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Existence and Uniqueness of the Solution to Holland's Equations for a Class of Multicolumn Distillation Systems

Abstract: The Holland equations for a multiunit system of distillation columns, interconnected so as to contain recycle loops, can be expressed as a matrix generalization of the Holland equations for a single complex column. Proof of the existence and uniqueness of a positive solution for the single-column case has previously been given. In this paper a proof is given for the case of a multiunit system containing one recycle loop.

Introduction

A brief description of the use and action of single distillation columns appears in a previous paper in this journal [1]. In practice these units are usually employed in systems (batteries) of from two to twenty interconnected columns. Batteries of units are able to accomplish separations of chemical compounds beyond the capability of a single unit. Units interconnected as depicted in Fig. 1 form a distillation train. The solution of equations determining flow rates in a multicolumn train may be obtained by successively applying to each unit the convergence acceleration procedure proposed by Holland [2] for single columns or some of the other methods that have appeared in the literature [3 – 10]. To whatever extent units comprising a battery are not in a train the interconnections form recycle loops. A very simple example is depicted by Fig. 2. To handle such situations Nartker, Srygley and Holland [11,12] extended Holland's single-column convergence acceleration technique to multicolumn systems, and Petryschuk and Johnson [13,14] subsequently successfully employed this extension to the simulation of a "light-ends" separation system (light ends are low-molecular-weight distillates.)

Familiarity with the approaches of Holland [2] and Nartker, et al. [12] is assumed. Subsequent nomenclature is similar to that used in these two references. In Ref. 15 a procedure different from that of Nartker, et al. [11,12] was presented for computing the Holland correction factors for a battery of distillation units, and an application of the new procedure was given. The advantages of the new approach are that only the Holland correction factors for single columns are invoked, it being unnecessary to introduce an additional set as is done by Nartker, et al. [12], and that the present development permits the existence and uniqueness of a positive solution to be proven, at least for the case of an arbitrary number of units comprising a single feedback loop. Presentation of this proof occupies the greater part of this paper. As in the case of Newton's method it is desirable to know sufficient conditions for a unique solution to exist even if these are not necessary conditions.

Preliminary considerations

The only streams considered here are those entering or leaving a unit (distillation column), and the term "product stream" refers to any of the several streams that may leave a unit. As Nartker [11] has pointed out, the component flow rate in any product stream may be employed as the denominator of product ratios [i.e., $(w_s/b)_{in}$ or $(d/b)_{in}$, as defined in the next section] for that unit, and it is not necessary that the same product stream supply the denominator for each component in a particular unit.

In practice one selects these denominators so as to avoid attempting to divide by zero; however, for notational convenience, herein all such denominators pertaining to the same unit are assumed to be flow rates in the same stream from that unit, and that stream is referred to as a "base stream." For definiteness the bottom stream from a unit is arbitrarily chosen as the base stream for that unit. Recall that during any particular iteration in

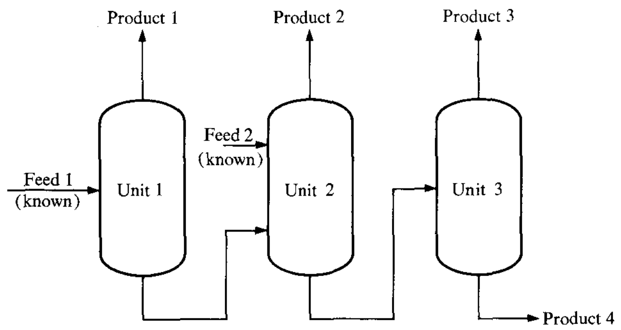

Figure 1 Configuration of a distillation train.

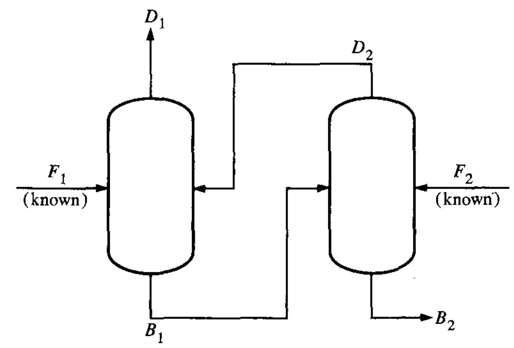

Figure 2 Distillation system with feedback.

the computation the product ratios are known positive constants fixed by the choice of the base stream for each unit.

An "external" feed stream is one that originates outside of the system, and in all cases the complete state (i.e., the flow rates of each component in the stream) of each external feed is assumed to be known. Further, the total flow rate of every feed stream is assumed to be specified. As shown previously [1], if other specifications are used, a positive solution to the Holland equations may not exist.

The term "recycle loop" denotes a succession of streams from one unit to another and through which some portion of a product stream from a particular unit might eventually return as feed to that unit without leaving the system. When a set of recycle loops is considered, the individual recycle loops are independent if and only if each contains at least one stream not in any other recycle loop of the set.

It is, of course, necessary to employ a set of independent stream flow specifications to determine the Holland correction factors. These independent specifications are further restricted in value by the condition that all dependent specifications must be positive (i.e., the specifications must be mutually consistent).

It is convenient to employ the convention that for any pair of matrices or vectors α and β the expressions $\alpha > \beta$, $\alpha \geq \beta$, $\alpha = \beta$, $\alpha \leq \beta$ and $\alpha < \beta$ denote that the particular relation is true for corresponding elements of α and β. The symbol "0" denotes a scalar zero or a vector or matrix, each element of which is zero.

Holland equations

The formulation given by Nartker, et al. [11,12] employs two sets of correction factors, one for the individual units and the other for the system as a whole. The former set is the same as that used by Holland [2] in solving single units. It is possible to express all the Holland equations for a multicolumn system in terms of the Holland factors for the individual units and so avoid the introduction of a set of redundant variables. For a system of of N units and $S + N$ product streams ($S \geq N$), the feed-rate vector of the ith component is defined by

$$f_i \equiv (f_{i1}{}^0, f_{i2}{}^0, \cdots, f_{in}{}^0, \cdots, f_{iN}{}^0)^T, \tag{1a}$$

where $f_{in}{}^0$ is the specified feed rate of component i into unit n.

The vector of total flow rates of bottom streams is

$$b_i \equiv (b_{i1}, b_{i2}, \cdots, b_{in}, \cdots, b_{iN})^T, \tag{1b}$$

where b_{in} is the flow rate of component i in the bottom stream from unit n.

The total flow rate of all components in the bottom stream is defined by the vector

$$c_B \equiv (B_1, B_2, \cdots, B_n, \cdots, B_N)^T, \tag{1c}$$

where

$$B_n = \sum_{i=1}^{I} b_{in},$$

and I denotes the total number of components.

The vector of Holland correction factors is denoted by

$$\Theta \equiv (\theta_1, \theta_2, \cdots, \theta_s, \cdots, \theta_S)^T, \qquad (\theta_s \geq 0). \tag{1d}$$

The superscript T in the above equations and throughout the paper denotes a transpose.

The material balances for the ith component in a system can be expressed by the matrix-vector equation

$$\Psi_i b_i = f_i, \tag{2}$$

where Ψ_i is an N by N matrix expressing the dependence of the component material balances upon the system configuration and the correction factors.

For the system configuration given by Fig. 2, one defines

$$\Psi_i \equiv \begin{bmatrix} \left(\frac{d}{b}\right)_{i1} \theta_1 + 1 & -\left(\frac{d}{b}\right)_{i2} \theta_2 \\ -1 & \left(\frac{d}{b}\right)_{i2} \theta_2 + 1 \end{bmatrix}, \tag{3}$$

where d_{in} is the flow rate of component i in the distillate from unit n. From Eqs. (2) and (3) the material balances for Fig. 2 are

$$\left(\frac{d}{b}\right)_{i1} \theta_1 b_{i1} + b_{i1} - \left(\frac{d}{b}\right)_{i2} \theta_2 b_{i2} = f_{i1}^{\,0} ,$$

$$-b_{i1} + \left(\frac{d}{b}\right)_{i2} \theta_2 b_{i2} + b_{i2} = f_{i2}^{\,0} . \qquad (4)$$

Since in this case the specifications of B_1 and B_2 are independent it is convenient to use them for evaluating the θ's. The Holland equations for the base streams are represented by the vector $g_B \equiv (\Gamma_1, \Gamma_2, \cdots, \Gamma_s)^T$ and are expressed by the matrix-vector equation

$$g_B \equiv c_B - \sum_i b_i = c_B - \sum_i \Psi_i^{-1} f_i = 0 . \qquad (5)$$

Equation (5) strongly resembles the set of Holland equations for a single complex column [1,2] except for the fact that the individual equations cannot be written separately, since the explicit expression for Ψ_i^{-1} is not usually available. One notes, however, that Ψ_i is diagonally dominant (by columns) and is an M-matrix. Consequently $\Psi_i^{-1} \geq 0$ as shown by Varga [16]. Equation (5) is solved by Newton-Raphson iteration.

Existence and uniqueness

The following proofs will be carried out in the non-negative orthant of the space spanned by the set of Holland correction factors $(\Theta^T, \theta_{S+1})^T$. Note that the vector Θ as defined in Eq. (1) lies in an S-dimensional hyperplane of this space. A familiarity with the corresponding proofs by induction for the case of a single complex column as given in Ref. 1 is presumed. The proof here will be developed by induction and be restricted to a system of an arbitrary number N of units comprising exactly one feedback loop; that is, each unit supplies exactly one feed stream to another unit and each unit receives exactly one such feed. Figure 1 would depict such a system for $N = 3$ if product 4 therein were fed back to unit 1. The units are numbered so that unit n feeds unit $n + 1$ and unit N feeds unit 1. The number and arrangement of external feeds and of product streams leaving the system are unrestricted. The latter are numbered consecutively starting with unit 1 and with all such product streams from unit n numbered prior to any in unit $n + 1$. The specified total flow rate of the sth such stream is denoted by W_s, the flow rate of component i in this stream by w_{is}, and the total number of such streams is $S + 1$. N of the product specifications are not independent, and all specifications must be mutually consistent. The base stream from a unit is that stream which feeds another unit in the system. For notational convenience only the bottom stream from each unit is taken as the base stream.

One now defines

$c_W \equiv (W_1, W_2, \cdots, W_s, \cdots, W_S)^T$, the Holland vector of specified flow rates in all non-base product streams;

$\Phi \equiv$ the diagonal matrix having θ_s as its sth diagonal element;

$\Lambda_i \equiv (\lambda_{sn})_i$ as a rectangular constant matrix having S rows and N columns with exactly one nonzero element in each row such that

$$\lambda_{sni} \equiv \begin{cases} (w_s/b)_{in} & \text{if stream } s \text{ is supplied by unit } n \\ 0 & \text{otherwise;} \end{cases}$$

$\Psi_i \equiv (\psi_{mn})_i$ as an N by N square matrix with

$$\psi_{mni} \equiv \begin{cases} -1 \text{ if } n = m + 1 \text{ or if } m = 1 \text{ and } n = N \\ 1 + \sum_* (w_*/b)_{in}\, \theta_* \text{ if } m = n \\ 0 \text{ otherwise ,} \end{cases}$$

where the subscript * denotes a value of s for which stream s leaves unit n. With the foregoing definitions Eq. (5) remains unaltered, and one verifies that the material balance relations must constitute a positive solution to Eq. (6) together with one component of Eq. (5). That is,

$$g_W \equiv c_W - \Phi \sum_i \Lambda_i b_i = c_W - \Phi \sum_i \Lambda_i \Psi_i^{-1} f_i = 0 , \qquad (6)$$

$$\Gamma_n = 0 \text{ for some } n . \qquad (7)$$

The reader is reminded that all work is restricted to the first orthant. Existence and uniqueness of the solution to Eq. (6) together with one component of Eq. (5) will be proven by showing the following:

A. $g_B \leq 0$ only in a bounded region, and $\partial g_B/\partial\theta_0 > 0$ and $\partial g_B/\partial\Theta > 0$;
B. If $g_W = 0$ in the region where $g_B \leq 0$, then $g_W = 0$ on a continuous curve which passes into the region where $g_B \geq 0$, and the tangent to which has positive direction cosines;
C. $g_W = 0$ does occur in the region where $g_B \leq 0$.

Existence and uniqueness will then follow from conditions A, B, C and the continuity of all functions.

• *Condition A*

The above definitions for the ψ_{mni} show that for the class of configuration being considered the graph of Ψ_i is strongly connected. Varga [16] shows that this property, in addition to those that Ψ_i has already been shown to possess, insures that $\Psi_i^{-1} > 0$ rather than merely ≥ 0. Thus, in the present case Eq. (2) yields

$$b_i = \Psi_i^{-1} f_i > 0 , \qquad (8)$$

and since the only nonzero element of $\partial\Psi_i/\partial\theta_s$ is positive,

$$\frac{\partial b_i}{\partial \theta_s} = -\Psi_i^{-1} \frac{\partial \Psi_i}{\partial \theta_s} \Psi_i^{-1} f_i$$

$$= -\Psi_i^{-1} \frac{\partial \Psi_i}{\partial \theta_s} b_i < 0 . \qquad (9)$$

Equations (5) and (9) show that

$$\partial g_B / \partial \theta_s > 0 . \qquad (10)$$

Equation (10) shows immediately that

$$\left. \frac{\partial \theta_r}{\partial \theta_s} \right|_{\Gamma_n = 0} = - \frac{\partial \Gamma_n / \partial \theta_s}{\partial \Gamma_n / \partial \theta_r} < 0 . \qquad (11)$$

On the θ_s-axis, θ_s is the only nonzero θ, and hence all material fed to the unit must leave via stream s if Eq. (4) is to be satisfied. Consequently, if stream s leaves unit n

$$(w_s/b)_{in}\, \theta_s\, b_{in} = u^T f_i , \qquad (12)$$

where

$$u^T \equiv (1, 1, \cdots, 1) .$$

Equations (5) and (12) then show that on the θ_s-axis

$$\Gamma_n \equiv B_n - \sum_i b_{in} = \Gamma_n^{\,0}(\theta_s)$$

$$\equiv B_n - \frac{1}{\theta_s} \sum_i \left(\frac{w_s}{b} \right)_{in}^{-1} u^T f_i . \qquad (13)$$

Since Eq. (13) always has a positive root $\theta_s^{\,+}$ the surface $\Gamma_n = 0$ must intersect the θ_s-axis when stream s leaves unit n. Further, in view of Eq. (10), only one such intersection is possible. In general no single surface $\Gamma_n = 0$ intersects each axis, although every θ_s-axis is intersected at a finite point $\theta_s = \theta_s^{\,+}$ by some surface $\Gamma_k = 0$. Equations (11) and (13) show these surfaces to have the general shape depicted in Fig. 3. One sees that any solution to Eqs. (5) must lie in the region R^+ for which

$$0 \le \theta_s \le \theta_s^{\,+} , \qquad (s = 1, 2, \cdots, S+1) .$$

• *Condition B*

Some knowledge of the surfaces defined by $g_W = 0$ in Eq. (6) is now required. If G_s denotes an arbitrary element of g_W, then Eq. (6) shows

$$G_s \equiv W_s - \theta_s \sum_i \left(\frac{w_s}{b} \right)_{in} b_{in} , \qquad (14)$$

where stream s leaves unit n. Equation (6) does not involve G_{S+1} or W_{S+1}, and for this specification,

$$G_{S+1} \equiv W_{S+1} - \theta_{S+1} \sum_i \left(\frac{w_{S+1}}{b} \right)_{iN} b_{iN} . \qquad (15)$$

Then, in view of Eq. (9) and since $\theta_s > 0$,

$$\frac{\partial G_s}{\partial \theta_p} = -\theta_s \sum_i \left(\frac{w_s}{b} \right)_{in} \frac{\partial b_{in}}{\partial \theta_p} > 0 , \qquad p \ne s . \qquad (16)$$

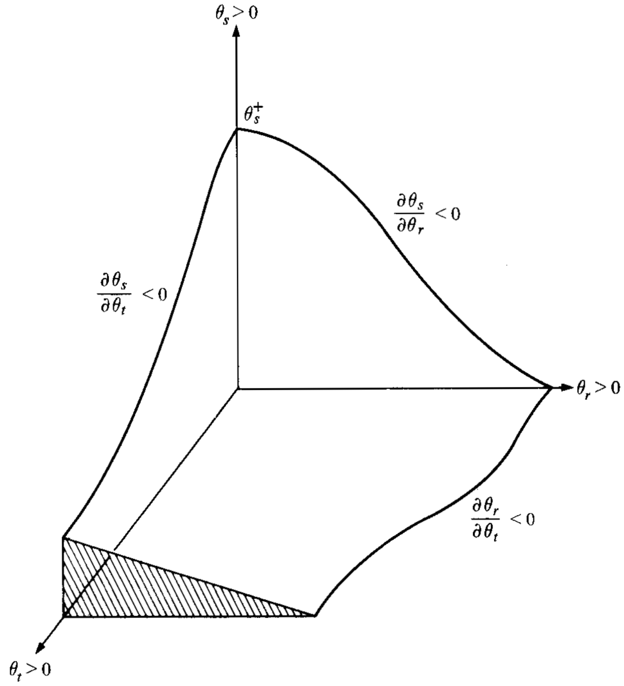

Figure 3 Section of possible level surface of $\Gamma_k = 0$.

The overall component material balance is

$$\sum_s \left(\frac{w_s}{b} \right)_{i*} \theta_s b_{i*} = u^T f_i , \qquad (17)$$

where the subscript * denotes the number of the unit supplying stream s. Differentiating Eq. (17) with respect to θ_p yields

$$\left(\frac{w_p}{b} \right)_{i*} b_{i*} + \sum_s \left(\frac{w_s}{b} \right)_{i*} \theta_s \frac{\partial b_{i*}}{\partial \theta_p} = 0 . \qquad (18)$$

Then Eqs. (14), (16), and (18) produce

$$\frac{\partial G_p}{\partial \theta_p} = - \sum_i \left(\frac{w_p}{b} \right)_{in} b_{in} - \theta_p \sum_i \left(\frac{w_p}{b} \right)_{in} \frac{\partial b_{in}}{\partial \theta_p}$$

$$= \sum_{s; p \ne s} - \frac{\partial G_s}{\partial \theta_p} < 0 \qquad (19)$$

except on the θ_p-axis, where $\partial G_p / \partial \theta_p = 0$. Equation (17) shows b_i is not defined at the origin. The physical explanation is, of course, that for all θ's equal to zero, material is being fed to the system but none is being withdrawn.

The induction assumption will subsequently be used to show that the S surfaces $g_W = 0$ intersect in a continuous curve in the first orthant of the $S+1$ dimensional space spanned by $(\Theta^T, \theta_{S+1})^T$. Denote this curve by

$$\Theta^*(\theta) \equiv [\theta_1^*(\theta), \theta_2^*(\theta), \cdots, \theta_S^*(\theta), \theta]^T ,$$

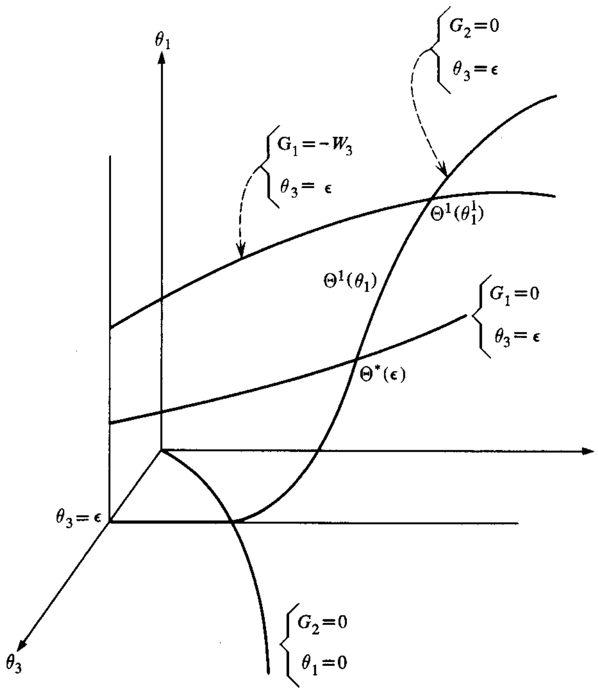

Figure 4 Path exhibiting point of $\Theta^*(\theta_{S+1})$.

where θ denotes θ_{S+1} , the subscript $S+1$ being suppressed for notational convenience.

The behavior of $\Theta^*(\theta)$ is now to be investigated. Let η_s be proportional to the cosine of the angle between the tangent to $\Theta^*(\theta)$ and the θ_s-axis. Now at any point $\Theta^*(\theta)$ must be orthogonal to the normal to each surface given by $g_W=0$. This is expressed by

$$\sum_s \eta_s\, \partial G_p/\partial\theta_s = 0\,, \qquad (p=1\,,2\,,\cdots,S)\,.$$

The number of unknowns η_s is $S+1$; consequently, one sets $\eta_{S+1}=1$ so that the above relations become

$$\frac{\partial g_W}{\partial\Theta}\,\eta = -\frac{\partial g_W}{\partial\theta_{S+1}}\,, \tag{20}$$

where

$$\eta \equiv (\eta_1\,,\eta_2\,,\cdots,\eta_S)^T\,.$$

Equation (19) shows that $\partial g_W/\partial\Theta$ is diagonally dominant by columns; in fact the sum of all elements in column p is precisely $-\partial G_{S+1}/\partial\theta_p$ [1]. Furthermore, Eqs. (16) and (19) show that all off-diagonal elements of this matrix are positive. Varga [16] shows that these conditions are sufficient to insure that $(\partial g_W/\partial\Theta)^{-1}$ exists and contains only negative elements, since the graph of $\partial g_W/\partial\Theta$ is strongly connected. Thus, in view of Eq. (16),

$$\eta = -\left(\frac{\partial g_W}{\partial\Theta}\right)^{-1}\frac{\partial g_W}{\partial\theta_{S+1}} > 0\,. \tag{21}$$

Hence, as one proceeds along $\Theta^*(\theta)$ in the direction of increasing θ the distance from the origin steadily increases. In particular, within the finite region R^+ each element of g_W is a continuous function of each θ_s , and hence for each s, $\eta_s/\sqrt{\eta^T\eta}$ exceeds some finite positive minimum η_s^- [17]. Consequently, two points on $\Theta^*(\theta)$ and further apart than θ_s^+/η_s^- cannot both be in R^+. It would be sufficient if η_s^- existed for only one s. One observes that $\Theta^*(\theta)$ cannot intersect any coordinate hyperplane except perhaps that of $\theta_{S+1}=0$, since for $\theta_s=0$, $G_s=W_s>0$.

• *Condition C*

The induction assumption is invoked at this point to show the existence of $\Theta^*(\theta)$ and that $g_B[\Theta^*(0)]<0$. Proof in the case of $N=1$, as required for induction, is deferred to a subsequent paragraph. The assumed solution for the case of $N-1$ units will be the solution for the present system of N units when each θ corresponding to a stream from unit N is zero, hence all material entering unit N leaves through the base (bottom) stream and is thus fed to unit 1. This is the same as if unit N were absent, unit $N-1$ fed unit 1 and all external feeds to unit N were external feeds to unit 1 entering on the same stage as the feed from unit $N-1$. The specification for the bottom stream from unit n will be $B_n+1+\sum_i f_{iN}{}^0$, $n\neq N$. All remaining specifications will be the same as in the case of N units except for the stream denoted by $s=1$. The specification for this stream will be

$$W_1{}^0 = W_1 + W_{S+1}\,. \tag{22}$$

At least one specification of the type given by Eq. (22) is necessary for the total specified output to equal the total feed (mutual consistency). To avoid confusion regarding the dimension of the space under consideration it is assumed that stream $S+1$ is the only non-base stream leaving unit N. This restriction is removed subsequently.

In the absence of a recycle loop the existence of $\Theta^*(\theta)$ follows trivially from the induction assumption [1]. Here, because of the necessity for Eq. (22), the solution for $N-1$ units does not lie on $\Theta^*(\theta)$ but rather on $\theta_{S+1}=0$ and the intersection of the surfaces given by

$$G_1 = W_1 - \theta_1\sum_i\left(\frac{w_1}{b}\right)_{i1} b_{i1} = W_1 - W_1{}^0 = -W_{s+1} < 0\,,$$

$$G_s = 0\,, \qquad (s=2\,,3\,,\cdots,S)\,. \tag{23}$$

Thus the curve represented by Eqs. (23) exists. It is continuous since the implicit function theorem [18] holds except at the origin. Hence it intersects the hyperplane $\theta_{S+1}=\varepsilon$ for sufficiently small values of $\varepsilon>0$. Denote this intersection by $\Theta^1(\theta_1{}^1)$, where

$$\Theta^1(\theta_1) \equiv [\theta_1\,,\theta_2{}^*(\theta_1)\,,\cdots,\theta_S{}^*(\theta_1)\,,\varepsilon]^T\,.$$

The procedure to exhibit a point on $\Theta^*(\theta)$ is depicted in Fig. 4 for the case $N = 3 = S + 1$. Roughly, one proceeds from the point $\Theta^1(\theta_1^{\ 1})$ along the curve $\Theta^1(\theta_1)$ defined by

$$G_s = 0 , \qquad (s = 2, 3, \cdots, S) , \tag{24a}$$

$$\theta_{S+1} = \varepsilon , \tag{24b}$$

until this curve intersects the surface $G_1 = 0$ [point $\Theta^*(\varepsilon)$ on Fig. 4]. In this way the path is always on the surface given by Eqs. (24a). What must be shown is that the last intersection actually exists; certainly no corresponding intersection exists in the hyperplane for $\theta_{S+1} = 0$. To this end select an ε small enough so that $g_B[\Theta^1(\theta_1^{\ 0})] < 0$ for the original specifications for N units. This is always possible since from the solution for $N - 1$ units

$$\Gamma_n = B_n - \sum_i b_{in} = B_n - (B_n + 1 + \sum_i f_{in}^{\ 0}) < 0 ,$$

$$(n = 1, 2, \cdots, N - 1) ,$$

$$\Gamma_N = B_N - \sum_i b_{i(N-1)} = B_N - (B_{N-1} + 1 + \sum_i f_{iN}^{\ 0}) < 0 .$$

In the hyperplane defined by $\theta_{S+1} = \varepsilon$ one verifies

a. $\Theta^1(\theta_1)$ exists because $\Theta^1(\theta_1^{\ 1})$ is a point on it and remains in the hyperplane by definition;
b. the origin does not lie in the hyperplane; hence all functions are continuous [19] there and the implicit function theorem holds [18];
c. $G_s[\Theta^1(\theta_1)] = 0$ by definition, and $G_s = W_s > 0$ for $\theta_s = 0$, $(s = 1, 2, \cdots, S)$, in view of b); hence $\Theta^1(\theta_1)$ intersects no coordinate hyperplane except possibly that of $\theta_1 = 0$;
d. $\Theta^1(\theta_1)$ is subject to the analysis and results of subsection B but in one less dimension and with θ_1 taking the role of θ_{S+1} and the region $0 < (\Theta, \varepsilon) \leq \Theta^1(\theta_1^{\ 1})$ replacing R^+;
e. $\Theta^1(0)$ exists as the intersection of $\Theta^1(\theta_1)$ and the coordinate hyperplane $\theta_1 = 0$, due to d);
f. $G_1[\Theta^1(0)] = W_1 > 0$ in view of c) and e) while $G_1[\Theta^1(\theta_1^{\ 1})] = -W_{S+1} < 0$ by construction; hence, in view of b), $G_1[\Theta^1(\theta_1^*)] = 0$ exists for $0 < \theta_1^* < \theta_1^{\ 1}$ and is unique by virtue of d);
g. $\Theta^*(\theta)$ exists since $\Theta^1(\theta_1^*) \equiv \Theta^*(\varepsilon)$ is a point on it; $\Theta^*(\theta)$ intersects $\theta_{S+1} = \varepsilon$ at no other point since $G_1[\Theta^1(\theta_1^*)] = 0$ is unique.

With the existence of $\Theta^*(\theta)$ now established one observes that Eqs. (10) and (21) together with item d) and the construction of $\Theta^1(\theta_1^{\ 1})$ show

$$g_B[\Theta^*(\varepsilon)] < g_B[\Theta^1(\theta_1^{\ 1})] < 0$$

so that $\Theta^*(\varepsilon)$ lies within R^+. As shown in subsection B, $\Theta^*(\theta)$ must leave this region, but since $\eta > 0$ it cannot intersect any coordinate hyperplane. Consequently $\Theta^*(\theta)$ intersects at least one of the hyperplanes $\theta_s = \theta_s^{\ +}$, $(s = 1, \cdots, S + 1)$. Let $\theta_p = \theta_p^{\ +}$ denote the first such intersection, and suppose unit n supplies stream p. Then

$$g_W = 0 ,$$

$$\Gamma_n = 0 \tag{25}$$

comprises the set of $S + 1$ independent Holland equations for the system of N units. Since at $\theta_p = \theta_p^{\ +}$, $\Gamma_n > 0$ while $\Gamma_n[\Theta^*(\varepsilon)] < 0$ and Γ_n is continuous, $\Theta^*(\theta)$ intersects the surface $\Gamma_n = 0$. Since $\eta > 0$, Eq. (10) shows that only one such intersection is possible. Thus Eqs. (25) possess exactly one positive solution $\Theta^*(\theta_{S+1}^*) > 0$. Because of requirement that all specifications be mutually consistent, there is also

$$g_B[\Theta^*(\theta_{S+1}^*)] = 0 ,$$

$$G_{S+1}[\Theta^*(\theta_{S+1}^*)] \equiv W_{S+1} - \theta_{S+1}^* \sum_i \left(\frac{w_{S+1}}{b}\right)_{in} b_{in} = 0 , \tag{26}$$

and G_{S+1} is now to be included in g_W.

The existence and uniqueness of the positive solution to the Holland equations when unit N has an arbitrary number of side draws is now shown by further induction from a system of N units and $S + N$ product streams to a system of N units and $S + N + 1$ product streams, where the N units are arranged in exactly one recycle loop as before, and thus $S + 1$ product streams leave the system. The argument is the same as that just presented with minor simplifications due to the fact that no additional units are considered.

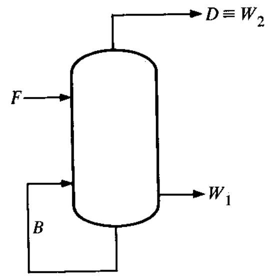

Figure 5 Single unit with feed loop.

Single unit recycle loop

Proof by induction requires the desired result to be established when only one unit is present. Physically this corresponds to a "feedaround" such as is often present in crude distillation units. It will suffice to carry out the proof for a unit with one side draw as depicted in Fig. 5. The base stream constitutes the feedaround and for nota-

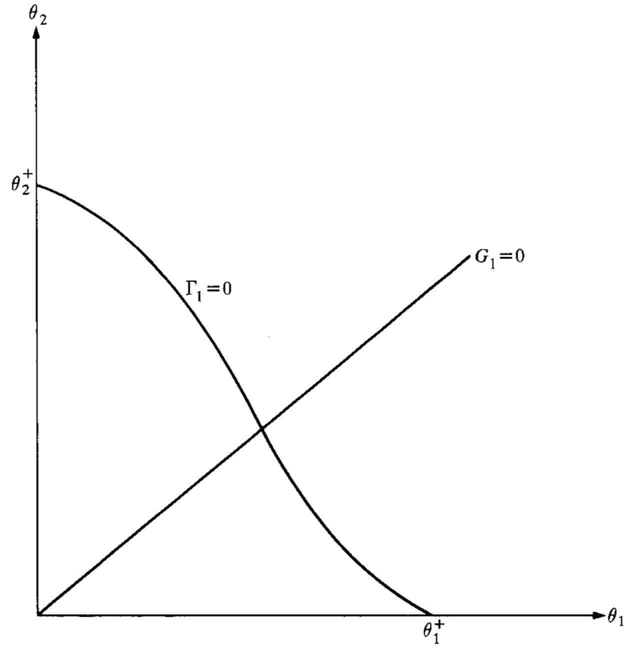

Figure 6 Solution for a single unit.

tional convenience and consistency is arbitrarily taken as the bottom stream. The Holland equations for this system are

$$\Gamma_1 \equiv B - \sum_i \frac{f_i}{(d/b)_i\, \theta_2 + (w_1/b)_i\, \theta_1},$$

$$G_1 \equiv W_1 - \theta_1 \sum_i \frac{(w_1/b)_i f_i}{(d/b)_i\, \theta_2 + (w_1/b)_i\, \theta_1}. \qquad (27)$$

In the second of Eqs. (27) set $\theta_2/\theta_1 = \beta$ and obtain

$$G_1(\beta) = W_1 - \sum_i \frac{(w_i/b)_i f_i}{(d/b)_i\, \beta + (w_1/b)_i}, \qquad (28)$$

showing G_1 to be constant on radial lines from the origin, as depicted in Fig. 6. The origin itself is excluded from consideration since the functions are undefined there. As before, $\theta_1 = \theta_2 = 0$ corresponds to feed entering the unit without any product being withdrawn. Define

$$\beta^+ \equiv \max_i \left[(w_1/b)_i/(d/b)_i\right] (F/W_1)\,.$$

Then Eq. (28) shows $G_1(0) = W_1 - F < 0$, $G_1(\beta^+) > 0$ and G_1 is a continuous function of β. Thus $G(\beta_0) = 0$ has a solution [19]. Since $dG_1/d\beta > 0$ this solution is unique. The required result is then available from the first of Eqs. (27) as

$$\theta_1 = \frac{1}{B} \sum_i \frac{f_i}{(d/b)_i \beta_0 + (w_1/b)_i}$$

Summary

For the special case of exactly one recycle loop with an arbitrary number of columns the existence of a unique positive solution to the Holland equations has been proven. One sees that the only point at which the limitation to exactly one recycle loop was crucial was in obtaining Inequality (9).

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