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Hopscotch Difference Methods for Nonlinear Hyperbolic Systems

Abstract: In a recent series of papers, one of the authors has developed and demonstrated properties of a computational algorithm for solving partial differential equations. This process, known as the hopscotch algorithm, has been studied particularly with reference to the efficient integration of parabolic and elliptic problems. In the present paper attention is directed to the application of the technique to the numerical integration of first-order nonlinear hyperbolic systems. While maintaining the properties of the hopscotch process as applied to parabolic problems, it is shown that one of the novel schemes generated by this approach has an added bonus, namely, maximum stability for a variable choice of damping or pseudoviscous term. This property should be of particular value in the solution of problems with shocks. A class of hopscotch Lax-Wendroff schemes is also studied.

Introduction

Over the past few years, several finite difference methods have been devised for the solution of nonlinear hyperbolic systems of the form $(\partial u/\partial t) + (\partial f/\partial x) = 0$, where u is an N-vector function of the unknowns $(u_1, u_2, \cdots, u_N)$ and f is a nonlinear function of the components of u. In the main, such methods have been tested exhaustively on smooth solutions although increasing attention is being paid to the schemes' capabilities of representing shock-like phenomena that arise in the solutions of nonlinear hyperbolic systems (see Burstein [1], Burstein and Rubin [2], Emery [3], van Leer [4]).

The explicit methods that have been considered fall into two distinct categories, namely, first-order methods with added viscosity of the type first introduced by Von Neumann and Richtmyer [5] and second-order methods, which have no additional viscosity terms and are of a higher order of accuracy than the former methods. Van Leer has recently stated [4] that first-order methods with an optimal pseudoviscosity term seem to perform better on problems with discontinuities than do the higher-order accurate methods. There appears to be a great deal of truth in this statement. The best second-order method at present appears to be that due to Burstein and Rubin as described in Ref. 2, where several numerical experiments are reported on the use of their method on problems with shock-like phenomena.

We mention also in connection with shocks the recent work of Rusanov [6], who considers third-order accurate explicit methods for the solution of nonlinear hyperbolic systems and recognizes the better dissipative properties of third-order accurate schemes compared to second-order methods. Following on from this work, Burstein and Mirrin [7] have considered other third-order methods of a form similar to that of Rusanov and have considered the application of these methods to shock problems. However, these third-order methods are time-consuming in comparison with the first- and second-order methods and we feel that, as yet, it is not obvious that such methods have anything to offer over and above the first-order methods with *optimal* pseudoviscosity.

In the present paper, we will consider both first-order and second-order methods. These schemes are similar in nature to the hopscotch scheme introduced by Gourlay [8] for the solution of parabolic differential equations.

First-order method

We consider the solution of the nonlinear system

$$(\partial u/\partial t) + (\partial f/\partial x) = 0 \tag{1}$$

subject to the initial conditions

$$u(x, 0) = g_1(x), \qquad \alpha \leq x \leq \beta \tag{2}$$

and some suitable boundary conditions

$$B(u,a,b,t) = g_2(t). \tag{3}$$

(The precise form of the boundary conditions will not concern us because we will not consider their influence. The development of the schemes and the stability analysis will be carried through as if we were studying the initial value problem.)

To solve equations (1) subject to conditions (2) and (3) we superimpose, in the usual way, a rectilinear grid with mesh spacing $\Delta_x = h$ in the x-direction and $\Delta_t = k$ in the time direction. We assume the mesh ratio $p = k/h$ is constant. We represent a grid point by (i, n), where $(ih,nk) = (x,t)$ and represent the value of the unknown function u at the point (i,n) by $u_i^n = u(i,n) = u(ih,nk)$.

The difference operators are defined in the usual way as

$\delta_x u_i^n = u_{i+\frac{1}{2}}^n - u_{i-\frac{1}{2}}^n$, the central difference operator,

$\mu_x u_i^n = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n)$, the average difference operator,

and

$$H_x u_i^n = u_{i+1}^n - u_{i-1}^n .$$

Using this notation we can then define the familiar first-order method due to Lax [9] as

$$u_i^{n+1} = u_i^n - (p/2)H_x f_i^n + \sigma\delta_x^2 u_i^n , \qquad (4)$$

where σ is a coefficient of the pseudoviscous term $\delta_x^2 u_i^n$ and is chosen to obtain the best possible shock resolution. This method is subject to the well-known local stability requirement, obtained by applying the von Neumann analysis to the linearized scheme:

$$p|\lambda| \leq \sqrt{2\sigma} \qquad \text{and } 0 \leq \sigma \leq \tfrac{1}{2},$$

where $|\lambda|$ is the maximum modulus eigenvalue of the Jacobian matrix A of partial derivatives of f with respect to u.

For good shock resolution it is found that a fairly small value of σ is needed, requiring a small value of p to be chosen. This choice leads to an increase in the number of steps required to carry through the computation. Since we are dealing with explicit schemes, the theoretical upper bound on $p|\lambda|$, given by the Courant-Friedrichs-Lewy condition, is $p|\lambda| \leq 1$.

Ideally we would like to be able to compute with a difference scheme with a value of $p|\lambda|$ close to the upper limit and yet have considerable freedom in the amount of pseudoviscosity (i.e., the value of a parameter σ) employed.

In this section we show that a hopscotch version of the above Lax scheme enjoys precisely these characteristics. It is stable up to the C-F-L limit *for any positive value of* σ and is still explicit. This freedom permits an arbitrary variation in the amount of viscosity introduced during the computation.

Consider the following difference scheme:

$$u_i^{n+1} + \theta_i^{n+1}[(p/2)H_x f_i^{n+1} - \sigma\delta_x^2 u_i^{n+1}] = u_i^n - \theta_i^n[(p/2)H_x f_i^n - \sigma\delta_x^2 u_i^n] , \qquad (5)$$

where the function

$$\theta_i^n = \begin{cases} 1 & \text{if } n+i \text{ is odd} \\ 0 & \text{if } n+i \text{ is even} . \end{cases}$$

For $n + i$ odd it is clear that Eq. (5) is equivalent to Eq. (4). However for $n + i$ even, Eq. (5) becomes the implicit scheme

$$u_i^{n+1} + [(p/2)H_x f_i^{n+1} - \sigma\delta_x^2 u_i^{n+1}] = u_i^n$$

which, at first sight, appears to require the solution of a nonlinear tridiagonal system.

For a fixed value of n, the hopscotch implementation proceeds as follows. Apply scheme (5) for those values of i with $(n + i)$ odd, thereby calculating every alternate point explicitly. If we now apply scheme (5) for those values of i with $(n + i)$ even and make use of the points we have just calculated explicitly, the over-all method becomes computationally explicit.

Application of method (5) over a time interval $2k$, that is, two applications of (5), gives

$$u_i^{n+1} + \theta_i^{n+1}[(p/2)H_x f_i^{n+1} - \sigma\delta_x^2 u_i^{n+1}] = u_i^n - \theta_i^n[(p/2)H_x f_i^n - \sigma\delta_x^2 u_i^n] ;$$

$$u_i^{n+2} + \theta_i^{n+2}[(p/2)H_x f_i^{n+2} - \sigma\delta_x^2 u_i^{n+2}] = u_i^{n+1} - \theta_i^{n+1}[(p/2)H_x f_i^{n+1} - \sigma\delta_x^2 u_i^{n+1}] . \qquad (6)$$

With the difference method written in this manner, it becomes clear that a "fast" version of the algorithm can be derived similar to that described in [8] for the parabolic differential equations. By rewriting the right-hand side as

$$2u_i^{n+1} - \{u_i^{n+1} + \theta_i^{n+1}[(p/2)H_x f_i^{n+1} - \sigma\delta_x^2 u_i^{n+1}]\} \qquad (7)$$

and using Eq. (6), Eq. (7) can be written as

$$u_i^{n+2} + \theta_i^{n+2}[(p/2)H_x f_i^{n+2} - \sigma\delta_x^2 u_i^{n+2}] = 2u_i^{n+1} - \{u_i^n - \theta_i^n[(p/2)H_x f_i^n - \sigma\delta_x^2 u_i^n]\} . \qquad (8)$$

For the points for which $\theta_i^{n+2} = \theta_i^n = 0$, Eq. (8) reduces to the simple relation $u_i^{n+2} = 2u_i^{n+1} - u_i^n$. Note that this relation does not apply at all grid points $(n + i)$ with i fixed and thus that the inherent linear instability of the two-term recursion will not manifest itself.

The values at remaining points at $t = (n + 2)k$ are calculated using these points and (8) with $\theta_i^{n+2} = \theta_i^n = 1$. The computational details are similar to those reported in [8] and are not repeated.

The stability of the difference scheme (6) and (7) may be studied away from boundaries and from the initial line by using the equivalence of the process to a three-level scheme on two interlacing grids (as mentioned in [10]). Thus (6) and (7) are equivalent to the three-level scheme

$$(1 + 2\sigma)u_i^{n+1} = (1 - 2\sigma)u_i^{n-1} - pH_x f_i^n + 4\sigma\mu_x u_i^n .$$

The stability of this scheme is studied using the standard von Neumann approach to its linearization. The condition for stability thus amounts to requiring that the roots ρ of the quadratic equation

$$(1 + 2\sigma)\rho^2 + \{2ip\lambda \sin\theta - 4\sigma\cos\theta\}\rho - (1 - 2\sigma) = 0$$

have modulus less than or equal to one for all θ.

With the results of Miller [11], it is straightforward to verify that the conditions for stability are

$$p|\lambda| \leq 1 \quad \text{and} \quad \sigma \geq 0 .$$

The freedom in the available values of σ allows considerable control to be exercised in the amount of second-order viscosity introduced, without reducing the possible step length. This control is in marked contrast to the Lax scheme. Experiments confirm the conclusions of the above paragraphs.

Hopscotch Lax-Wendroff method

The two-step version of the Lax-Wendroff Method [12] formulated by Richtmyer [13] is well known. Just as the Lax method (4) formed the basis for the hopscotch Lax method, the two-step Lax-Wendroff method forms the basis of the hopscotch Lax-Wendroff Method. To derive the required scheme consider the difference equations given by

$$\left.\begin{aligned} u_{i\pm\frac{1}{2}}^{*n+1} &= \mu_x u_{i\pm\frac{1}{2}}^n - (p/2)\delta_x f_{i\pm\frac{1}{2}}^n \\ u_i^{n+1} &= u_i^n - p\delta_x f_i^{*n+1} \end{aligned}\right\} n + i \text{ odd} \tag{9}$$

$$\left.\begin{aligned} u_{i\pm\frac{1}{2}}^{*n+1} &= \mu_x u_{i\pm\frac{1}{2}}^{n+1} - (p/2)\delta_x f_{i\pm\frac{1}{2}}^{n+1} \\ u_i^{n+1} &= u_i^n - p\delta_x f_i^{*n+1} \end{aligned}\right\} n + i \text{ even}, \tag{10}$$

where $f_i^{*n+1} \equiv f(u_i^{*n+1})$, and where u_i^{*n+1} is an intermediate or auxiliary solution. These equations can be written in terms of the function θ_i^n as

$$\begin{aligned} u_{i\pm\frac{1}{2}}^{*n+1} = \theta_i^{n+1}[\mu_x u_{i\pm\frac{1}{2}}^{n+1} - (p/2)\delta_x f_{i\pm\frac{1}{2}}^{n+1}] \\ + \theta_i^n[\mu_x u_{i\pm\frac{1}{2}}^n - (p/2)\delta_x f_{i\pm\frac{1}{2}}^n] \end{aligned} \tag{11}$$

$$u_i^{n+1} = u_i^n - (p/2)\delta_x f_i^{*n+1}, \tag{12}$$

where it is understood that the order of the calculation is that given by Eq. (9) followed by Eq. (10). That is, formulae (11) and (12) are solved first for $\theta_i^{n+1} = 0$ and then with $\theta_i^n = 0$. Careful consideration of these equations will indicate that the method is truly implicit for half the points u_i^{n+1}; that is for $\theta_i^{n+1} = 1$. A local expansion of the truncation error will indicate that the method is second-order accurate. However, in order to solve Eqs. (11) and (12) an additional feature must be added to the method, namely that half the points must be calculated iteratively. This is performed as follows. Write Eqs. (11) and (12) in the form

$$\begin{aligned} {}^{(j)}u_{i\pm\frac{1}{2}}^{*n+1} &= \theta_i^{n+1}[\mu_x\, {}^{(j)}u_{i\pm\frac{1}{2}}^{n+1} - (p/2)\delta_x\, {}^{(j)}f_{i\pm\frac{1}{2}}^{n+1}] \\ &\quad + \theta_i^n[\mu_x u_{i\pm\frac{1}{2}}^n - (p/2)\delta_x f_{i\pm\frac{1}{2}}^n] \\ {}^{(j+1)}u_i^{n+1} &= u_i^n - (p/2)\delta_x\, {}^{(j)}f_i^{*n+1}, \end{aligned} \tag{13}$$

where the superscript j applies only to those points for which $\theta_i^{n+1} = 1$. The case $j = 1$ is solved by using an average of the u_i^{n+1} calculated from Eqs. (11) and (12) with $\theta_i^{n+1} = 0$, namely ${}^{(1)}u_i^{n+1} = \frac{1}{2}(u_{i+1}^{n+1} + u_{i-1}^{n+1})$. It was found in practice that j need take only the values up to either two or three.

The stability condition of the hopscotch Lax-Wendroff method may be derived by appealing to the result for the hopscotch Lax scheme. If we set $\sigma = p^2\lambda^2/2$ in the quadratic equation then that equation governs the stability of the hopscotch Lax-Wendroff scheme. It follows automatically that this scheme is stable for $p|\lambda| \leq 1$.

Generalized hopscotch Lax-Wendroff method

In [14], [15], it was found expedient to generalize the two-step Lax-Wendroff method in order to obtain a class of methods that gave better results than the usual two-step formulation. The expense in so doing however, was that the region of stability was reduced. The difference method of [14] can be written as

$$u_i^{*n+1} = \mu_x u_i^n - 2ap\delta_x f_i^n$$

$$u_i^{n+1} = u_i^n - p[(1 - \tfrac{1}{4}a)\delta_x f_i^n + \tfrac{1}{4}a\delta_x f_i^{*n+1}],$$

where a is a parameter. It was found for stability that

$$a \geq \tfrac{1}{4}, \text{and } p|\lambda| \leq \frac{1}{2\sqrt{a}}$$

have to be satisfied simultaneously.

This method can be used to generate a generalized hopscotch Lax-Wendroff method, in much the same way as was done previously. (The details are omitted). A similar iterative process is required to obtain the solution. From empirical evidence it appears that this scheme is stable for $\frac{1}{4} \leq a \leq \frac{1}{2}$ up to the Courant-Friedrichs-Lewy limit. However in view of the disappointing behavior of this scheme in practice, it would seem to be of little value to verify the stability conditions analytically. The generalized hopscotch Lax-Wendroff method cannot be recommended for practical computation.

Numerical experiments

The model problem considered was the smooth solution problem:

$$(\partial u/\partial t) + (\partial/\partial x)\,(\tfrac{1}{2}u^2) = 0, \quad u(x,0) = x, \quad u(0,t) = 0$$

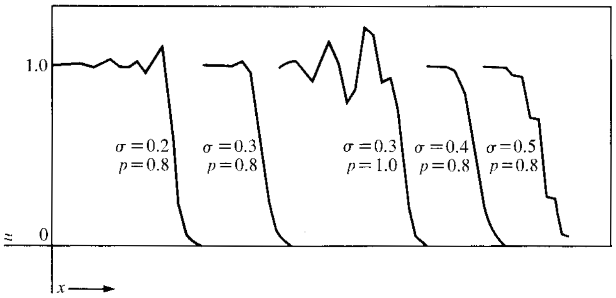
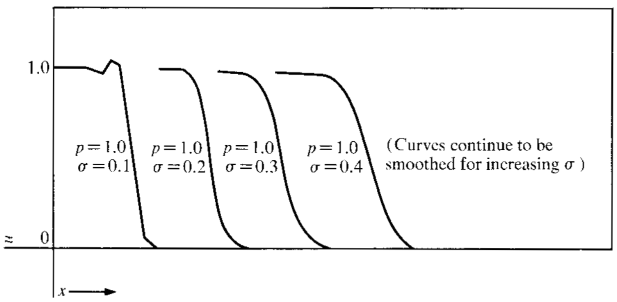


Figure 1 Lax method at 50 time steps, $h = 0.01$.

Figure 2 Hopscotch Lax method at 50 time steps, $h = 0.01$.



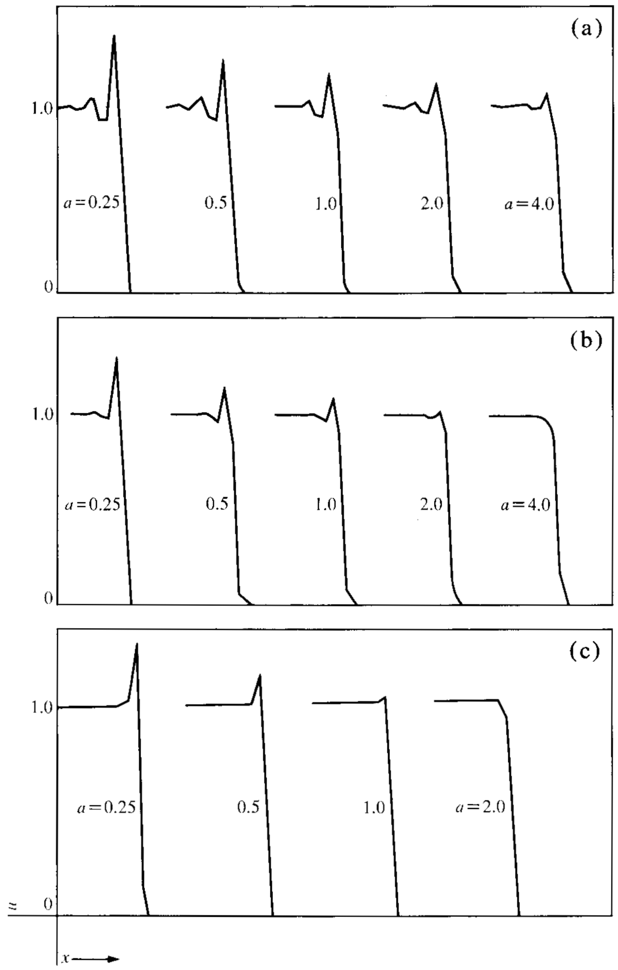


Figure 3 Generalized second-order method for $h = 0.01$, 100 time steps: (a) $p = 0.5$; (b) $p = 0.7$; (c) $p = 0.95$.

with theoretical solution $u = [x/(1+t)]$. We call this problem "P.1" for future reference. We also performed a series of experiments on the shock problem comprising a simple step function of size unity, namely

$$(\partial u/\partial t) + (\partial/\partial x)(\tfrac{1}{2}u^2) = 0$$

$$u(x,0) = \begin{cases} 1 & 0 \le x \le \alpha \\ 0 & \alpha < x \le 1 \,; \end{cases}$$

$$u(0,t) = 1 \,,$$

where in the experiments α was taken to be 0.1. For reference purposes we call this problem "P.2."

For the model problem P.1, we compared the first-order method (6) and (7) with the solution of P.1 for a series of p and σ . The prediction of the stability criterion for the method is borne out by these experiments, namely for all positive σ with $p|\lambda| \le 1$ stable results were obtained.

For problem P.2 the shock resolution for method (6) and (7) was certainly as good as, if not better than, that obtained from the Lax method. Moreover it was possible to take maximum steps with the hopscotch Lax scheme and then adjust σ to give good resolution. With the Lax scheme one is required to choose σ and step size together, to satisfy stability.

The experiments on P.1 and P.2 for the second-order methods was less encouraging, although the shock resolution in P.2 was as good as that obtainable from standard second-order schemes. For P.2 the results were poorer than those obtained for (6) and (7).

Sample graphs of the behavior of the Lax, hopscotch Lax, and generalized Lax-Wendroff method (generalized second-order method) are given in Figs. 1 to 3. It may be seen that the shock front representation for hopscotch Lax is better than that for the Lax scheme. The shock fronts propagate at the right speed in the computations.

Conclusions

Novel hopscotch schemes for the integration of nonlinear hyperbolic systems have been derived. A first-order scheme, the hopscotch Lax method has very desirable stability/pseudoviscosity properties that should prove useful in the computation of shock problems. In comparison, the advantages of a hopscotch formulation of second-order methods are of much less importance.

Generalization of the novel methods to a higher number of space dimensions is an extremely straightforward task and follows from the discussion given in Ref. 8.

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