the shock wave structure.

The effects of dispersion on the propagation of waves in a nonlinear medium have attracted considerable attention [3]. The approximation of dispersion functions by finite polynomials results in Burger's equation and in the Korteweg and de Vries equation, which have been studied extensively [3]. An approximation of the dispersion function by a polynomial gives rise to a strong dispersion, especially at high frequencies, and always prevents the "overtaking" phenomenon, but the effects of the original exact dispersion are still open to question. In the third section we consider a transmission line in which the dispersion results from a shunt branch, where the nonlinear capacitance is in series with a parallel RC combination. The dispersion can be controlled by varying a parameter of the system and the resulting effects then analyzed. We observe from the solution that the dispersion must be strong enough to balance the contraction effect that is due to the nonlinearity of the system. Otherwise, overtaking will occur, resulting in multivalued fields and showing that a critical dispersion is required for the existence of a single-valued, steady state, moving wave profile in this nonlinear system.

For a nonlinear dispersive wave system we may define various wave velocities. These include the low- and high-frequency wave velocities of very small signals, assuming that the nonlinear capacitance is charged to a specified spatially uniform bias level. We can also invoke shock velocities related to the overall, or average, capacitance associated with the shock transmission at low frequencies. The critical dispersion, which depends on the nonlinearity of the system, can be expressed as a relation among some of these wave velocities. Because such wave velocity concepts apply to many systems, it is hoped that our present results can eventually be generalized to a larger class of nonlinear dispersive systems. As an illustration, we also consider in the fourth section the case of a transmission line with a dispersive series inductance and find that the result is identical to that obtained in the third section if the results are expressed in terms of the wave velocities mentioned here.

In our subsequent discussions, for simplicity, we focus on systems in which the shock forms on the leading edge of a pulse. Because the modifications required for shocks on a trailing edge are trivial, they are not explicitly mentioned at every step.

Shock wave structure has been considered by a number of authors who were motivated by various physical problems, including, e.g., the classical case of hydrodynamic shocks [21]. Khokhlov [13] specifically considered nonlinear transmission lines and found a shock thickness that agrees with our own results of the next section. G. B. Whitham in a comprehensive paper [22] discussed many aspects of shock wave structure, emphasizing magnetohydrodynamics, and anticipated some of the concepts presented throughout this paper. In particular he emphasized the view of shock structure as an equilibrium between the nonlinear behavior and the dispersion. He furthermore pointed out, as we shall, that as parameters are varied in some of these systems the steady state profile can make a transition from single-valued behavior to multivalued behavior. As we show later in more detail, the multivalued profiles are, of course, nonphysical and reflect on the accuracy of the equations leading to them.

Shock wave structure for simple relaxation time

To facilitate the discussion for the general case in the next section, we consider here a special case for which an explicit solution can be given, showing the detailed structure of a shock wave. A transmission line that has a nonlinear capacitance with a simple series resistance is shown in Fig. 1. This transmission line is a special case of lines analyzed by Landauer and Thomas [20] as a model for the study of TEM wave propagation in a ferroelectric material at a temperature just above the transition point. Riley [6] has also given experimental and theoretical analyses of a line in which the nonlinear capacitance has a resistance in series with it. We present here an analytic solution for a steady state shock wave in this system, so that the velocity and the detailed structure of a shock wave can be related to the characteristics of the medium.

From Fig. 1, the basic equations governing the fields can easily be obtained by invoking Kirchhoff's law of voltage and current:

$$\partial V/\partial x = -L(\partial I/\partial t), \tag{1}$$

$$\partial I/\partial x = -\partial Q/\partial t, \text{ and} \tag{2}$$

$$V = U(Q) + R(\partial Q/\partial t), \tag{3}$$

where V, I, and Q are the transmission line voltage, current, and charge, respectively; L is the inductance per unit length; R^{-1} is the conductance per unit length, which characterizes the damping mechanism; and U is the voltage across the nonlinear capacitor and is assumed to be a single-valued function of the charge Q.

For a wave of constant profile, we assume a solution of the form

$$Q = Q(\theta); \qquad \theta = x - ut, \tag{4}$$

where u is the constant shock wave velocity characterizing the charge variation as well as the variation of voltage V and current I. With this assumption Eqs. (1) and (2) become $V' = uLI'$ and $I' = uQ'$, respectively, where the primes denote differentiation with respect to the argument of the function. These two equations can readily be integrated to yield

$$V = u^2LQ. \tag{5}$$

In obtaining the last equation, we have set the integration constant equal to zero, which is correct for a shock transition propagating along an initially uncharged line. With the notation $U(Q) = (1/C_0)\,[Q + n(Q)]$, where C_0 is the linear or small-signal capacitance of the transmission line and $n(Q)$ represents the nonlinear behavior of the capacitance, Eqs. (3) and (5) can be combined to yield

$$\alpha Q = n(Q) + \tau_0(\partial Q/\partial t). \tag{6}$$

In Eq. (6) we have invoked the following notation:

$$\alpha = u^2/v_0^{\,2} - 1,$$

$$v_0^{\,2} = 1/LC_0, \text{ and } \tau_0 = RC_0,$$

in which α represents the deviation of the shock wave velocity from the linear characteristic wave velocity v_0, and will be related to the shock strength later; and τ_0 is the relaxation time for a small signal on the uncharged line. In terms of α, Eq. (5) can be written

$$V = (1/C_0)\,(1 + \alpha)\,Q. \tag{7}$$

Physically, Eq. (6) states that the nonlinearity and the relaxation offset each other's effect, resulting in a modification of the linear capacitance, as is evident from Eq. (7). For the functional form given in Eq. (4), Eq. (6) becomes

$$u\tau_0 dQ/d\theta = n(Q) - \alpha Q, \tag{8}$$

which determines the detailed waveform of the steady state shock.

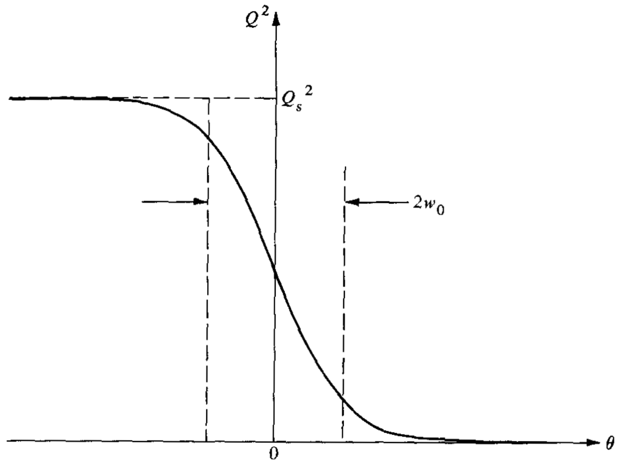

Figure 2 Typical shock wave form: $Q^2(x - ut) = \frac{1}{2}Q_s^{\,2}[1 - \tanh\,(x - ut)/w_0]$.

The waveform depends on the nonlinearity of the medium. For example, for the lowest order nonlinearity in a medium with a center of inversion, the cubic case: $n(Q) = \eta\, Q^3$, where η is the nonlinear constant. In that case Eq. (8) can be integrated to yield

$$Q^2(x - ut) = \tfrac{1}{2}\, Q_s^{\,2}[1 - \tanh(x - x_0 - ut)/w_0], \tag{9}$$

where Q_s in the shock wave amplitude and x_0 is the integration constant that determines the initial position of the shock wave. The shock velocity and (half) width, u and w_0, are related to the shock amplitude by

$$u = v_0(1 + \eta\, Q_s^{\,2})^{\frac{1}{2}}, \text{ and} \tag{10}$$

$$w_0 = u\tau_0/\eta\, Q_s^{\,2}. \tag{11}$$

A typical waveform of the shock given by Eq. (9) is plotted in Fig. 2, which agrees with Riley's numerical calculations and with his experimental results [6]. From Eqs. (10) and (11) we observe that a larger shock amplitude gives a higher shock velocity and a narrower shock transition. The shock width w_0 is linearly proportional to the relaxation time τ_0. Hence the experimental measurements of the shock width may be used for the direct determination of the relaxation process in a nonlinear material.

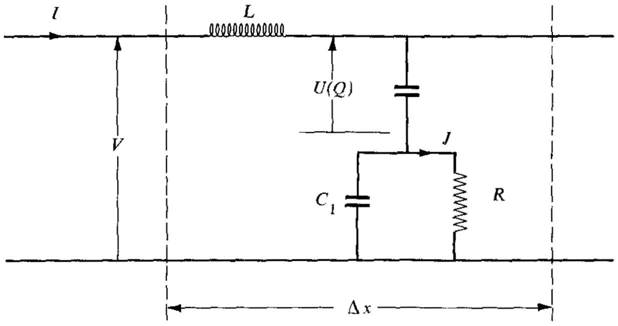

Figure 3 Dispersive nonlinear capacitance transmission line.

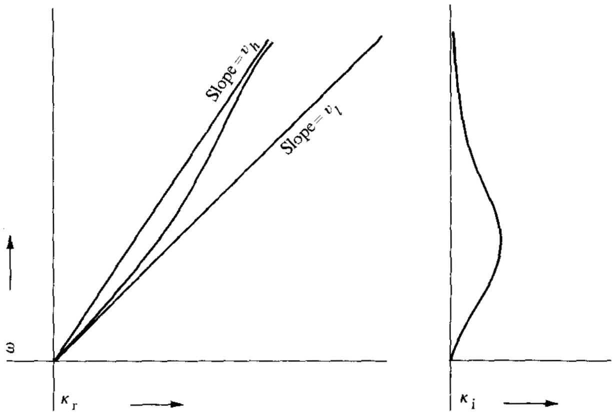

Figure 4 Dispersion curves of the transmission line of Fig. 3; κ_r and κ_i are the real and the imaginary parts of the propagation constant, respectively. There are additional solutions to Eq. (16) not shown here, for which ω is purely imaginary and κ is real. These additional modes correspond to a relaxation process, rather than to propagation.

Equation (8) may also be integrated for other types of nonlinearity; e.g., for the case of quadratic nonlinearity, $n(Q) = \eta Q^2$, the shock width is $w_0 = 2\mu \tau_0/\eta Q_s$. Khokhlov [13] treats essentially the same problem as we discuss here. He considers the quadratic nonlinearity and sinusoidal excitation rather than a single voltage transition. He finds the same shock width that we quote here for the quadratic nonlinearity.

Dispersive nonlinear capacitance transmission line

In this section we introduce a more complicated dispersion process in which a linear capacitance is added in parallel with the resistance, as shown in Fig. 3. Roughly speaking, low- and high-frequency currents pass through the resistive and capacitive branches, respectively, in the lower portion of the shunt circuit. A low frequency wave thus "sees" a capacitance equal to the nonlinear capacitance, but a high frequency wave sees a capacitance equal to that of the nonlinear and linear capacitances in series. Consequently, the low frequency capacitance is larger than the high frequency capacitance, and a high frequency wave propagates at a higher velocity than does a low frequency wave.

Two extreme cases of this transmission line are easily understood. When $C_1 \to 0$, the nonlinear capacitance has a simple relaxation time and this is the case we have analyzed in the preceding section, with the conclusion that a single-valued, steady state shock wave exists. The other extreme occurs when $C_1 \to \infty$, in which the resistance is short-circuited and the transmission line is lossless and dispersionless. In this case it has been shown that a wave continues to distort as it propagates along the line [5]. This eventually results in multivalued fields which are considered physically unacceptable. As is shown in the next section, the value of $1/C_1$ can be taken as a measure of the dispersion of the transmission line. Thus, the effect of linear dispersion on the shock wave structure can be studied by varying the value of C_1. In particular, the two special cases discussed here suggest that there must exist an intermediate value of C_1 above which no single-valued, steady state shock wave can exist. In other words, a minimum (or critical) dispersion of the nonlinear transmission line is required for a steady state shock wave to remain single-valued and we determine here this critical dispersion.

With reference to Fig. 3, the governing equations can easily be obtained by invoking Kirchhoff's laws of voltage and current:

$$\partial V/\partial x = -L(\partial I/\partial t), \tag{12}$$

$$\partial I/\partial x = -\partial Q/\partial t, \tag{13}$$

$$V = U(Q) + RJ, \text{ and} \tag{14}$$

$$J + \tau_1(\partial J/\partial t) = \partial Q/\partial t; \qquad \tau_1 = RC_1, \tag{15}$$

where V and I are the transmission line voltage and current, Q and U are the charge and voltage associated with the nonlinear capacitance, J is the current flowing through the resistance, and τ_1 is the relaxation time associated with the capacitance C_1.

• *Linear dispersion*

In the absence of nonlinearity in the capacitance we have $U(Q) = (1/C_0)Q$, where C_0 is the linear part of the nonlinear capacitance. In this case, we obtain the dispersion relation

$$\kappa^2 = \frac{\omega^2}{v_0^2[1 + j\omega\tau_0/(1 + j\omega\tau_1)]}, \tag{16}$$

in which $v_0^2 = 1/LC_0$ and $\tau_0 = RC_0$,

where κ is the propagation constant, ω is the frequency, v_0 is the characteristic velocity of low frequency waves, and τ_0 is the linear relaxation time of the nonlinear capacitance when it is uncharged. The real and imaginary parts of κ are plotted as functions of real values of ω in Fig. 4. It is evident from Fig. 4 that v_l and v_h are respectively the low- and high-frequency wave velocities and are given by $v_l = v_0$, and

$$v_h = v_0 \left(\frac{C_0 + C_1}{C_1}\right)^{\frac{1}{2}}.$$

Equivalently, we can define the low- and high-frequency capacitances of the transmission line as, respectively,

$C_l = C_0$, and

$$C_h = C_0 C_1/(C_0 + C_1), \tag{17}$$

where C_h is the total capacitance of C_0 and C_1 in series, as can be seen directly in Fig. 3. It is also evident in Fig. 4 that the wave velocity increases monotonically from v_l to v_h. Keeping C_0 and v_0 fixed, we can control C_h or v_h by varying the value of C_1; the smaller the value of C_1, the smaller the value of C_h. With this smaller value comes a stronger dispersion.

- *Steady state shock wave*

For a steady state shock wave propagating with a constant profile, we assume once again a solution of the form $Q = Q(\theta)$; $\theta = x - ut$, and similarly for V, I, and J in Eqs.(12) through(15). In terms of θ these equations become the ordinary differential equations

$$V_\theta = uL\, I_\theta, \tag{18}$$

$$I_\theta = u\, Q_\theta, \tag{19}$$

$$V = U(Q) + RJ, \text{ and} \tag{20}$$

$$J - u\,\tau_1 J_\theta = -\, uQ_\theta. \tag{21}$$

The subscript θ denotes differentiation with respect to θ. Equations (18) and (19) can readily be integrated to yield

$$V = u^2 LQ. \tag{22}$$

In arriving at the last equation, we have again set the integration constant equal to zero. This is appropriate in the special case of a shock transition starting from an initially uncharged line and leaving a charge density Q_s after its passage. Eliminating V and J from Eqs. (20) through (22) we obtain

$$\frac{dQ}{d\theta} = -\frac{f(Q)}{u[1 - \tau_1 f'(Q)]}, \tag{23}$$

with

$$f(Q) = (1/R)[u^2 L\, Q - U(Q)]. \tag{24}$$

Equation (23) determines the shock wave for a given function U.

As an example, let us consider again the nonlinear voltage-charge relation, $U(Q) = (1/C_0)\ [Q + \eta\, Q^3]$, as in the preceding section. In this case, Eq. (23) becomes

$$\frac{d\theta}{dQ} = -3u\tau_1 \frac{\alpha + Q^2}{Q(\Delta - Q^2)}, \tag{25}$$

with

$$\alpha = (1/3\eta)\ [(\tau_0/\tau_1) - \eta\Delta],$$

$$\Delta = (1/\eta)\ [(u^2/v_0^2) - 1]. \tag{26}$$

Equation (25) can then be integrated to yield

$$(\theta - \theta_0)/w = (1 - \delta)\ln|\Delta - Q^2| - \ln|Q^2|, \tag{27}$$

where θ_0 is an integration constant that specifies the initial position of the shock and w and δ are constants defined by $w = 3u\tau_1\alpha/2\Delta$ and $\delta = -\Delta/\alpha$.

For the nonlinear coefficient $\eta > 0$, Eq. (27) is plotted for different values of α in Fig. 5. Since $Q^2 > 0$, the branches below $Q^2 = 0$ are irrelevant and may be ignored. Thus, for the charge distribution of a steady state shock profile, we must choose the branches within the range $0 \leq Q^2 \leq \Delta$. These branches represent a transition between an uncharged state, $Q^2 = 0$, and a charged state, $Q^2 = Q_s^2 = \Delta$. The branches above $Q^2 = \Delta$ give rise to unbounded solutions, unrelated to an initially uncharged line, and hence must be rejected. The portion of the shock profiles which is relevant, between $Q^2 = 0$ and $Q^2 = \Delta$, is shown in the figure in heavier lines. We observe that for the case $\eta > 0$, the profile of a steady state shock wave is single-valued if $\alpha > 0$ and double-valued if $\alpha < 0$. The value of α is related to the dispersion of the system and it is from this effect of α on the steady state shock profile that we determine the critical dispersion required for the existence of a single-valued shock.

We now relate α to the dispersion of the system. For a shock transition between two states, $Q = 0$ and $Q = Q_s$, from Eq. (26), we obtain

$$\Delta = Q_s^2 \text{ and} \tag{28}$$

$$\alpha = (1/3\eta)\ [\tau_0/\tau_1 - \eta Q_s^2]. \tag{29}$$

As shown by Eq. (16), τ_0/τ_1 is a measure of the dispersion in the system. The value of α thus, changes with τ_0/τ_1, assuming that Q_s is held fixed. For $\eta > 0$, α has the range $\alpha > \alpha_{min} = -\,{}^1\!/\!{}_3 Q_s^2$. In this range we recall that $\alpha = 0$ is the critical value that divides single-valued and multivalued steady state shock waves. Right at the critical value of α no bounded solution exists. Therefore the condition for the existence of a bounded, single-valued, steady state shock wave is $\alpha > 0$, or, after invoking Eq. (29),

$$\tau_0/\tau_1 = C_0/C_1 > \eta Q_s^2. \tag{30}$$

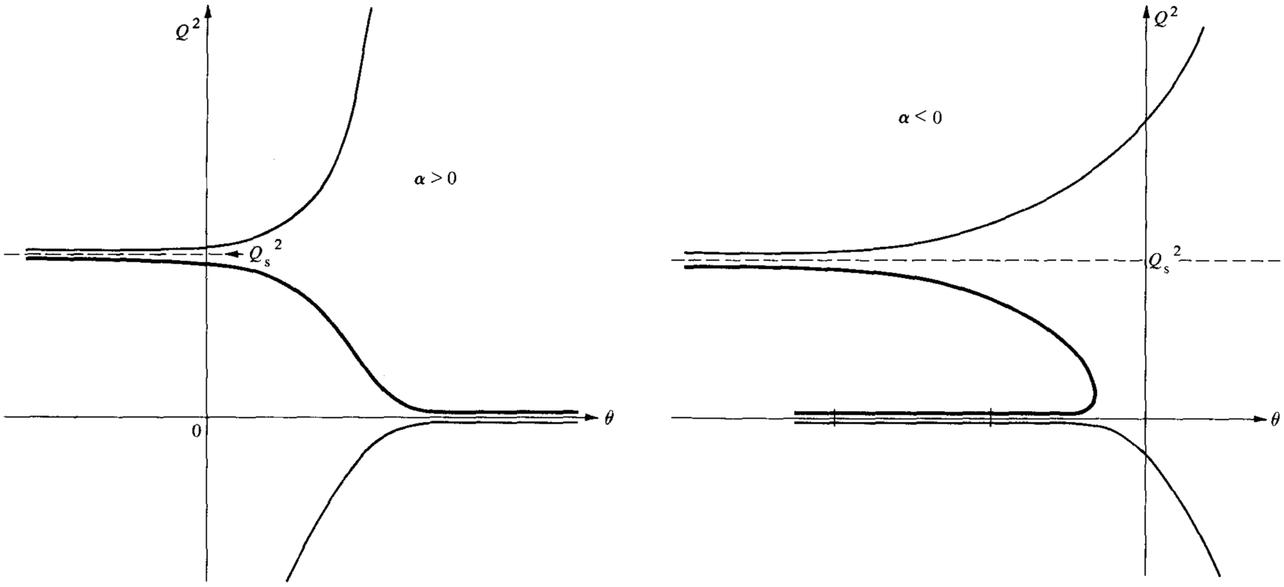

Figure 5 Complete curves of the shock wave solutions for different values of α.

Since $1/C_1$ is a measure of the dispersion in the system, Eq. (30) specifies the minimum or critical dispersion needed to prevent a wave from becoming multivalued.

The multivalued steady state shock wave is nonphysical and arises because our equations do not accurately represent the behavior of the system. This situation is analogous to the condition presented by the transmission line of Fig. 1 for the case $R = 0$, in which we find the over-taking phenomenon and multivalued solutions. The latter disappear, however, when we go to a more accurate treatment, applicable in the presence of fast changes. Thus, taking $R \neq 0$ in Fig. 1 leads to a single-valued wave with a finite width. One similarly expects that, in the case of Fig. 3, allowing C_1 itself to be associated with a finite relaxation time of its own (i.e., a series resistance) will avoid multivalued wave forms. Alternatively one can take the view proposed by Whitham [22] that the multivalued behavior in the steady state profile is avoided by the formation of discontinuities. But that leads again to questions about the real profile of such a discontinuity and therefore to the need for a better physical representation than exists in Fig. 3. Using Whitham's approach we would then find a discontinuous initial rise in Q, followed by a subsequent continuous portion, which is still a solution of Eq. (23) and is therefore represented in Fig. 5(b). The leading edge shock would terminate on the upper branch of the heavy curve in Fig. 5(b) at a point selected so that the velocity of the leading edge shock equals that of the remaining wave form.

Returning to the general case, Eq. (23) can be put in the form

$$\frac{d\theta}{dQ} = -\frac{1 - \tau_1 f'(Q)}{uf(Q)}. \tag{23a}$$

Before and after shock passage $dQ/d\theta$ vanishes. Hence Q as a function of θ must vary between the two roots of the equation $f(Q) = 0$. For the shock transition between $Q = 0$ and $Q = Q_s > 0$, we have, from Eq. (24), $f(Q_s) = (1/R)\,[u^2 LQ_s - U(Q_s)] = 0$. From this we obtain the shock wave velocity

$$u = (LC_s)^{-\frac{1}{2}}, \tag{31}$$

where C_s is the shock capacitance defined by

$$C_s = Q_s/U(Q_s).$$

For $0 < Q < Q_s$, we observe that $f(Q) > 0$. In order for the steady state shock wave to be single-valued, we must have $d\theta/dQ < 0$, which requires, from (23a),

$$1 - \tau_1 f'(Q) > 0,$$

or, after invoking Eq. (24),

$$1 - C_1[Lu^2 - U'(Q)] > 0. \tag{32}$$

Substituting Eq. (31) into (32) we obtain

$$\frac{1}{C_1} > \frac{1}{C_s} - \frac{1}{C_d(Q)}, \tag{33}$$

with the differential capacitance $C_d(Q) = 1/[dU(Q)/dQ]$.

Equation (33) must be satisfied for all values of Q within the range of the shock transition. For $U(Q) = (1/C_0)\,[Q + \eta Q^3]$ with $\eta > 0$, $1/C_d(Q)$ is smallest when $Q = 0$. Therefore Eq. (33) is most demanding at $Q = 0$ and it can easily be shown that this condition is identical to that given in Eq. (30).

When the relation $1/C_1 = (1/C_h) - (1/C_l)$ is invoked, in which C_h and C_l are the high- and low-frequency capacitances, respectively, as defined in Eq. (17), Eq. (33), can now be written as

$$(1/C_h) - (1/C_l) > (1/C_s) - (1/C_d),$$

or, in terms of wave velocities,

$$v_h^2 - v_l^2 > v_s^2 - v_d^2, \tag{34}$$

where v_h and v_l are the high- and low-frequency wave velocities, respectively, and v_s and v_d are the shock velocity (i.e., the velocity of the steady state profile) and the velocity for small, low frequency signals. This condition states that the velocity difference resulting from the dispersive character of a system must be large enough to overcome the velocity difference resulting from the nonlinearity of the system.

We observe that condition (34) is most demanding for the part of the shock in which v_d has its smallest value, i.e., on the uncharged line ahead of the moving shock. There $v_d = v_l$ and Eqs. (34) simplifies to

$$v_h^2 > v_s^2, \tag{35}$$

requiring high frequency components on the uncharged line to be able to pass ahead of the shock. High frequency components at some higher level of charge will, of course, have even higher velocities and will therefore also run through and ahead of the shock. In the case of a shock forming at the trailing edge of a wave, Eq. (34) applies without modification; Eq. (35) applies if it is understood that v_h is the small-signal, high frequency velocity ahead of the shock.

Since similar wave velocities appear in any nonlinear dispersive system, it will be interesting to see whether this condition on the wave velocities applies to other systems as well. For this purpose, we consider another type of dispersion in the next section.

Dispersive inductance transmission line

Instead of a frequency dispersion in the nonlinear capacitance itself, we now consider the case of frequency dispersion in the linear inductance, as shown in Fig. 6. The governing equations of the system are

$$\partial V/\partial x = -L_0(\partial I/\partial t) - L_1(\partial J/\partial t), \tag{36}$$

$$\partial I/\partial x = -\partial Q/\partial t, \tag{37}$$

$$I = J + \tau(\partial J/\partial t); \qquad \tau = L_1/R, \text{ and} \tag{38}$$

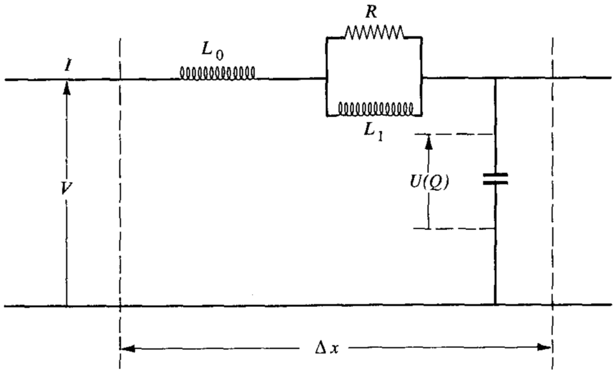

Figure 6 Dispersive inductance transmission line.

$$V = U(Q), \tag{39}$$

where V, I, and Q are the transmission line voltage, current, and charge, respectively; J is the current flowing in the L_1 branch; τ is the relaxation time; and U is a single-valued function of Q which characterizes the voltage-charge relationship of the capacitance.

For a steady state shock wave, we assume that $Q = Q(\theta)$, with $\theta = x - ut$, and similarly for V, I, and J. Equations (4.1) through (4.3) can then be integrated to yield

$$\frac{d\theta}{dQ} = \frac{u\tau[f'(Q) + u^2L_1]}{f(Q)},$$

in which

$$f(Q) = U(Q) - u^2LQ; \qquad L = L_0 + L_1.$$

As in the preceding section, for the shock transition between two constant states, $Q = 0$ and $Q = Q_s$, we obtain, from $f(Q_s) = 0$, the shock wave velocity: $u = 1/\sqrt{LC_s}$; $C_s = Q/U(Q)$. The condition that $d\theta/dQ < 0$ requires that $f'(Q) + u^2L_1 > 0$, from which, with a little algebra, we can obtain,

$$1/(C_dL_0) > 1/(C_sL),$$

where $1/(C_dL_0)$ is the high frequency velocity on a uniformly charged line. Its lowest value along the shock is $1/(C_0L_0) = v_h^2$, i.e., the high frequency value on the unbiased line ahead of the shock. We therefore once again obtain

$$v_h^2 > v_s^2, \tag{35}$$

as in the preceding section. We emphasize the fact that the equivalence between Eqs.(35) and(33) depends on the use of wave velocities. If expressed instead in terms of transmission line parameters the equivalence disappears.

Physical considerations

Our result, in the form of Eq. (35), has a form which suggests a greater generality and also invites physical analysis. At this point we admittedly depart from a strict analytical basis and enter the realm of speculation. In the process of self-steepening, leading to shock formation, higher frequency components are generated. At a particular point when the waveform has become vertical, just before the onset of over-taking, we have very high frequencies. Clearly, if such high frequency components move faster than the shock, they will move ahead of the shock and prevent the wave form from ever reaching the stage at which derivatives do in fact become unbounded. We can thus also see, at least on an intuitive level, that our monotonic steady state profiles are stable solutions. A very spread-out transition will sharpen up, whereas a transition having very steep portions will lose those portions, since they will move through the transition and ahead of it. After separating from the shock, the high frequency components will be attenuated. Clearly there is a need for a broadly applicable analytic version of the concepts suggested here.

Acknowledgment

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