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Turán Formulae and Highest Precision Quadrature Rules for Chebyshev Coefficients

Abstract: Expansions of functions in series of Chebyshev polynomials are frequently used in numerical analysis. The coefficients occurring in the expansion are definite integrals; the purpose of this paper is to investigate numerical integration formulae for the coefficients of highest degree of precision.

Introduction

To each function $f(x)$, integrable on $(-1, 1)$, there corresponds a Fourier-Chebyshev series,

$$\sum_{n=0}^{\infty}{}' A_n T_n(x), \tag{1}$$

where

$$A_n = A_n(f) = \frac{2}{\pi}\int_{-1}^{1} f(x) T_n(x) \frac{dx}{\sqrt{1-x^2}}, \qquad n = 0, 1, 2, \cdots. \tag{2}$$

The stroke on the summation sign means that the term for $n = 0$ is to be halved, and $T_n(x) = \cos n\theta$, with $x = \cos\theta$, is the Chebyshev polynomial (of the first kind). Chebyshev expansions are widely used in numerical analysis (cf. Fox and Parker [1]) and consequently methods of computing the coefficients, (2), are of interest. Our purpose is to describe a sequence of quadrature rules for evaluating (2), one for each $n = 1, 2, \cdots$, each of which is of highest degree of precision (that is, exact for a polynomial of degree as high as possible). Some of our formulae are related to ones given by Turán [2] and the simple form in which they emerge here enables us to answer a question of Turán [2]. Indeed, in the particular case that interests us we obtain a generalization of Turán's formulae.

The second section of this paper is devoted to the derivation of this generalization. In the third section, application is made of the Turán formulae to the evaluation of Chebyshev coefficients, and some bounds for degree of best polynomial approximation using our methods are obtained. Finally, a section is devoted to the evaluation of Chebyshev coefficients using only function values and not derivatives.

Generalization of some Turán formulae

Let us put

$$\xi_j = \cos[(2j-1)\pi/2n], \qquad j = 1, \cdots, n.$$

The ξ_j are the zeros of $T_n(x)$. We recall the generating function for the Chebyshev polynomial (Cf. Erdélyi [3])

$$G(z, t) = (1-t^2)/(1-2zt+t^2) = 2\sum_{j=0}^{\infty}{}' t^j T_j(z), \tag{3}$$

valid for $|t| < 1$ and $-1 \le z \le 1$. If we put

$$\alpha(t) = (1-t^2)/-2t; \quad \beta(t) = (1+t^2)/2t \tag{4}$$

then we can write

$$G(z, t) = \alpha/(z-\beta).$$

• *Lemma 1*

If (1) converges for $-1 \le x \le 1$ and

$$f(x) = \sum_{j=0}^{\infty}{}' A_j T_j(x), \qquad -1 \le x \le 1, \tag{5}$$

then

$$\frac{1}{n}\sum_{i=1}^{n} f(\xi_i) = \sum_{j=0}^{\infty}{}' (-1)^j A_{2jn}. \tag{6}$$

Proof

Put $Lf = \sum_{i=1}^{n} f(\xi_i)$; then

$$L_z G(z, t) = \alpha \sum_{i=1}^{n} \frac{1}{\xi_i - \beta} = -\alpha\, T_n'(\beta)/T_n(\beta) ,$$

in view of the identity

$$\frac{p'(x)}{p(x)} = \sum_{i=1}^{n} \frac{1}{x - x_i} ,$$

where the x_i are the zeros of an arbitrary $p \in P_n$ (P_n denotes the set of polynomials of degree at most n). The subscript z on the linear functional L indicates that it operates on functions of z. Now

$$r(t) = -\alpha\, T_n'(\beta)/T_n(\beta)$$

is a rational function of t [via (4)] and its value on the unit circle in the complex t-plane, $t = e^{i\theta}$, $0 \le \theta \le 2\pi$ is

$$r(e^{i\theta}) = -\frac{in \sin n\theta}{\cos n\theta} ,$$

hence

$$L_z G(z, t) = r(t) = n(1 - t^{2n})/(1 + t^{2n}) = 2n \sum_{j=0}^{\infty}{}' (-1)^j t^{2jn} , \qquad |t| < 1 . \tag{7}$$

If we apply L_z to both sides of (3) we obtain, in view of (7),

$$L_z T_k(z) = \begin{cases} (-1)^j n , k = 2jn , & j = 0, 1, 2, \cdots , \\ 0 , k \ne 2jn , & j = 0, 1, 2, \cdots . \end{cases}$$

The lemma now follows upon applying L_z to both sides in (5).

Remark 1
This result is also an easy consequence of the orthogonality property of the Chebyshev polynomials on the set $\{\xi_1, \cdots, \xi_n\}$.

Remark 2
If $f \in P_{2n-1}$ then (6) becomes

$$\frac{1}{n} \sum_{i=1}^{n} f(\xi_i) = A_0/2 = 1/\pi \int_{-1}^{1} f(x) \frac{dx}{\sqrt{1 - x^2}} ,$$

Gaussian quadrature with respect to the weight function $(1 - x^2)^{-\frac{1}{2}}$.

We next recall the notion of the *divided differences* of a function. If g has a continuous kth derivative on $[-1, 1]$ and $x_1, \cdots, x_s$ are distinct points of that interval then

$$g(\overbrace{x_1, x_1, \cdots, x_1}^{m_1}, \overbrace{x_2, \cdots, x_2}^{m_2}, \cdots, \overbrace{x_s, \cdots, x_s}^{m_s}) ,$$

where $m_j \le k + 1$, $j = 1, \cdots, s$, the divided difference of g with respect to the x_i with multiplicity m_i, $i = 1, \cdots, s$, is the leading coefficient of the unique $p \in P_l$,

$$l = \sum_{i=1}^{s} m_i - 1 ,$$

which satisfies

$$p^{(j)}(x_i) = g^{(j)}(x_i) , \quad i = 1, \cdots, s ; \quad j = 0, \cdots, m_i - 1 .$$

• *Lemma 2*
If $f \in C^r([-1, 1])$ and

$$f(x) = \sum_{j=0}^{\infty}{}' A_j T_j(x) , \qquad -1 \le x \le 1 ,$$

then

$$L_r f = f'(\xi_1^r, \xi_2^r, \cdots, \xi_n^r) = \frac{n 2^{rn-1}}{(r-1)!} \sum_{j=0}^{\infty}{}' (-1)^j (j+1) \cdots (j+r-1) \times (2j + r) A_{(2j+r)n} ,$$

(where ξ_i^r is shorthand for $\overbrace{\xi_i, \cdots, \xi_i}^{r}$).

Proof
First we observe that

$$g(\xi_1, \cdots, \xi_n) = 2^{n-1} \sum_{i=1}^{n} \frac{g(\xi_i)}{T_n'(\xi_i)} ,$$

and, therefore, if we put $h(z) = (z - \beta)^{-1}$

$$h(\xi_1, \cdots, \xi_n) = 2^{n-1} \sum_{i=1}^{n} \frac{1}{\xi_i - \beta} \frac{1}{T_n'(\xi_i)} = -\frac{2^{n-1}}{T_n(\beta)} .$$

Hence

$$h'(\xi_1, \cdots, \xi_n) = 2^{n-1} \left[\frac{1}{T_n(\beta)}\right]' ,$$

and by continuity we obtain

$$h'(\xi_1^r, \cdots, \xi_n^r) = 2^{(n-1)r} \left[\frac{1}{T_n^r(\beta)}\right]' .$$

Thus

$$L_{r,z} G(z, t) = \alpha 2^{r(n-1)} \left[\frac{1}{T_n^r(\beta)}\right]' = -\alpha\, 2^{r(n-1)}\, r \frac{T_n'(\beta)}{T_n^{r+1}(\beta)} ,$$

a rational function of t, whose form we determine by its values when $t = e^{i\theta}$. If $t = e^{i\theta}$

$$L_{r,z} G(z, t) = -2^{r(n-1)} \frac{inr \sin n\theta}{(\cos n\theta)^{r+1}} ,$$

so that

$$L_{r,z}G(z,t) = nr2^{nr}\, t^{nr}\,(1-t^{2n})/(1+t^{2n})^{r+1}$$
$$= \frac{n2^{nr}}{(r-1)!}\sum_{j=0}^{\infty}(-1)^j(j+1)\cdots(j+r-1)$$
$$\times\,(2j+r)t^{(2j+r)n}. \qquad (8)$$

The lemma now follows as in the conclusion of the proof for the case in Lemma 1.

Let Γ_n denote the lemniscate in the complex z-plane defined by $|T_n(z)| = 1$. For each positive $\rho < 1$, let C_ρ be the ellipse in the z-plane with foci $(\pm 1, 0)$ defined by

$$z = \beta(t) = \frac{t+1/t}{2}, \qquad |t| = \rho. \qquad (9)$$

The mapping in (9) gives a 1-to-1 conformal mapping of the unit disc, $|t| < 1$, onto the extended z-plane with the segment $[-1, 1]$ deleted. Put

$$\tau_n = \{t/|t| < 1,\quad |T_n[\beta(t)]| > 1\}.$$

Observe that the open segment $-1 < t < 1$ is contained in τ_n.

• *Lemma 3*

The equation

$$\int_{-1}^{1} f(x)\,\frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{n}\left[\sum_{i=1}^{n} f(\xi_i) + \sum_{j=1}^{\infty}\alpha_j f'(\xi_1^{2j},\cdots,\xi_n^{2j})\right] \qquad (10)$$

is satisfied by $f(x) = G(x, t)$ for $t \in \tau_n$, where

$$\alpha_j = (-1)^j\,\frac{\binom{-\frac{1}{2}}{j}}{2j4^{(n-1)j}}, \qquad j = 1, 2, \cdots. \qquad (11)$$

Proof

In view of (3),

$$\int_{-1}^{1} G(x,t)\,\frac{dx}{\sqrt{1-x^2}} = \pi.$$

But the right-hand side of (10) for $f(x) = G(x, t)$ is, according to (7) and (8), equal to

$$\frac{\pi}{n}\left[n\frac{1-t^{2n}}{1+t^{2n}} + n\frac{1-t^{2n}}{1+t^{2n}}\sum_{j=1}^{\infty}(-1)^j\binom{-\frac{1}{2}}{j}\left(\frac{t^n+t^{-n}}{2}\right)^{-2j}\right]$$
$$= \pi\frac{1-t^{2n}}{1+t^{2n}}\sum_{j=0}^{\infty}(-1)^j\binom{-\frac{1}{2}}{j}\left[\frac{1}{T_n(\beta)}\right]^{2j}$$
$$= \pi\frac{1-t^{2n}}{1+t^{2n}}\left(1-\frac{1}{T_n^2(\beta)}\right)^{-\frac{1}{2}} = \pi, \qquad (12)$$

where we have used the identity

$$T_n(\beta) = (t^n + t^{-n})/2,$$

and the fact that the infinite series in (12) is a convergent binomial expansion since $t\in\tau_n$.

Let R be a region in the z-plane containing Γ_n strictly in its interior, and let $A(R)$ denote the set of functions analytic on R.

• *Theorem 1*

If $f\in A(R)$ then f satisfies (10) with the coefficients, α_j, given by (11).

Proof

There exists $\delta > 0$ such that $\Gamma : |T_n(z)| = 1 + \delta$, (a simple closed curve), is contained in R and Γ_n is strictly inside Γ. Then if $-1 \le x \le 1$ we have, by the Cauchy integral formula,

$$f(x) = \frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{\zeta - x}\,d\zeta.$$

Let us put

$$If = \int_{-1}^{1} f(x)\frac{dx}{\sqrt{1-x^2}},$$

and let Q denote any of the bounded linear functionals. $I, L, L_r, r = 1, 2, \cdots$. Then

$$Qf = \frac{1}{2\pi i}\int_{\Gamma} f(\zeta)\; Q_x\left(\frac{1}{\zeta - x}\right)d\zeta,$$

and, in particular

$$\left|If - \frac{\pi}{n}\left[\sum_{i=1}^{n} f(\xi_i) + \sum_{j=1}^{m}\alpha_j f'(\xi_1^{2j},\cdots,\xi_n^{2j})\right]\right|$$
$$= \left|If - \frac{\pi}{n}\left[Lf + \sum_{j=1}^{m}\alpha_j L_{2j}f\right]\right|$$
$$= \left|\frac{1}{2\pi i}\int_{\Gamma} f(\zeta)\left\{I_x\frac{1}{\zeta-x} - \frac{\pi}{n}\left[L_x\frac{1}{\zeta-x} + \sum_{j=1}^{m}\alpha_j L_{2j,x}\frac{1}{\zeta-x}\right]\right\}d\zeta\right|$$
$$\le \frac{1}{2\pi}\int_{\Gamma}|f(\zeta)|\;|\alpha(t)|^{-1} \times\left[\left|\frac{\pi}{n}\right|\left|\sum_{j=m+1}^{\infty}\alpha_j L_{2j,x}G(x,t)\right|\right]d\zeta, \qquad (13)$$

where t is in the pre-image of Γ under the mapping β^{-1}, and we have used Lemma 3. As is evident from (12), given $\epsilon > 0$, there exists m_0 such that for $m \ge m_0$,

$$\left|If - \frac{\pi}{n}\left[\sum_{i=1}^{n} f(\xi_i) + \sum_{j=1}^{m}\alpha_j f'(\xi_1^{2j},\cdots,\xi_n^{2j})\right]\right| \le CM\epsilon,$$

where M is a bound on f on Γ, and C is a constant bounding $|\alpha(t)|^{-1}$ and other functions of t appearing in the final sum in (13). This concludes the proof.

Remark 1
The somewhat unusual condition that $f \in A(R)$ can be replaced by a more familiar but less precise requirement in the hypothesis of the Theorem 1. Let E_ρ be the inside of the ellipse C_ρ defined above. Then if we require that f be analytic in E_ρ for some ρ satisfying

$$\rho < (\sqrt{2} - 1)^{1/n}, \tag{14}$$

the conclusion of Theorem 1 holds. To see this we observe that for $z \in C_\rho$, $|T_n(z)| \geq \frac{1}{2}(\rho^{-n} - \rho^n)$ and hence if (14) holds, $|T_n(z)| > 1$, so that E_ρ can serve as a region R.

Remark 2
This theorem is our generalization of Turán's formula, namely, if $f \in P_{2kn-1}$ then the hypotheses of the theorem are satisfied and we obtain

$$\int_{-1}^{1} f(x) \frac{dx}{\sqrt{1-x^2}} = (\pi/n)\left[\sum_{i=1}^{n} f(\xi_i) + \sum_{j=1}^{k-1} \alpha_j f'(\xi_1^{2j}, \cdots, \xi_n^{2j})\right]. \tag{15}$$

Turán [2] showed that for any weight function, $w(x)$, there is a unique quadrature formula

$$\int_{-1}^{1} f(x) w(x) dx = \sum_{j=1}^{n} [\lambda_j^{(0)} f(x_j) + \lambda_j^{(1)} f'(x_j) + \cdots + \lambda_j^{[2(k-1)]} f^{[2(k-1)]}(x_j)] \tag{16}$$

valid for $f \in P_{2kn-1}$, $k = 1, 2, \cdots$. This formula is obtained by choosing the nodes $x_1, \cdots, x_n$ to be the zeros of the unique polynomial which minimizes

$$\int_{-1}^{1} [p(x)]^{2k} w(x) dx$$

among all $p \in P_n$ with leading coefficient 1, and integrating the Hermite interpolating polynomial of degree at most $(2k-1)n - 1$, which agrees with f and its first $2(k-1)$ derivatives at $x_1, \cdots, x_n$. Thus in the case $w(x) = (1-x^2)^{-1/2}$, (16) and (15) must be identical. As a further remark, then, we see that

$$\int_{-1}^{1} [x^n + a_{n-1}x^{n-1} + \cdots + a_0]^p \frac{dx}{\sqrt{1-x^2}}$$

is minimized for $p = 2k$, $k = 1, 2, \cdots$, by the Chebyshev polynomial of degree n normalized so that its leading coefficient is 1. It is known that this is the case for all real $p \geq 1$.

Turán [2] asks whether, even for $k = 2$, the weights $\lambda_j^{(0)}, \lambda_j^{(1)}, \lambda_j^{(2)}$ are positive for $w(x) \equiv 1$. He shows that $\lambda_j^{(2)}$ are positive but states that the status of the $\lambda_j^{(0)}$ and $\lambda_j^{(1)}$ is an open question. When $w(x) = (1-x^2)^{-\frac{1}{2}}$ our format, (15), of the Turán formula enables us to determine the weights easily for $k = 2$. When $k = 2$, (15) reads

$$\int_{-1}^{1} f(x) \frac{dx}{\sqrt{1-x^2}} = \pi/n \left[\sum_{i=1}^{n} f(\xi_i) + \frac{1}{4^n} f'(\xi_1^2, \cdots, \xi_n^2)\right]. \tag{17}$$

In order to evaluate the divided difference $f'(\xi_1^2, \cdots, \xi_n^2)$, we note that

$$p(x) = \sum_{j=1}^{n} \left[h(\xi_j) \frac{1-\xi_j x}{n^2} \left(\frac{T_n(x)}{x-\xi_j}\right)^2 + h'(\xi_j) \frac{1-\xi_j^2}{n^2(x-\xi_j)} T_n^2(x)\right]$$

satisfies $p \in P_{2n-1}$, $p(\xi_j) = h(\xi_j)$, $p'(\xi_j) = h'(\xi_j)$, $j = 1, \cdots, n$. Therefore $f'(\xi_1^2, \cdots, \xi_n^2)$ is the leading coefficient of p, which yields

$$f'(\xi_1^2, \cdots, \xi_n^2) = (4^{n-1}/n^2) \sum_{j=1}^{n} [(-\xi_j) f'(\xi_j) + (1-\xi_j^2) f''(\xi_j)]. \tag{18}$$

Thus, in this case, the Turán weights $\lambda_j^{(0)}$ and $\lambda_j^{(2)}$ are are all positive but the $\lambda_j^{(1)}$ are not.

We end this section by briefly describing another expansion, similar to (10), but based on the points $n_j = \cos j\pi/n$, $j = 0, 1, \cdots, n$. It is well known that

$$\int_{-1}^{1} \frac{f(x) dx}{\sqrt{1-x^2}} = \frac{\pi}{n} \sum_{j=0}^{n}{}'' f(\eta_j) \tag{19}$$

for $f \in P_{2n-1}$. The double stroke on the summation sign indicates that the first and last terms are to be halved. Equation (19) is the highest-precision quadrature formula of the Lobatto type corresponding to the weight $w(x) = 1/\sqrt{1-x^2}$ (cf. Krylov [4]). The method used to prove Theorem 1 also can be used to prove the following theorem.

• *Theorem 2*
Let f be analytic inside $\{z: |T_n^2(z) - 1| = 1 + \epsilon\}$ for some $\epsilon > 0$. Then

$$\int_{-1}^{1} \frac{f(x) dx}{\sqrt{1-x^2}} = \frac{\pi}{n} \sum_{j=0}^{n}{}'' f(\eta_j) + \sum_{j=1}^{\infty} (-1)^j \alpha_j f'(\eta_0^j, \eta_1^{2j}, \cdots, \eta_{n-1}^{2j}, \eta_n^j)\Big] \tag{20}$$

where α_j are given by (11).

Remark 3
Formula (20) also has the property that any partial sum of the series is a quadrature formula of highest degree of precision.

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}dx = \frac{\pi}{n}\Big[\sum_{j=0}^{n}{}'' f(\eta_j)$$
$$+ \sum_{j=1}^{k-1} \alpha_j (-1)^j f'(\eta_0^j, \eta_1^{2j}, \cdots, \eta_{n-1}^{2j}, \eta_n^j)\Big]$$

is exact for $f \in P_{2kn-1}$, $k = 1, 2, \cdots$. This is a quadrature formula of the Turán type where the nodes $+1$ and -1 are prescribed, what might be called a "Lobattomized" Turán formula.

Quadrature rules for Chebyshev coefficients involving derivatives and other applications of the Turán formula

Let us next apply the formula (17) to the evaluation of A_n. We obtain, in view of (17), for $f \in P_{3n-1}$,

$$A_n(f) = \frac{2}{n}\Big[\sum_{j=1}^{n} f(\xi_j)T_n(\xi_j)$$
$$+ \frac{1}{4n^2}\Big(\sum_{j=1}^{n} \{(-\xi_j)[f(\xi_j)T_n'(\xi_j) + T_n(\xi_j)f'(\xi_j)]$$
$$+ (1-\xi_j^2)[f(\xi_j)T_n''(\xi_j) + 2f'(\xi_j)T_n'(\xi_j)$$
$$+ T_n(\xi_j)f''(\xi_j)]\}\Big)\Big],$$

which, since $T_n(\xi_j) = 0$, $(1-x^2)T_n''(x) - xT_n'(x) = -n^2T_n(x)$ and $T_n'(\xi_j) = (-1)^{j-1}n(1-\xi_j^2)^{-\frac{1}{2}}$, simplifies to

$$A_n(f) = \frac{1}{n^3}\sum_{j=1}^{n} (1-\xi_j^2)f'(\xi_j)T_n'(\xi_j)$$
$$= \frac{1}{n}\sum_{j=1}^{n} \frac{f'(\xi_j)}{T_n'(\xi_j)}. \tag{21}$$

• *Theorem 3*

The quadrature rule

$$A_n(f) = \frac{1}{n}\sum_{j=1}^{n} \frac{f'(\xi_j)}{T_n'(\xi_j)} = \frac{2^{1-n}}{n} f'(\xi_1, \cdots, \xi_n) \tag{22}$$

is exact for $f \in P_{3n-1}$, $n = 1, 2, \cdots$. Moreover, there is no quadrature formula

$$A_n(f) = \sum_{i=1}^{n} [a_i f(x_i) + b_i f'(x_i)] \tag{23}$$

exact for $f \in P_{3n}$, for any choice of a_i, b_i and x_i.

Proof

Equation (22) follows immediately from (21). If

$$\omega(x) = (x - x_i) \cdots (x - x_n), \tag{24}$$

then for $f = \omega^2 T_n \in P_{3n}$ the right-hand side of (23) is zero but the left-hand side is positive.

Remark 1

Equation (22) is the only formula of the type

$$A_n(f) = \sum_{j=1}^{n} b_j f'(x_j) \tag{25}$$

exact for $f \in P_{3n-1}$. For if (25) exists, consider

$$p_i(x) = \int_0^x \frac{\omega(t)}{(t-x_i)} T_n(t)dt, \qquad i = 1, \cdots, n,$$

where ω is defined by (24).

Since $p_i(x) \in P_{2n}$, and when (22) and (25) are applied with $f = p_i$, we have

$$0 = b_i \omega'(x_i) T_n(x_i).$$

Because b_i and $\omega'(x_i)$ are not zero, x_i must be a zero of T_n. This argument can be repeated for each $i = 1, \cdots, n$.

We do not know (when $n \geq 4$) if (22) is the only rule of the form (23) that is exact for $f \in P_{3n-1}$ with the x_i satisfying $-1 \leq x_i \leq 1$, but it is not unique if nodes outside of $[-1, 1]$ are permitted.

Remark 2

Note that (22) is a *sequence* of formulae, one for each $n = 1, 2, \cdots$. The evaluation of $A_0(f)$ can be accomplished by (17).

In order to present another application of our methods, we recall some notation and results about best uniform approximation by polynomials. If $f \in C([-1, 1])$ and p^* is its best uniform approximation out of P_n then

$$E_n(f) = \|f - p^*\| = \max_{-1 \leq x \leq 1} |f(x) - p^*(x)|.$$

If X denotes the set of distinct real numbers $\{x_1, \cdots, x_k\}$ and $p_X{}^*$ is the best approximation of f on X out of P_{k-2} then

$$\max_{i=1,\cdots,k} |f(x_i) - p_X^*(x_i)| = \|f - p_X^*\|_X = E_{k-2}^X(f).$$

It is well known (cf. Rivlin [5]) that

$$E_{k-2}^X(f) = |f(x_1, \cdots, x_k)/g(x_1, \cdots, x_k)|,$$

where g is any function satisfying $g(x_i) = (-1)^i$, $i = 1, \cdots, k$. Also, if $X \subset [-1, 1]$,

$$E_{k-2}^X(f) \leq E_{k-2}(f).$$

Choose $X = T = \{\xi_1, \cdots, \xi_k\}$, the zeros of $T_k(x)$; then

$$g(\xi_1, \cdots, \xi_k) = 2^{k-1}\sum_{i=1}^{k} \frac{(-1)^i}{T_k'(\xi_i)} = -2^{k-1}\sum_{i=1}^{k} \frac{1}{|T_k'(\xi_i)|}$$
$$= -\frac{2^{k-1}}{k}\sum_{i=1}^{k} \sin\frac{(2i-1)\pi}{2k} = -\frac{2^{k-1}}{k \sin\frac{\pi}{2k}}.$$

Thus for $k \geq 2$,

$$E_{k-2}^T(f) = \frac{k \sin(\pi/2k)}{2^{k-1}} |f(\xi_1, \cdots, \xi_k)|.$$

A now familiar procedure yields

$$G(\xi_1, \cdots, \xi_k; t) = 2^{k-1} t^{k-1} \frac{1-t^2}{1+t^{2k}}$$

$$= 2^{k-1} \sum_{j=0}^{\infty} (-1)^j (t^{(2j+1)k-1} - t^{(2j+1)k+1}),$$

and hence if

$$f(x) = \sum_{j=0}^{\infty}{}' A_j T_j(x), \qquad -1 \leq x \leq 1$$

we have for $k \geq 2$,

$$E_{k-2}^T(f) = \frac{k \sin(\pi/2k)}{2} |(A_{k-1} - A_{k+1})$$

$$- (A_{3k-1} - A_{3k+1}) + \cdots|. \tag{27}$$

A simple consequence of (27) is therefore the weaker result

$$E_n(f) \geq \tfrac{1}{2} |(A_{n+1} - A_{n+3}) - (A_{3n+5} - A_{3n+7}) + \cdots|. \tag{28}$$

If

$$\eta_j = \cos j\pi/n, \qquad j = 0, \cdots, n,$$

so that the η_j are the extrema of $T_n(x)$ on $[-1, 1]$, i.e.,

$$T_n(\eta_j) = (-1)^j, \qquad j = 0, \cdots, n,$$

we put $U = \{\eta_0, \cdots, \eta_n\}$. Since

$$f(\eta_0, \cdots, \eta_n) = \frac{2^{n-1}}{n} \sum_{j=0}^{n}{}'' (-1)^j f(\eta_j), \tag{29}$$

where the two strokes on the summation sign mean that the first and last summands are to be halved, we see that $E_{n-1}^U(f) = |f(\eta_0, \cdots, \eta_n)|/2^{n-1}$. But

$$G(\eta_0, \cdots, \eta_n; t) = 2^n \sum_{j=1}^{\infty} t^{(2j-1)n},$$

and hence if

$$f(x) = \sum_{j=0}^{\infty}{}' A_j T_j(x), \qquad -1 \leq x \leq 1,$$

we obtain

$$f(\eta_0, \cdots, \eta_n) = 2^{n-1} \sum_{j=1}^{\infty} A_{(2j-1)n}, \tag{30}$$

and recover the well-known result of de La Vallée Poussin,

$$E_n^U(f) = \left| \sum_{j=1}^{\infty} A_{(2j-1)(n+1)} \right|. \tag{31}$$

Highest precision quadrature formulae for Chebyshev coefficients using only function evaluations

• *Theorem 4*

The quadrature rule

$$A_n(f) = \frac{1}{n} \sum_{j=0}^{n}{}'' (-1)^j f(\eta_j) = 2^{1-n} f(\eta_0, \cdots, \eta_n) \tag{32}$$

is exact for $f \in P_{3n-1}$, $n = 1, 2, \cdots$. Moreover, when $n > 1$ there is no quadrature formula

$$A_n(f) = \sum_{j=0}^{n} c_j f(x_j) \tag{33}$$

that is exact for $f \in P_{3n}$ for any choice of c_j and distinct x_j, and (32) is the only rule of the form (33) exact for $f \in P_{3n-1}$.

Proof

Equation (32) is an immediate consequence of (30) and (29). Suppose (33) is exact for $f \in P_{3n-1}$. Put $\Omega(x) = (x - \eta_0) \cdots (x - \eta_n)$, $\omega(x) = (x - x_0)(x - x_1) \cdots (x - x_n)$, and consider $f(x) = \Omega(x)\omega(x)(x - x_i)^{-1} \in P_{2n+1}$. According to (32) and (33), we have

$$0 = A_n(f) = c_i f(x_i) = c_i \Omega(x_i) \omega'(x_i).$$

Since $c_i \omega'(x_i) \neq 0$ we must have $\Omega(x_i) = 0$. The same argument for each i leads to $\omega = \Omega$. The weights in (32) are, clearly, uniquely determined. Moreover, (30) is not exact for $f \in P_{3n}$ as $f(x) = (1 - x^2)[(x - \eta_1) \cdots (x - \eta_{n-1})]^2 T_n(x) \in P_{3n}$ shows.

Remark

When $n = 1$ the highest degree of precision is 4, and the unique quadrature formula, exact for $f \in P_4$, is

$$\frac{2}{\pi} \int_{-1}^{1} f(x) x \frac{dx}{\sqrt{1-x^2}} = \frac{2}{3} \sqrt{3} f\left(\frac{\sqrt{3}}{2}\right) - \frac{2}{3} \sqrt{3} f\left(-\frac{\sqrt{3}}{2}\right).$$

Let us consider quadrature formulae of highest precision

$$A_n(f) = \sum_{j=0}^{k} c_j f(x_j). \tag{34}$$

We have just seen that for $k = n > 1$, (32) is the unique formula of highest precision. Let $h(k)$ be the largest integer such that there exists a formula (34) exact for $f \in P_h$. Then for $n > 1$ we obtained $h(n) = 3n - 1$. We wish to conclude by examining the function $h(k)$ as k varies.

• *Theorem 5*

1) $h(k) = k - 1$, $k = 1, \cdots, n - 1$.

2) a) $h[(2m - 1)n] = (4m - 1)n - 1$, $\qquad n > 1$,

b) $h(2mn - 1) = (4m + 1)n + 1$,

for $m = 1, 2, \cdots$.

3) a) If $2mn - 1 < k < (2m+1)n$,
$m = 1, 2, \cdots$
$h(k) < 2mn + k$.
b) If $(2m-1)n < k < 2mn - 1$,
$m = 1, 2, \cdots$,
$h(k) < (2m-1)n + k + 1$.

Proof
1) If $h(k) \geq k$ then

$$0 = A_n(f) = \sum_{i=0}^{k} c_i f(x_i)$$

for any $f \in P_k$. In particular, choose $f_j \in P_k$ such that

$$f_j(x_i) = \delta_{ij},$$

then $c_j = 0$ and there is no quadrature formula of the indicated kind. However, given any set of distinct nodes $x_0, \cdots, x_k$, the k linear equations in $k+1$ unknowns

$$\sum_{i=0}^{k} c_i x_i^j = 0, \qquad j = 0, \cdots, k-1$$

have a nontrivial solution, and, indeed, no $c_i = 0$. For if, say, $c_q = 0$, the homogeneous system

$$\sum_{\substack{i=0\\ i\neq q}}^{k} c_i x_i^j = 0, \qquad j = 0, \cdots, k-1$$

has only the trivial (zero) solution since its matrix is of Vandermonde type. Thus $h(k) = k - 1$.

2a) Let $\eta_j^{(2mn)} = \cos(j\pi/2mn)$, $j = 0, \cdots, 2mn$. There is a quadrature formula of Lobatto type

$$\int_{-1}^{1} g(x) \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2mn} \sum_{j=0}^{2mn}{}'' g(\eta_j^{(2mn)}) \tag{35}$$

which is exact for $g \in P_{4mn-1}$. [Cf. (19).] Since each zero of $T_n(x)$,

$$\xi_i = \cos(2i-1)\frac{\pi}{2n}, \qquad i = 1, \cdots, n$$

is found among the $\eta_j^{(2mn)}$, $j = 0, \cdots, 2mn$, we obtain from (35)

$$A_n(f) = \sum_{j=0}^{(2m-1)n} b_j f(\lambda_j), \tag{36}$$

[where the λ_j are the η_j which are not zeros of $T_n(x)$] exact for $f \in P_{(4m-1)n-1}$. If there is a formula

$$A_n(f) = \sum_{j=0}^{(2m-1)n} c_j f(x_j)$$

exact for $f \in P_{(4m-1)n}$, then we put $\Omega(x) = (x-\lambda_0)\cdots(x-\lambda_{(2m-1)n})$ and $\omega(x) = (x-x_0)\cdots(x-x_{(2m-1)n})$. Thus

$$f(x) = \Omega(x)\frac{\omega(x)}{x-x_i} \in P_{(4m-1)n-1}, \qquad \text{since } n > 1,$$

Hence $A_n(f) = 0$ by (36) and therefore

$$c_i\omega'(x_i)\Omega(x_i) = 0,$$

which implies that $\Omega = \omega$. But (36) cannot be exact for $f \in P_{(4m-1)n}$ as the choice

$$f(x) = (1-x^2)[(x-\lambda_1)\cdots(x-\lambda_{(2m-1)n-1})]^2 T_n(x) \in P_{(4m-1)n}$$

demonstrates.

2b) $h(2mn-1) < (4m+1)n$ since $\omega^2 T_n \in P_{(4m+1)n}$. By Gaussian quadrature (cf. Remark 2 following Lemma 1),

$$\int_{-1}^{1} g(x)\frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{(2m+1)n}\sum_{j=1}^{(2m+1)n} g(\xi_j^{[(2m+1)n]}) \tag{37}$$

is exact for $g \in P_{2(2m+1)n-1}$, where $\xi_j^{[(2m+1)n]}$ are the zeros of $T_{(2m+1)n}(x)$. Hence

$$A_n(f) = \sum_{j=0}^{2mn-1} d_j f(\mu_j), \tag{38}$$

where the μ_j are the zeros of $T_{(2m+1)n}$ which are not also zeros of T_n, is exact for $f \in P_{(4m+1)n-1}$.

3a) Let

$$A_n(f) = \sum_{j=0}^{k} c_j f(x_j)$$

be exact for $f \in P_{h(k)}$. Let $\Omega(x) = (x-\mu_0)\cdots(x-\mu_{2mn-1}) \in P_{2mn}$ and $\omega(x) = (x-x_0)\cdots(x-x_k) \in P_{k+1}$, then

$$f(x) = \Omega(x)\frac{\omega(x)}{x-x_i} \in P_{2mn+k},$$

and in view of (38) we conclude, in a fashion that is by now familiar, that $h(k) < 2mn + k$.

3b) The argument resembles **3a)** and we omit it. •

Remark
This theorem gives the surprising information that the addition of nodes to a quadrature formula may result in *reducing* the highest degree of precision.

Conclusion
Let us now place our results in a broader context. The existing literature seems to be exclusively concerned with approximations of the Chebyshev coefficients as a means of approximating a partial sum of a Chebyshev expansion. With that aim a single formula applicable to all of the coefficients of interest is desirable. We, however, provide a different formula for each coefficient since our object is to study the highest degree of precision formulae for each coefficient. A good survey of the state of the art of Chebyshev coefficient evaluation is given in Cooper [6]. The formula (32) is mentioned there, but no attention is given to the highest degree of precision problem.

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