

Optimum Storage Allocation for a File in Steady State

Abstract: A file of fixed-length records in auxiliary storage using a key-to-address transformation to assign records to addresses is considered. The file is assumed to be in steady state, that is that the rates of additions to and of deletions from the file are equal.

The loading factors that minimize file maintenance costs in terms of storage space and additional accesses are computed for different bucket sizes and different operational conditions.

Introduction

This paper describes a study of auxiliary storage allocation for a key-to-address transformation file organization [1-3]. In this organization, there is a primary storage area divided into buckets. Each bucket can contain s records of the file. The transformation is applied to the key of a record to be loaded and it produces the address of the bucket into which that record will be stored. If more records are assigned to a bucket than it can hold, the excess records are stored in a separate overflow area. The overflow records belonging to a particular bucket are organized in a chain with address pointers.

To retrieve a record with a given key, a single access to auxiliary storage is sufficient if that record is stored in the primary storage area. If it is in the overflow area, one or more additional accesses—following the pointers from bucket to record and from record to record—are required. The number of accesses depends on the position of the record in the chain.

The average length of the chains and, therefore, the average number of additional accesses depends on the bucket size and the storage capacity that is provided in the primary area relative to the number of records to be stored.

Studies [3-5] have been made of the optimal storage allocation for a file that does not change after initial loading. Lum, et al. [3], made performance studies using actual files and different key-to-address transformations. As performance indicators, the authors use the average number of accesses per record and the number of overflow records. Under certain assumptions, formulae can

be derived for these quantities [4]. Using these formulae it is possible to compute, for each bucket size, the load factors (that is, the ratio between the number of records to be loaded and the capacity of primary storage) that makes file-handling costs minimal [5]. These costs consist of storage cost for primary and overflow area, plus the cost for additional accesses.

In the study reported in this paper, a different situation is analyzed. A file is considered to which records are added and from which records are deleted.

Olsen [6] points out the difference between this situation and the initial loading problem. He considers a steady-state situation in which the number of additions and deletions per time period are equal, and he derives a formula for the average overflow. In this paper, a method is developed to compute also the average number of additional accesses. With these results, the optimal loading for the steady-state situation can be computed in a similar fashion to that for the initial loading model [5].

Analytical approach

The file contains n records. The primary storage area consists of b buckets, each of size s . The average loading—number of records assigned to a bucket—is thus: $m = n/b$. The load factor is

$$\ell = \frac{n}{bs} = \frac{m}{s}.$$

It is assumed that the overflow area has unlimited capacity. All records have equal probability to be de-

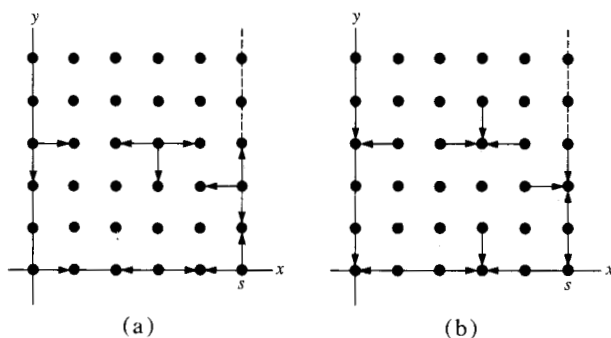


Figure 1 Buckets entering and leaving state (x, y) .

leted from the file. When a record is added, we assume that all buckets have equal probability of being selected by the key-to-address transformation.

At a given moment, a bucket will contain $x \leq s$ records, while y records ($0 \leq y < \infty$) assigned to that bucket reside in overflow storage. The bucket is said to be in the state (x, y) . It is possible that $x < s$, while $y > 0$ because records deleted from a bucket are not replenished from the overflow chain. When a record is added, it goes into the assigned bucket when $x < s$. It is attached to the overflow chain when $x = s$.

The first objective is to find the frequency function $u(x, y; m)$ or for simplicity $u(x, y)$ for the variables x and y for each bucket in the steady-state situation for a given average loading m . This steady-state condition will not be reached until the rates of additions and deletions are equal. A further condition is that the average number of buckets in the file with state (x, y) remains the same.

In the next section, these conditions are expressed mathematically giving a set of equations from which $u(x, y; m)$ can be computed for every ℓ .

If $u(x, y)$ is available, then it is possible to compute:

1. The average overflow per bucket,

$$\bar{i} = \sum_{y=0}^{\infty} y \sum_{x=0}^s u(x, y), \text{ and}$$

2. The average number of additional accesses,

$$\bar{a} = \frac{1}{m} \sum_{i=1}^{\infty} i \sum_{x=0}^s \sum_{y=0}^{\infty} u(x, y).$$

The first formula requires no further explanation. The second one can be understood if we observe that just one record requiring exactly i additional accesses is found in each chain of length i or greater. Thus, the average number of records that require i additional accesses is

$$b \sum_{x=0}^s \sum_{y=i}^{\infty} u(x, y).$$

Therefore, the average number of additional accesses per record taken over the whole file is

$$\frac{b}{n} \sum_{i=1}^{\infty} i \sum_{x=0}^s \sum_{y=i}^{\infty} u(x, y).$$

With $m = n/b$ this gives the formula for $\bar{a}$. This is also the average number of accesses required to retrieve a record under the condition that all records are equally in demand.

In the same way as in [5] a minimum can now be sought for the cost function for file operation:

$$C(b, s; \alpha) = \bar{i}c_s + \alpha n c_p + \alpha n \bar{a} c_a$$

where

$\bar{i} = b(s + \bar{i})$ = total average storage used for the file,

c_s = cost of storing one record during the unit of time,

α = file activity (the fraction of records accessed during the unit of time),

c_p = cost of an access to the primary area, and

c_a = cost of an additional access.

We assume that c_p is constant. The value of the variable part of C will be compared with nc_s , the nominal cost for storing n records.

The ratio between variable and nominal cost is the relative cost function

$$R = \frac{\bar{i}c_s + \alpha n \bar{a} c_a}{nc_s} = \frac{s + \bar{i}}{m} + \gamma \bar{a},$$

where

$$\gamma = \frac{\alpha c_a}{c_s}.$$

The factor γ is dependent on the application and the environment in which the application runs [5].

Minima are computed for a range of values of γ and bucket size s . The results are tabulated at the end of the paper.

Difference equations for $u(x, y)$

Difference equations for $u(x, y)$ can be found by considering the steady-state condition. The rate of buckets leaving the state (x, y) must be equal to the rate of buckets entering the state (x, y) . Consider $u(x, y)$ on the domain $0 \leq x \leq s$, $0 \leq y < \infty$. The transitions that cause a bucket to leave or to enter the state (x, y) are shown in Fig. 1(a) and 1(b), respectively. As a matter of fact, different situations exist for interior and boundary points of the domain. For instance, the state (x, y) of a bucket with $0 < x < s$, $y > 0$ will change to state

- $(x - 1, y)$ when a record is deleted from that bucket,
- $(x + 1, y)$ when a record is added to that bucket, and
- $(x, y - 1)$ when a record from the overflow chain is deleted.

Transitions to $(x, y + 1)$ cannot occur because additions to the overflow chain are not made while there is room in the bucket.

The same state can be entered from

- $(x - 1, y)$ when a record is added to a bucket,
- $(x + 1, y)$ when a record is deleted from a bucket, and
- $(x, y + 1)$ when an overflow record is deleted.

The rate of deletion of records out of buckets in a certain state is proportional to the fraction of records in those buckets. The rate of addition of records to buckets in a certain state is proportional to the fraction of buckets in that state.

Assuming a negative exponential distribution for interarrival and interdeparture times, this gives the following equilibrium condition:

$$\begin{aligned} & \frac{bxu(x, y) + byu(x, y)}{n} + u(x, y) \\ &= \frac{b(x + 1)u(x + 1, y) + b(y + 1)u(x, y + 1)}{n} \\ &+ u(x - 1, y). \end{aligned}$$

The equation can be explained as follows: $bxu(x, y)$ is the number of records in buckets with state (x, y) and $byu(x, y)$ is the number in the overflow chains belonging to those buckets. Therefore, the first term on the left-hand side is the rate of leaving the state (x, y) due to deletions. The second one is the rate of leaving the state (x, y) due to additions. The right-hand side gives, in a similar fashion, the rate of entering the state (x, y) .

After rearranging and using $m = n/b$,

$$\begin{aligned} (x + y + m)u(x, y) &= (x + 1)u(x + 1, y) \\ &+ (y + 1)u(x, y + 1) \\ &+ mu(x - 1, y). \end{aligned} \quad (1)$$

It is easy to see that this equation is also valid for $0 < x < s$ and $y = 0$. In the same way

$$\begin{aligned} (y + m)u(0, y) &= u(1, y) \\ &+ (y + 1)u(0, y + 1) \quad \text{for } y \geq 0, \end{aligned} \quad (2)$$

$$\begin{aligned} (s + y + m)u(s, y) &= (y + 1)u(s, y + 1) \\ &+ mu(s - 1, y) + mu(s, y - 1) \\ &\quad \text{for } y > 0, \end{aligned} \quad (3)$$

$$(s + m)u(s, 0) = u(s, 1) + mu(s - 1, 0). \quad (4)$$

$u(x, y)$ as stationary probability distribution of a Markov chain

The following discrete time Markov chain $\{X_n\}$ is considered.

X_k is the state of a particular bucket after the k th file event. A file event is an addition or deletion of a record. The transitions that may occur in the state space ($0 \leq x \leq s, 0 \leq y < \infty$) were shown in Figs. 1(a) and 1(b). The process is Markovian because it is assumed that the outcome X_k is dependent only on the state X_{k-1} .

In an actual steady state condition of the file, the total number of records may fluctuate; however, changes are relatively small for a large n . In considering one bucket, the total is assumed to be a constant and, as a consequence, the transition probabilities are considered to be constant.

To set up the transition probability matrix M , the possible states are ordered with the first coordinate moving most rapidly:

$$\begin{aligned} & (0, 0); (1, 0); (2, 0); \dots; (s, 0); (0, 1); (1, 1) \\ & \dots; (s, 1); (0, 2) \dots \end{aligned}$$

Now, the infinite matrix can be considered to be composed of partitions E_{pq} ($0 \leq p < \infty, 0 \leq q < \infty$) of order $(s + 1, s + 1)$. For each y only the partitions E_{yy-1} , E_{yy} and E_{yy+1} contain nonzero elements. Rows and columns of elements and submatrices are numbered starting with zero. The x th ($x = 0, 1, 2, \dots, s$) row of E_{yy-1} contains the probabilities of transition from (x, y) to $(i, y - 1)$. Only when $i = x$ is this probability nonzero and $E_{yy-1}(x, x) = y/n$, that is, the probability of deleting a record from the overflow chain.

The x th ($x = 0, 1, \dots, s$) row of E_{yy} contains the probabilities of transition from (x, y) to (i, y) . These are nonzero when $i = x - 1$; $i = x$ or $i = x + 1$ for $i = 1, 2, \dots, s - 1$. In row 0 and row s , only the last two or the first two of these three elements occur. The probabilities are

$$E_{yy}(x, x - 1) = \frac{x}{n} \text{ for deleting a primary record,}$$

$$\begin{aligned} E_{yy}(x, x) &= 1 - \frac{x + y}{n} - \frac{1}{b} \text{ for staying in state} \\ &\quad (x, y), \text{ and} \end{aligned}$$

$$E_{yy}(x, x + 1) = \frac{1}{b} \text{ for adding a record to the bucket.}$$

Finally, E_{yy+1} will contain the probabilities for going from (x, y) to $(x, y + 1)$. Only if $x = s$ is the matrix element nonzero and $E_{yy+1}(s, s) = 1/b$, the probability for adding a record to the overflow chain of a full bucket.

The structure of the matrix is indicated in Fig. 2 for $s = 4$. The crosses indicate places where a nonzero element occurs.

If the stationary probability distribution of the Markov chain $\vec{u} = u(0, 0); u(1, 0); u(2, 0); \dots; u(s, 0); u(1, 1); \dots$ exists, then:

$$\vec{u}M = \vec{u}.$$

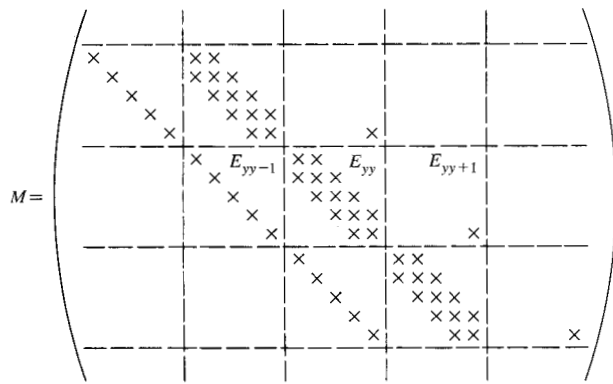


Figure 2 Matrix with $s = 4$.

The condition for the column corresponding with the state (x, y) for $0 < x < s$ is:

$$\begin{aligned} & u(x-1, y) E_{yy}(x-1, x) + u(x, y) E_{yy}(x, x) \\ & + u(x+1, y) E_{yy}(x+1, x) + u(x, y+1) E_{y+1y}(x, x) \\ & = u(x, y), \quad \text{or} \\ & u(x-1, y) \frac{1}{b} + u(x, y) \left(1 - \frac{x+y}{n} - \frac{1}{b}\right) \\ & + u(x+1, y) \frac{x+1}{n} + u(x, y+1) \frac{y+1}{n} = u(x, y). \end{aligned}$$

After multiplying by n and rearranging, Eq. (1) is found again.

In similar fashion the Eqs. (2), (3), and (4) can be derived from the matrix formulation.

Some properties of $u(x, y)$

From the difference equation for $u(x, y)$, it is possible to derive several properties. The following notation for marginal distributions is used:

$$f(x) = \sum_{y=0}^{\infty} u(x, y) \quad \text{and} \quad g(y) = \sum_{x=0}^s u(x, y).$$

Lemma 1: The marginal distribution $f(x)$ has a truncated Poisson distribution; i.e.,

$$f(x) = e^{-m} \frac{m^x}{x!} / \sum_{x=0}^s e^{-m} \frac{m^x}{x!}.$$

Proof: Summing Eq. (1) over all values of y for a fixed x gives:

$$\begin{aligned} & xf(x) + mf(x) + \sum_{y=0}^{\infty} y u(x, y) \\ & = (x+1) f(x+1) + \sum_{y=0}^{\infty} (y+1) u(x, y+1) \\ & + mf(x-1). \end{aligned}$$

Since

$$\sum_{y=0}^{\infty} y u(x, y) = \sum_{y=0}^{\infty} (y+1) u(x, y+1),$$

the following identity holds for $0 < x < s$:

$$xf(x) + mf(x) = (x+1) f(x+1) + mf(x-1). \quad (5)$$

In the same way from Eq. (2) for $x = 0$:

$$mf(0) = f(1), \quad (6)$$

and from Eqs. (3) and (4) for $x = s$:

$$mf(s-1) = sf(s). \quad (7)$$

Substituting Eq. (6) into Eq. (5) for $x = 1$ gives

$$mf(1) = 2 \cdot f(2).$$

By induction, it can then be proved that for every $0 \leq x < s$:

$$mf(x) = (x+1) f(x+1). \quad (8)$$

Clearly, $f(x) = (m^x/x!)k$ is a solution of Eq. (8) for every k . With the condition $\sum_{x=0}^s f(x) = 1$, this gives

$$\begin{aligned} f(x) &= \frac{m^x}{x!} / \sum_{x=0}^s \frac{m^x}{x!} \\ &= \frac{e^{-m} \frac{m^x}{x!}}{\sum_{x=0}^s e^{-m} \frac{m^x}{x!}}. \end{aligned}$$

These are the frequencies for a truncated Poisson distribution with parameter m and range $0 \leq x \leq s$, Q.E.D.

This result was obtained by Olsen [6] by directly formulating the difference equation for $f(x)$.

Introducing the notation $P(r) = (e^{-m} m^r)/r!$ for the Poisson distribution and $Q(s) = \sum_{r=s}^{\infty} P(r)$, then

$$f(x) = \frac{P(x)}{1 - Q(s+1)}, \quad 0 \leq x \leq s.$$

The functions $P(r)$ and $Q(s)$ are available in tabulated form (for example in [7]).

Lemma 2: The total number of records assigned to a bucket has a Poisson distribution.

Proof: Let $\sum_{x+y=r} u(x, y) = A(r)$. It will be proved that: $A(r) = P(r)$ for $r = 0, 1, 2, \dots$.

For $0 < x+y=r < s$ the summation of Eq. (1) gives

$$\begin{aligned} \sum_{x+y=r} (r+m) u(x, y) &= \sum_{x+y=r} (x+1) u(x+1, y) \\ &+ \sum_{x+y=r} (y+1) u(x, y+1) + m \sum_{\substack{x+y=r \\ x>0}} u(x-1, y) \quad \text{or} \end{aligned}$$

$$\begin{aligned}
(r+m)A(r) &= (r+1)u(0, r+1) \\
&+ (r+1) \sum_{x=1}^r u(x, r+1-x) + (r+1)u(r+1, 0) \\
&+ mA(r-1), \\
(r+m)A(r) &= (r+1)A(r+1) + mA(r-1).
\end{aligned}$$

For $r=0$ we have $mA(0) = A(1)$. By using this in the equality for $r=1$, it follows that:

$$mA(1) = 2A(2)$$

and by induction, we get

$$mA(r) = (r+1)A(r+1).$$

For each r considered, this identity is also satisfied by $P(r) = e^{-m} m^r / r!$.

For $x+y \geq s$ the summation over $x+y=r$ with $0 \leq x \leq s$ gives:

$$\begin{aligned}
(r+m) \sum_{x+y=r} u(x, y) &= \sum_{x+y=r} (x+1) u(x+1, y) \\
&+ \sum_{x+y=r} (y+1) u(x, y+1) + m \sum_{x+y=r} u(x-1, y) \\
&+ mu(s, r-s-1).
\end{aligned}$$

If y is eliminated from the right-hand side and the substitution of $(x+1)$ by x is made in the first sum and of $(x-1)$ by x in the third sum, it follows that:

$$\begin{aligned}
(r+m) \sum_{x+y=r} u(x, y) &= (r+1) \sum_{x=0}^s u(x, r-x+1) \\
&+ m \sum_{x=0}^s u(x, r-x-1) \text{ or again:}
\end{aligned}$$

$$(r+m)A(r) = (r+1)A(r+1) + mA(r-1),$$

which, as before, is also satisfied by $P(r) = e^{-m} m^r / r!$. This concludes the proof.

This result is easily understood if we make the following observation. At any time, the n records in the file are related to the b buckets by the key-to-address transformation. It is therefore assumed that each bucket has an equal probability to be selected for addition of a record.

Thus, $r = x + y$ has the binominal probability distribution with $p = 1/b$ and n is very large. The limit for $n \rightarrow \infty$ with $m = n/b$ remains constant and is the Poisson distribution.

Lemma 3: $mu(s, y-1) = yg(y)$ for $y > 0$.

Proof: Summing Eqs. (1), (2), and (3) over $0 \leq x \leq s$ gives

$$\begin{aligned}
\sum_{x=0}^s (x+y+m) u(x, y) &= \sum_{x=0}^{s-1} (x+1) u(x+1, y) \\
&+ (y+1) \sum_{x=0}^s u(x, y+1) + m \sum_{x=1}^s u(x-1, y) \\
&+ mu(s, y-1) \text{ or}
\end{aligned}$$

$$yg(y) + mu(s, y) = (y+1)g(y+1) + mu(s, y-1).$$

For $y=0$: $mu(s, 0) = g(1)$.

Using this with the previous result for $y=1$ gives:

$$mu(s, 1) = 2g(2)$$

and then by induction for every $y > 0$,

$$mu(s, y-1) = yg(y).$$

Q.E.D.

Computation of $\bar{i}$ and $\bar{a}$

With the results obtained in the previous section, the computation of $\bar{i}$ and $\bar{a}$ can be simplified.

$$\begin{aligned}
\bar{i} &= \sum_{y=0}^{\infty} y \sum_{y=1}^s u(x, y) = \sum_{y=1}^{\infty} yg(y) = m \sum_{y=1}^{\infty} u(s, y-1) \\
&= mf(s) = \frac{mP(s)}{1-Q(s+1)}
\end{aligned} \tag{9}$$

Hence, the computation of the average overflow can be done without explicitly knowing $u(x, y)$:

$$\bar{a} = \frac{1}{m} \sum_{i=1}^{\infty} i \sum_{x=0}^s \sum_{y=i}^{\infty} u(x, y) = \frac{1}{m} \sum_{i=1}^{\infty} i \sum_{y=i}^{\infty} g(y).$$

Interchanging the order of summation gives:

$$\sum_{y=1}^{\infty} g(y) \sum_{i=1}^y i = \frac{1}{2m} \sum_{y=1}^{\infty} y(y+1) g(y).$$

Here, again, using Lemma 3 results in

$$\bar{a} = \frac{1}{2} \sum_{y=1}^{\infty} (y+1) u(s, y-1).$$

Therefore, only the distribution on the right boundary of the state space is required to compute $\bar{a}$.

Computation of $u(x, y)$

The stationary distribution is a solution of the infinite set of equations

$$\vec{u}M = \vec{u}$$

with the condition that the sum of the elements of $\vec{u}$ is one.

It is clear that $\lim_{y \rightarrow \infty} u(x, y) = 0$ uniform in x , because for each ϵ we can find a y' such that

$$u(x', y') \leq \sum_{x+y=x'+y'} u(s, y) = P(x'+y') \leq P(y') < \epsilon.$$

Therefore, to find a numerical solution, the infinite state space is replaced by a finite state space. A transition from a state (x, y) to $(x, y+1)$ only occurs for $x=s$. In the finite state space, this transition from the largest $y=y'$ is replaced by a transition recurring to the same state. In the matrix M , this means that in the partition $E_{y'y'}$ the element $E_{y'y'}(s, s)$ is increased by $1/b$ and that

the finite square matrix $M_{y'}$ of order $(1 + y')(s + 1)$ is considered.

For numerical computation, it is more appropriate to reorder rows and columns in such a way that the matrix corresponds with the order

$$\vec{u} = \{u(0, 0); u(0, 1); u(0, 2) \cdots u(0, y'); u(1, 0); u(1, 1) \cdots; u(s, y')\}.$$

Now, partitions G_{pq} ($0 \leq p \leq s$, $0 \leq q \leq s$) of order $(y' + 1)$ are considered. For a certain x with $0 < x < s$, only the partitions G_{xx-1} , G_{xx} and G_{xx+1} contain nonzero elements. G_{xx-1} contains the probability that a primary record is deleted. Thus $G_{xx-1} = (x/n) I$, where I stands for the unit matrix. G_{xx} contains in the main diagonal the probability of no change $1 - (1/b) - (x + y)/n$ and in the lower diagonal the probability that y is reduced by 1.

Thus, $G_{xx} = [1 - (1/b) - (x/n)]I + (1/n)(-S + SL)$ where S stands for a diagonal matrix with diagonal: $\{0, 1, 2, \cdots, y'\}$ and L is a lower diagonal matrix.

Finally, $G_{xx+1} = (1/b) I$ because it contains the probabilities that a record is added to the bucket.

As a matter of fact, G_{xx-1} does not exist for $x = 0$. Further, for $x = s$ the probabilities of going from (s, y) to $(s, y + 1)$ are added as an upper diagonal with elements $1/b$. And, finally, $G_{ss}(y', y')$ is $1/b$ larger than $G_{xx}(y', y')$ for $x < s$ corresponding with the transition of state (s, y) into itself. Or

$$G_{ss} = \left(1 - \frac{1}{b} - \frac{s}{n}\right) I + \frac{1}{n}(SL - S + U) + \frac{1}{b}R,$$

where U is an upper diagonal matrix and R has all zero elements except for $R(y', y') = 1$.

Now, finally, we will solve:

$$\begin{aligned}\vec{u} M_{y'} &= \vec{u} \quad \text{or} \\ \vec{u}(M_{y'} - I) &= 0.\end{aligned}$$

After multiplication by b , we partition the matrix of the system into partitions H_{pq} ($0 \leq p \leq s$, $0 \leq q \leq s$). It follows that:

$$H_{xx-1} = \frac{x}{m} I,$$

$$H_{xx} = -\left(1 + \frac{x}{m}\right)I + \frac{1}{m}P \quad \text{with } P = SL - S,$$

$$H_{xx+1} = I.$$

For $x = s$: $H_{ss} = -[1 + (s/m)]I + (1/m)Q + R$ with $Q = P + mU$. The determinant of the homogeneous system $\vec{u}\{H_{pq}\} = 0$ is zero because the sums of the rows of M and $M_{y'}$ are 1 (being Markov matrices). The required solution for $\vec{u}$ has the property that the sum of the elements is 1.

The specific configuration

$$\vec{u} \cdot \begin{bmatrix} H_{00} & H_{01} & 0 & \cdots & 0 \\ H_{10} & H_{11} & H_{12} & \cdots & 0 \\ & & \cdots & & \\ \cdots & H_{s-2s-2} & H_{s-1s-1} & H_{s-1s} & \\ & \cdots & H_{ss-1} & H_{ss} & \end{bmatrix} = 0$$

can be used to find a solution. By successive postmultiplications, the partitions in the upper diagonal are reduced to zero. The i^{th} transformation gives

$$H_{ii}^{(i)} = H_{ii} - H_{ii-1}(H_{i-1 i-1}^{(i-1)})^{-1} \cdot H_{i-1 i} \quad \text{for}$$

$$i = 1, 2, \cdots, s.$$

The lower diagonal partitions are transformed as:

$$H_{ii-1}^{(i)} = H_{ii-1}(H_{i-1 i-1}^{(i-1)})^{-1}.$$

If $\vec{u}$ is partitioned in $\{u^{(0)}, u^{(1)}, \cdots, u^{(s)}\}$ then $u^{(s)} = \{u(s, 0), u(s, 1), \cdots, u(s, y')\}$.

After the s th transformation, the partitioned matrix contains unity matrices at the main diagonal except for $H_{ss}^{(s)}$ and all zero partitions above the main diagonal. Thus the determinant is equal to $\det H_{ss}^{(s)}$. Since the original determinant was zero, $\det H_{ss}^{(s)} = 0$.

So the system $u^{(s)}H_{ss}^{(s)} = 0$ can be solved and the value of the elements in $u^{(s)}$ are fixed by the condition

$$\sum_{i=0}^{y'} u(s, i) = f(s) = \frac{P(s)}{1 - Q(s + 1)}.$$

If besides $\vec{i}$ only $\vec{a}$ is required, then the computation ends here (see previous section).

If the special forms of H_{ii-1} and $H_{i-1 i}$ are taken into consideration, it follows that

$$H_{ii}^{(i)} = H_{ii} - \frac{i}{m}(H_{i-1 i-1}^{(i-1)})^{-1} \quad \text{with}$$

$$H_{ii} = \frac{1}{m}P - \left(1 + \frac{1}{m}\right)I.$$

If the complete stationary distribution is required, it is observed that:

$$(H_{i-1 i-1}^{(i-1)})^{-1} = \frac{m}{i}\{H_{ii} - H_{ii}^{(i)}\} \quad \text{and}$$

$$u^{(i-1)} + u^{(i)}H_{ii-1}^{(i)} = 0 \quad \text{or}$$

$$\begin{aligned}u^{(i-1)} &= -u^{(i)}H_{ii-1}^{(i)} \\ &= -u^{(i)} \cdot H_{ii-1} \cdot (H_{i-1 i-1}^{(i-1)})^{-1} \\ &= -u^{(i)} \frac{i}{m} \cdot \frac{m}{i}\{H_{ii} - H_{ii}^{(i)}\} \\ &= -u^{(i)}(H_{ii} - H_{ii}^{(i)}).\end{aligned}$$

So starting with $i = s$ all partitions of $\vec{u}$ can be computed.

Table 1 Influence of y' .

	$s = 1$ $m = 1.589$ $\gamma = 0.500$ $y' = 2 + u$		$s = 1$ $m = 6.934$ $\gamma = 0.010$ $y' = 10 + u$		$s = 3$ $m = 2.069$ $\gamma = 2.000$ $y' = 3 + u$		$s = 3$ $m = 6.711$ $\gamma = 0.050$ $y' = 2 + u$		$s = 5$ $m = 4.058$ $\gamma = 1.000$ $y' = 5 + u$		$s = 10$ $m = 12.377$ $\gamma = 0.100$ $y' = 13 + u$		$s = 20$ $m = 32.508$ $\gamma = 0.010$ $y' = 33 + u$	
Increase														
u	Cost	$Q(y' + s)$	Cost	$Q(y' + s)$	Cost	$Q(y' + s)$	Cost	$Q(y' + s)$	Cost	$Q(y' + s)$	Cost	$Q(y' + s)$	Cost	$Q(y' + s)$
0	1.683	0.2138	1.052	0.0939	2.292	0.0192	1.166	0.1414	1.786	0.0089	1.233	0.0044	1.072	0.0006
1	1.710	0.0773	1.053	0.0504	2.300	0.0054	1.169	0.0793	1.789	0.0031	1.234	0.0022	1.073	0.0003
2	1.721	0.0231	1.053	0.0253	2.303	0.0013	1.171	0.0414	1.790	0.0010	1.234	0.0010	1.073	0.0002
3	1.725	0.0058	1.054	0.0119	2.303	0.0003	1.173	0.0201	1.790	0.0003	1.234	0.0005	1.073	0.0001
4	1.726	0.0013	1.054	0.0053	2.304	0.0001	1.173	0.0092	1.790	0.0001	1.234	0.0002	1.073	0.0001
5	1.726	0.0002	1.054	0.0021	2.304	0.0000	1.174	0.0039	1.790	0.0000	1.234	0.0001	1.073	0.0000
6	1.726	0.0000	1.054	0.0009	2.304	0.0000	1.174	0.0016	1.790	0.0000	1.234	0.0000	1.073	0.0000
7	1.726	0.0000	1.054	0.0003	2.304	0.0000	1.174	0.0006	1.790	0.0000	1.234	0.0000	1.073	0.0000
8	1.726	0.0000	1.054	0.0001	2.304	0.0000	1.174	0.0002	1.790	0.0000	1.234	0.0000	1.073	0.0000
9	1.726	0.0000	1.054	0.0000	2.304	0.0000	1.174	0.0001	1.790	0.0000	1.234	0.0000	1.073	0.0000

Numerical results

The formulae developed in this paper allow, for different values of s , the numerical computation of:

1. The frequency distribution $u(x, y; m)$ of x records in a bucket and y records in its overflow chain, and
2. The values of m that make the relative cost function R minimal for different application factors γ .

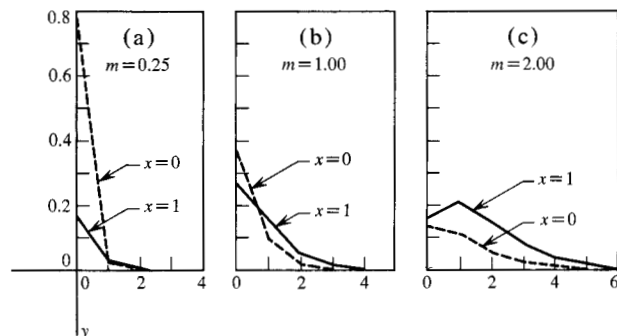
Programs were written in APL and FORTRAN. The final computations were made on the System 360 Model 91 and later on a Model 195. The influence of y' was investigated as follows. A first computation of some of the minima was made. Then, with the values of m found, the cost function was computed a number of times for increasing values of y' . The results are listed in Table 1. Also, listed are the values of $Q(y' + s)$.

The table indicates that the cost in three decimals remains the same for increasing y' when $Q(y' + s) \leq 0.0001$. In further computations, y' was fixed accordingly to the formula: $y' = 6 + 3k + (4 - k)[m]/2 - s$ (where $[m]$ is the entier of m). This selection of y' fulfills the condition: $Q(y' + s) \leq 0.0001$. Also $u(x, y)$ was computed for $s = 1, 2, 3, 5, 10, 20, 40$. For each s , the values of m used were $m = \ell s$ with the following values for the loading factor: $\ell = 0.25, 0.50, 0.75, 1.00, 1.25, 1.50, 1.75, 2.00$. In Fig. 3(a), 3(b), 3(c), some results for $s = 1$ are given and in Fig. 4(a), 4(b) some for $s = 5$.

The computations were made without explicitly using the properties of $u(x, y)$ derived previously and, further, the following properties were used as a check on the numerical results:

1. $\sum_{x+y=r} u(x, y) = P(r)$,
2. $\sum_{y=0}^{\infty} u(x, y) = P(x)/1 - Q(s + 1)$.

Figure 3 $u(x, y)$ for $s = 1$.



The agreement is always to five, but in most cases to six decimal places.

When m is small relative to s , then the chance for overflow records becomes negligible. Therefore, the values of $u(x, 0)$ will approach $P(x)$ when m becomes small. The numerical results are also in agreement with this conclusion. With a loading factor of 0.25 the maximum discrepancy for $s = 10$ is 0.000050. For $s = 20$ and $\ell = 0.25$, $u(x, 0)$ and $P(x)$ are equal to six decimal places.

The frequency distributions $u(x, y; m)$ were then used to compute average overflow and average number of additional accesses together with their standard deviations. These are listed in Tables 2 and 3. For comparison, the corresponding quantities for the initial loading [5] are also listed. In addition to the overflow percentage $(\bar{i}/m) 100$, also the utilization of primary storage is given $[(m - \bar{i})/s] 100$. Furthermore, the ratio between the space a record needs on the average and its length is given $(s + \bar{i})/m$.

Table 2 Average overflow.

Load Factor	Average Number Per Bucket	Overflow initial loading					Overflow steady state				
		Average Per Bucket	Stand. Dev.	Overflow Percentage	Utilization	Space Per Record	Average Per Bucket	Stand. Dev.	Overflow Percentage	Utilization	Space Per Record
Bucket size $s = 1$											
0.25	0.25	0.03	0.18	11.52	22.12	4.12	0.05	0.23	20.00	20.00	4.20
0.50	0.50	0.11	0.36	21.31	39.35	2.21	0.17	0.43	33.33	33.33	2.33
0.75	0.75	0.22	0.54	29.65	52.76	1.63	0.32	0.61	42.86	42.86	1.76
1.00	1.00	0.37	0.70	36.79	63.21	1.37	0.50	0.76	50.00	50.00	1.50
1.25	1.25	0.54	0.86	42.92	71.35	1.23	0.69	0.90	55.56	55.56	1.36
1.50	1.50	0.72	1.00	48.21	77.69	1.15	0.90	1.03	60.00	60.00	1.27
1.75	1.75	0.92	1.13	52.79	82.62	1.10	1.11	1.14	63.64	63.64	1.21
2.00	2.00	1.14	1.26	56.77	86.47	1.07	1.33	1.25	66.67	66.67	1.17
Bucket size $s = 5$											
0.25	1.25	0.00	0.06	0.18	24.96	4.00	0.01	0.11	0.73	24.82	4.01
0.50	2.50	0.06	0.34	2.48	48.76	2.02	0.17	0.51	6.97	46.51	2.07
0.75	3.75	0.32	0.84	8.63	68.53	1.42	0.66	1.04	17.66	61.75	1.51
1.00	5.00	0.88	1.42	17.55	82.45	1.18	1.42	1.53	28.49	71.51	1.28
1.25	6.25	1.70	1.98	27.21	90.98	1.07	2.36	1.94	37.75	77.81	1.18
1.50	7.50	2.72	2.43	36.22	95.67	1.03	3.40	2.30	45.30	82.05	1.12
1.75	8.75	3.85	2.79	43.98	98.03	1.01	4.50	2.60	51.41	85.03	1.09
2.00	10.00	5.04	3.08	50.43	99.14	1.00	5.64	2.87	56.40	87.21	1.06
Bucket size $s = 20$											
0.25	5.00	0.00	0.01	0.00	25.00	4.00	0.00	0.00	0.00	25.00	4.00
0.50	10.00	0.00	0.08	0.03	49.99	2.00	0.02	0.19	0.19	49.91	2.00
0.75	15.00	0.21	0.87	1.42	73.94	1.35	0.68	1.32	4.56	71.58	1.38
1.00	20.00	1.78	2.37	8.88	91.12	1.09	3.18	2.88	15.89	84.11	1.16
1.25	25.00	5.37	4.44	21.48	98.15	1.01	7.00	4.05	27.99	90.01	1.08
1.50	30.00	10.05	5.37	33.50	99.75	1.00	11.40	4.87	38.01	92.99	1.05
1.75	35.00	15.00	5.90	42.87	99.98	1.00	16.60	5.51	45.90	94.68	1.03
2.00	40.00	20.00	6.32	50.00	100.00	1.00	20.85	6.03	52.13	95.74	1.02

Figure 4 $u(x, y)$ for $s = 5$.

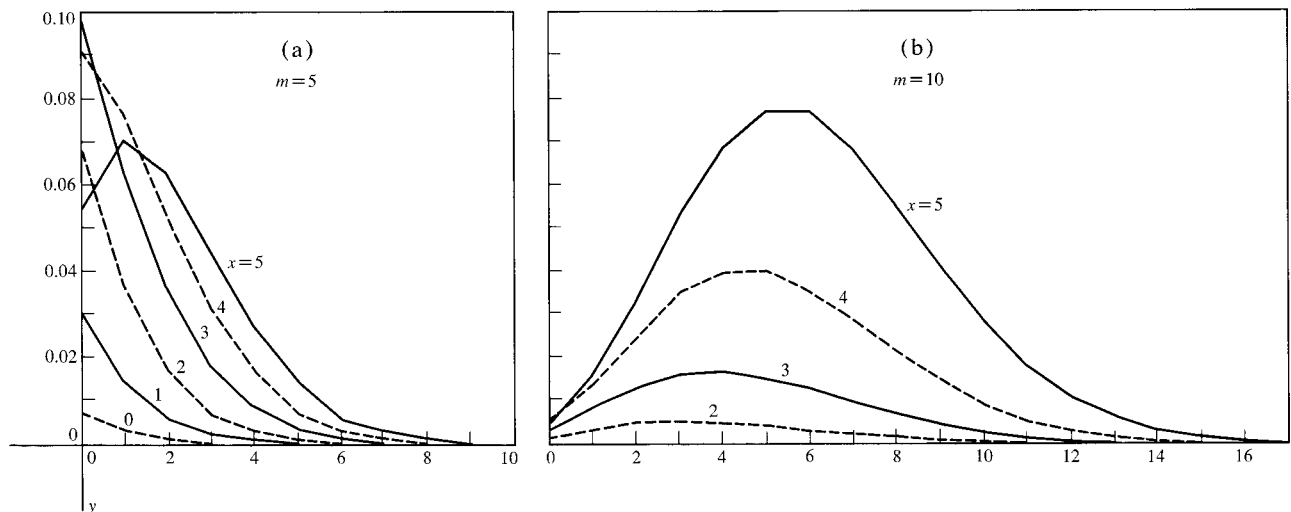


Table 3 Average additional accesses.

Load factor	Average number per bucket	Additional accesses			
		Initial Loading		Steady State	
		Average	Stand. dev.	Average	Stand. dev.
Bucket size $s = 1$					
0.25	0.25	0.13	0.36	0.21	0.44
0.50	0.50	0.25	0.52	0.38	0.59
0.75	0.75	0.38	0.65	0.53	0.70
1.00	1.00	0.50	0.76	0.67	0.80
1.25	1.25	0.63	0.87	0.80	0.89
1.50	1.50	0.75	0.97	0.92	0.98
1.75	1.75	0.88	1.06	1.04	1.07
2.00	2.00	1.00	1.15	1.17	1.16
Bucket size $s = 5$					
0.25	1.25	0.00	0.05	0.01	0.10
0.50	2.50	0.04	0.25	0.09	0.38
0.75	3.75	0.15	0.58	0.29	0.75
1.00	5.00	0.37	0.97	0.58	1.15
1.25	6.25	0.68	1.40	0.94	1.56
1.50	7.50	1.07	1.84	1.35	1.98
1.75	8.75	1.51	2.28	1.80	2.40
2.00	10.00	2.00	2.71	2.28	2.81
Bucket size $s = 20$					
0.25	5.00	0.00	0.00	0.00	0.00
0.50	10.00	0.00	0.03	0.00	0.07
0.75	15.00	0.03	0.35	0.10	0.55
1.00	20.00	0.31	1.24	0.54	1.58
1.25	25.00	1.08	2.64	1.45	2.99
1.50	30.00	2.33	4.27	2.75	4.59
1.75	35.00	3.93	5.99	4.35	6.26
2.00	40.00	5.75	7.71	6.15	7.94

Finally, the minima of the relative cost function were computed in the following way. A parabola was fitted through three points of the cost function. The abscissa: m_1, m_2, m_3 were first selected to be $m_1 = 0.9 m_2$; $m_3 = 1.1 m_2$ and:

$$m_2 = 0.13 - 0.76 \ln \gamma + (1.05 - 0.13\gamma) s.$$

The choice for m_2 coincides with the approximate value for m that gives minimum cost in the initial loading case [5].

The minimum of the parabola was taken as a new value of m_2 . The other two points were the previous central point and the equally distant point at the other side of the new m_2 . So for each approximation, $\bar{i}$, $\bar{a}$ and the value of $[(s+i)/m] + \gamma \bar{a}$ were computed for two new values of m , while for one value, the previous result was used. The process was stopped when $|m_3 - m_1| < 0.0005$.

The resulting values of m are given in Table 4 together with overflow, additional accesses and the minimum cost. Table 5 shows the optimal m in relation to γ and Table 6, the corresponding values of the minimum costs. Figure 5

Table 4 Minima of the relative cost function.

γ	m	Load factor	Overflow factor	Additional accesses	Minimum costs
Bucket size = 1					
2.000	0.850	0.850	0.460	0.586	2.808
1.000	1.175	1.175	0.540	0.758	2.149
0.500	1.589	1.589	0.614	0.966	1.726
0.100	3.009	3.009	0.751	1.654	1.248
0.050	3.899	3.899	0.796	2.084	1.157
0.010	6.936	6.936	0.874	3.565	1.054
Bucket size = 2					
2.000	1.446	0.723	0.299	0.408	2.499
1.000	1.893	0.946	0.382	0.580	2.019
0.500	2.447	1.224	0.465	0.801	1.683
0.100	4.301	2.151	0.636	1.590	1.260
0.050	5.441	2.721	0.697	2.101	1.169
0.010	9.303	4.651	0.808	3.910	1.062
Bucket size = 3					
2.000	2.067	0.689	0.221	0.316	2.304
1.000	2.604	0.868	0.296	0.474	1.922
0.500	3.262	1.087	0.376	0.689	1.640
0.100	5.426	1.809	0.557	1.503	1.261
0.050	6.739	2.246	0.626	2.051	1.174
0.010	11.195	3.732	0.757	4.056	1.066
Bucket size = 5					
2.000	3.378	0.676	0.143	0.221	2.065
1.000	4.058	0.812	0.204	0.354	1.790
0.500	4.880	0.976	0.275	0.547	1.573
0.100	7.500	1.500	0.453	1.348	1.254
0.050	9.054	1.811	0.527	1.916	1.175
0.010	14.326	2.865	0.680	4.111	1.070
Bucket size = 10					
2.000	6.918	0.692	0.075	0.132	1.785
1.000	7.863	0.786	0.115	0.228	1.615
0.500	8.971	0.897	0.167	0.379	1.471
0.100	12.379	1.238	0.317	1.088	1.234
0.050	14.361	1.436	0.390	1.642	1.168
0.010	20.915	2.091	0.556	3.949	1.074
Bucket size = 20					
2.000	14.575	0.729	0.039	0.079	1.568
1.000	15.898	0.795	0.062	0.143	1.464
0.500	17.406	0.870	0.095	0.253	1.371
0.100	21.854	1.093	0.205	0.822	1.203
0.050	24.380	1.219	0.266	1.311	1.152
0.010	32.505	1.625	0.422	3.522	1.073
Bucket size = 40					
2.000	30.904	0.773	0.019	0.047	1.409
1.000	32.748	0.819	0.033	0.090	1.344
0.500	34.821	0.871	0.052	0.166	1.284
0.100	40.738	1.018	0.126	0.602	1.168
0.050	43.956	1.099	0.170	0.996	1.130
0.010	54.051	1.351	0.298	2.937	1.067

shows curves for m as a function of s for different values of γ .

Finally, Table 7 shows the relative values of the cost function in the neighborhood of the minimum. The excess over the minimum is expressed as a percentage of the minimum.

Table 5 Optimum value of m

s	$\gamma = 2.000$	$\gamma = 1.000$	$\gamma = 0.500$	$\gamma = 0.100$	$\gamma = 0.050$	$\gamma = 0.010$
1	0.850	1.175	1.589	3.009	3.899	6.936
2	1.446	1.893	2.447	4.301	5.441	9.303
3	2.067	2.604	3.262	5.426	6.739	11.195
5	3.378	4.058	4.880	7.500	9.054	14.326
10	6.918	7.863	8.971	12.379	14.361	20.915
20	14.575	15.898	17.406	21.854	24.380	32.505
40	30.904	32.748	34.821	40.738	43.956	54.051

Table 6 Minimum cost

s	$\gamma = 2.000$	$\gamma = 1.000$	$\gamma = 0.500$	$\gamma = 0.100$	$\gamma = 0.050$	$\gamma = 0.010$
1	2.808	2.149	1.726	1.248	1.157	1.054
2	2.499	2.019	1.683	1.260	1.169	1.062
3	2.304	1.922	1.640	1.261	1.174	1.066
5	2.065	1.790	1.573	1.254	1.171	1.070
10	1.785	1.615	1.471	1.234	1.168	1.074
20	1.568	1.464	1.371	1.203	1.152	1.073
40	1.409	1.344	1.284	1.168	1.130	1.067

Table 8 (m for initial loading/ m for steady state) - 1.

s	$\gamma = 2.000$	$\gamma = 1.000$	$\gamma = 0.500$	$\gamma = 0.100$	$\gamma = 0.050$	$\gamma = 0.010$
1	0.039	-0.010	-0.059	-0.187	-0.253	-0.408
2	0.097	0.044	-0.008	-0.148	-0.218	-0.387
3	0.120	0.073	0.024	-0.116	-0.187	-0.363
5	0.135	0.098	0.055	-0.072	-0.139	-0.317
10	0.129	0.107	0.078	-0.018	-0.075	-0.241
20	0.108	0.096	0.079	0.018	-0.025	-0.161
40	0.085	0.078	0.069	0.033	0.006	-0.092

Table 9 Space per record for minimum.

s	$\gamma = 2.000$		$\gamma = 1.000$		$\gamma = 0.500$		$\gamma = 0.100$		$\gamma = 0.050$		$\gamma = 0.010$	
	a	b	a	b	a	b	a	b	a	b	a	b
1	1.469	1.636	1.269	1.391	1.149	1.243	1.035	1.083	1.018	1.052	1.004	1.018
2	1.463	1.682	1.279	1.439	1.161	1.282	1.040	1.101	1.021	1.065	1.005	1.023
3	1.439	1.672	1.273	1.448	1.162	1.296	1.041	1.110	1.022	1.071	1.005	1.025
5	1.396	1.622	1.256	1.436	1.159	1.300	1.043	1.120	1.023	1.079	1.005	1.029
10	1.328	1.520	1.224	1.387	1.146	1.282	1.043	1.125	1.024	1.086	1.006	1.034
20	1.263	1.410	1.189	1.320	1.129	1.244	1.042	1.120	1.024	1.086	1.006	1.037
40	1.205	1.313	1.155	1.254	1.109	1.200	1.040	1.108	1.023	1.080	1.006	1.038

a Initial Loading
 b Steady State

Table 7 Percentages over minimum costs at points ($s + j$, km).

j	k				
	0.8	0.9	1.0	1.1	1.2
	$s = 1 \quad \gamma = 2.000 \quad m = 0.850$				
2	60.1	43.6	30.8	20.6	12.4
1	18.8	9.9	3.4	-1.4	-4.9
0	1.7	0.4	0.0	0.3	1.1
	$s = 1 \quad \gamma = 0.010 \quad m = 6.936$				
2	6.4	4.5	3.3	2.5	1.9
1	2.8	1.9	1.4	1.0	0.8
0	0.2	0.1	0.0	0.0	0.1
	$s = 5 \quad \gamma = 1.000 \quad m = 4.058$				
2	24.4	13.3	5.5	0.2	-3.0
1	12.2	4.7	0.2	-2.1	-2.6
0	4.5	1.0	0.0	0.8	3.1
-1	3.3	4.0	6.5	10.4	15.3
-2	10.7	15.3	21.1	27.7	35.1
	$s = 10 \quad \gamma = 0.500 \quad m = 8.971$				
2	17.5	7.9	2.0	-1.1	-1.8
1	10.9	3.7	0.0	-1.0	0.2
0	5.9	1.3	0.0	1.1	4.2
-1	2.9	1.1	2.2	5.4	10.4
-2	2.3	3.3	6.8	12.1	18.9
	$s = 20 \quad \gamma = 0.100 \quad m = 21.854$				
2	10.4	3.6	0.4	-0.4	0.7
1	7.6	2.2	0.0	0.1	1.8
0	5.3	1.2	0.0	0.9	3.3
-1	3.5	0.6	0.4	2.1	5.1
-2	2.2	0.4	1.2	3.6	7.2
	$s = 20 \quad \gamma = 2.000 \quad m = 14.575$				
2	20.9	8.9	1.3	-1.6	0.6
1	16.0	5.5	0.0	-0.2	5.2
0	11.5	2.9	0.0	3.1	12.3
-1	7.6	1.6	1.9	8.7	22.2
-2	4.8	1.9	6.0	17.3	35.5
	$s = 40 \quad \gamma = 0.5 \quad m = 34.821$				
2	17.8	6.4	0.4	0.3	6.6
1	15.3	4.7	0.0	1.6	9.7
0	12.8	3.3	0.0	3.4	13.4
-1	10.5	2.1	0.4	5.7	17.7
-2	8.3	1.3	1.4	8.6	22.7

Analysis of results

The results in Tables 2 and 3 confirm the obvious supposition that the overflow and the number of additional accesses in the steady-state case are larger than in the case of initial loading. We observe that for $s = 1$, the overflow and utilization are the same. This means $\bar{i}/m = m - \bar{i}$ or $\bar{i}/m = m/(1 + m)$. This follows from Eq. (9) for $s = 1$: $i/m = [P(1)/(P(0) + P(1))]$.

Table 2 shows that for a file in steady state, optimal load factors are in general higher than actually used in practice. The same observation was made in [5]. A comparison between the optimal values of m for both cases is made in Table 8.

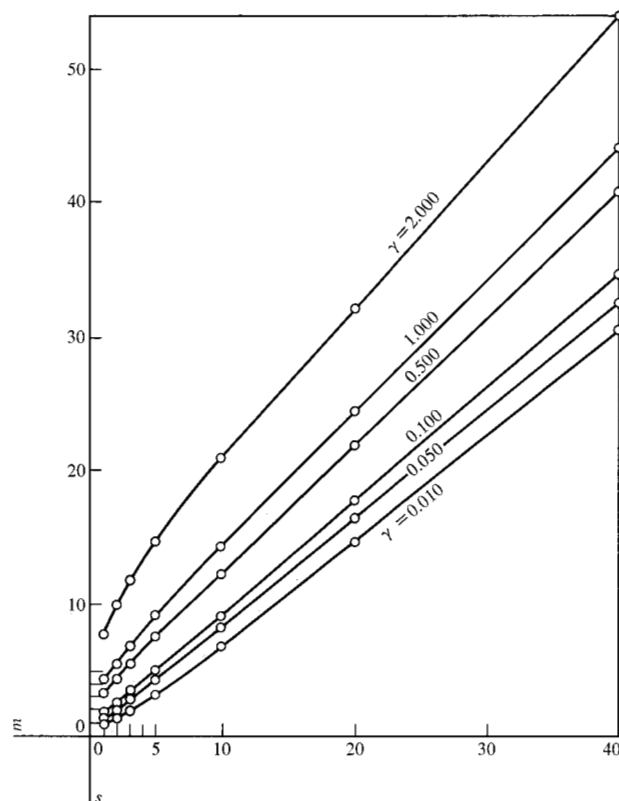


Figure 5 m as a function of s .

The values around the minimum (Table 7) show similar behavior to that for initial loading. When m is kept at the value for which the minimum is reached, then increasing as well as decreasing s gives higher cost. Lower values than in the minimum under consideration are only obtained by increasing both m and s at the same time.

For large values of γ the minimum cost decreases monotonically with increasing s (Table 6). However, for small values of γ , there is a slight initial cost increase before the cost starts to decrease with increasing bucket size. This behavior differs from the initial loading case [5]. There, the minimum always decreases with increasing bucket size.

Table 9 gives a comparison of the average space per record for optimal loading. For small values of γ this quantity increases with s . When additional accesses are expensive and/or frequent (γ is large), then the average space per record decreases with increasing bucket size in the case of initial loading. In the steady-state case, there is still an increase for small values of s . In all cases the values for steady state are larger than for initial loading.

Finally, Table 8 shows that the load factors that give minimum cost are higher for initial loading than for steady state when γ is large. The situation is reversed when γ is small.

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