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Some Numerical Results for Iterative Continuation in Nonlinear Boundary-value Problems

Abstract: In this communication we apply certain continuous analog iterative methods to two sample boundary-value problems. We show that the Euler-Newton and the second-order continuation of the Newton method are both useful algorithms for obtaining solutions to classes of nonlinear boundary-value problems. For a convergence analysis we rely upon a number of convergence theorems presented in earlier work. We indicate, through the numerical results, that the relaxed Newton methods are particularly useful in the iterative solution of "strongly" nonlinear problems where little information is available concerning "good starting values."

Introduction

In this communication we apply certain continuous analog iterative methods to two simple nonlinear boundary-value problems. More particularly, we consider the application of the Euler and the second-order continuation of Newton's method (see [1]). For a convergence analysis we rely upon a number of convergence theorems presented in [1]. (See also [2].) We indicate, through the numerical results, that the relaxed Newton algorithms are particularly useful in the iterative solution of "strongly" nonlinear problems where little information is available concerning "good starting values." We point out, in such cases, that the algorithms are considerably more useful than the (conservative) convergence theorems would indicate. This is very often the case in numerical analysis, however. In our case it is undoubtedly the result of applying certain (in general, rather poor) Runge-Kutta type error bounds in the derivation of the convergence conditions [1]. A more useful convergence result, based on a somewhat different approach, will appear in a forthcoming paper.

1. Two-point boundary value problem

In this section we present a brief review of the translation of a general two-point boundary-value problem (TPBVP) into an integral operator context (see, for example, [3] or [4]). We begin by considering a nonlinear TPBVP:

$$\dot{x}(t) = G(x,t), \quad g[x(0)] + h[x(1)] = c, \tag{1}$$

where G,g and h are "sufficiently smooth" vector-valued functions with $x,c \in R^p$ (R^p is a p-dimensional Euclidian space). We study a "related" linear TPBVP,

$$\dot{x} = A(t)x + k(t), \quad Bx(0) + Cx(1) = d, \tag{2}$$

in which we require the matrix set $\{A(t), B, C\}$ to be boundary compatible (this is equivalent to requiring that TPBVP (2) possess a unique solution for all $k(t)$ and c). Falb and DeJong prove in [4] (under appropriate conditions on G, g and h) that TPBVP (1) has the equivalent integral representation

$$x(t) = \Lambda(t)\ \{c - g[x(0)] - h[x(1)] + Bx(0) + Cx(1)\} + \int_0^1 \Gamma(t,s)\ \{G[x(s),s] - A(s)x(s)\}\ ds, \tag{3}$$

where Λ and Γ are Green's matrices for the linear problem (2).

By a similar analysis we can translate a typical nonlinear self-adjoint elliptic boundary-value problem (Dirichlet form),

$$Lu = f(u,x), \quad x \in D \subset R^p,$$

$$u(\partial D) = \phi(x), \tag{4}$$

into an equivalent integral representation of the form

$$u(x) = -\int_D G(x,\sigma)\, f[u(\sigma), \sigma]\ d\sigma - \int_{\partial D} G_\nu\ (\partial D,s)\ \phi\ (s)\ ds, \tag{5}$$

where G is the Green's function for the linear elliptic operator L and G_ν is the outward normal derivative of G. Equation (3) or (5) can now be viewed as a nonlinear operator map of an appropriate Banach space into itself. Thus we note that solving (3) or (5) is equivalent to solving an appropriately defined operator equation

$$F(z) = 0, \tag{6}$$

where $F = I - T$, with the nonlinear integral operator appearing on the right hand side of (3) or (5).

2. Two continuation algorithms for Newton's method

In [1] we developed a number of continuation algorithms based on Newton's method for iteratively solving problems of the form (2). The algorithms are obtained by simply translating certain basic function-theoretic continuation algorithms into a TPBVP context. It is clear that a similar translation in the context of Eq. (5) will reduce (4) to the problem of solving an appropriate sequence of linear elliptic problems.

We illustrate the translation by considering the first- and second-order relaxed Newton iteration (see [1]) in the two-point boundary-value problem context. We treat the Euler-Newton method,

$$\begin{aligned} x_{n+1} &= x_n - \tau_1 [F'_{x_n}]^{-1} F(x_0), \qquad n = 0,1,\cdots, N_1 - 1 \\ x_{n+1} &= x_n - [F'_{x_n}]^{-1} F(x_0), \qquad n = N_1, \cdots, \end{aligned} \tag{7}$$

and a Trapezoidal-Newton method,

$$\begin{aligned} &\left.\begin{aligned} y_n &= x_n - \tfrac{1}{2}\tau_2 [F'_{x_n}]^{-1} F(x_0) \\ x_{n+1} &= x_n - \tau_2 [F'_{y_n}]^{-1} F(x_0) \end{aligned}\right\}, \quad n = 0, 1, \cdots, N_2 - 1 \\ &x_{n+1} = x_n - [F'_{x_n}]^{-1} F(x_0), \qquad n = N_2, \cdots, \end{aligned} \tag{8}$$

where $\tau_1 = 1/N_1$ and $\tau_2 = 1/N_2$ and x_0 is some appropriate initial guess. The derivitives indicated are all taken in the Frechet sense.

Upon interpreting (8), for example, in the context of the TPBVP (3), we arrive at the Trapezoidal-Newton algorithm:

$$\dot{v}_n(t) = \left[\frac{\partial G}{\partial x}\right]_{x_n(t)} v_n(t) + \tfrac{1}{2}\tau_2 \{G[x_0(t), t] - \dot{x}_0(t)\}, \tag{9a}$$

$$\left[\frac{\partial g}{\partial x}\right]_{x_n(0)} v_n(0) + \left[\frac{\partial h}{\partial x}\right]_{x_n(1)} v_n(1)$$

$$= 1/2\ \tau_2 \{c - g[x_0(0)] - h[x_0(1)]\}, \tag{9b}$$

$$\dot{u}_n(t) = \left[\frac{\partial G}{\partial x}\right]_{y_n(t)} u_n(t) + \tau_2 \{G[x_0(t), t] - \dot{x}_0(t)\}, \tag{9c}$$

$$\left[\frac{\partial g}{\partial x}\right]_{y_n(0)} u_n(0) + \left[\frac{\partial h}{\partial x}\right]_{y_n(1)} u_n(1)$$

$$= \tau_2 \{c - g[x_0(0)] - h[x_0(1)]\}, \tag{9d}$$

for $n = 0,1, \cdots, N_2 - 1$, and

$$\dot{x}_{n+1}(t) = G[x_n(t), t] + \left[\frac{\partial G}{\partial x}\right]_{x_n(t)} [x_{n+1}(t) - x_n(t)], \tag{10a}$$

$$g[x_n(0)] + h[x_n(1) + \left[\frac{\partial g}{\partial x}\right]_{x_n(0)} [x_{n+1}(0) - x_n(0)]$$

$$+ \left[\frac{\partial h}{\partial x}\right]_{x_n(1)} [x_{n+1}(1) - x_n(1)] = c, \tag{10b}$$

for $n = N_2, \cdots,$

where $v_n = y_n - x_n$, $u_n = x_{n+1} - x_n$ and $\tau_2 = 1/N_2$. Note that Eqs. (10) represent quasilinearization and Eqs. (9) describe a "higher-order" method of "creeping" into the quasilinearization convergence region (from, perhaps, a poor initial guess).

An interpretation of either (7) or (8) in the context of (5) yields a similar sequence of linear elliptic boundary-value problems (Dirichlet). We illustrate the method later in the paper with one example in the TPBVP context and one example in the elliptic boundary-value problem context.

Remark 2.1: A typical convergence theorem for the algorithm (9)-(10) can be found in [1]. Convergence results for algorithms (7) and (8) in the elliptic boundary-value context are obtained simply by translating the main convergence theorems for continuation methods into the integral equation context, Eq. (5). The results obtained are almost identical (in structure) to those in [1].

Remark 2.2: An analysis of composite algorithms such as Continuation-SOR or Continuation-ADI for nonlinear problems of the form (4) will be presented in a future paper. (See, for example, [2] for a discussion of Newton-SOR, etc.) Branin[5] has obtained some very interesting numerical results for a special continuation algorithm used in the context of solving a set of nonlinear algebraic equations. Branin's method coupled with an appropriate "linear problem solver" offers promise (numerically) for problems of the form (4), where f is not "monotone" in u. A convergence analysis appears almost out of the question in such cases. This is not an unusual state of affairs in numerical analysis, however.

Remark 2.3: If shooting methods are employed to solve Eq. (1) and results of Antosiewicz[6] are used to interpret the shooting problem as a finite-dimensional root-finding problem of the form $F(y) = 0$, $y \in R^p$, algorithms (7) and (8) can then be interpreted in the "shooting context" to obtain good numerical results. Results along this line appear in Roberts and Shipman[7], although no exhaustive convergence study of this interesting "marriage" has appeared.

3. Application of the algorithms to specific examples

• *Example 1: A nonlinear oscillator problem*
Consider the nonlinear differential equation

$$\ddot{x}(t) + 6x(t) + \beta x^2(t) + \cos t = 0, \tag{11}$$

which describes an oscillator with a nonlinear restoring force. We wish to determine periodic solutions of (11) with period 2π, and so, we impose the boundary conditions

$$x(0) - x(2\pi) = 0 \text{ and } \dot{x}(0) - \dot{x}(2\pi) = 0. \tag{12}$$

The boundary-value problem (11) and (12) can be written in vector form as

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix} x_2(t) \\ -6x_1(t) - \beta x_1{}^2(t) - \cos t\end{bmatrix}, \tag{13a}$$

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} + \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1(2\pi)\\ x_2(2\pi)\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}. \tag{13b}$$

Thus, we define $G(\cdot,\cdot)$, $g(\cdot)$ and $h(\cdot)$, respectively, by

$$G(x,t) = \begin{bmatrix} x_2 \\ -6x_1 - \beta x_1{}^2 - \cos t\end{bmatrix},$$

$$g[x(0)] = \begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} \quad \text{and} \quad h[x(2\pi)] = \begin{bmatrix}-x_1(2\pi)\\ -x_2(2\pi)\end{bmatrix}.$$

We apply, for illustration, the Euler-Newton method to the solution of TPBVP (11). We generate, therefore, a sequence $\{u_n(\cdot)\}$, (where $u_n = x_{n+1} - x_n$) according to

$$\begin{bmatrix}\dot{u}_1{}^{(n)}\\ \dot{u}_2{}^{(n)}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -6 - 2\beta x_1{}^{(n)} & 0\end{bmatrix}\begin{bmatrix}u_1{}^{(n)}\\ u_2{}^{(n)}\end{bmatrix} + \tau_1\begin{bmatrix} x_2{}^{(0)} - \dot{x}_1{}^{(0)} \\ -6x_1{}^{(0)} - \beta x_1{}^{(0)2} - \cos t - \dot{x}_2{}^{(0)}\end{bmatrix}, \tag{14a}$$

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1{}^{(n)}(0)\\ u_2{}^{(n)}(0)\end{bmatrix} + \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}u_1{}^{(n)}(2\pi)\\ u_2{}^{(n)}(2\pi)\end{bmatrix} = \tau_1\begin{bmatrix}-x_1{}^{(0)} + x_1{}^{(0)}(2\pi)\\ -x_2{}^{(0)} + x_2{}^{(0)}(2\pi)\end{bmatrix}, \tag{14b}$$

for $n = 0, 1, \cdots, N_1 - 1$, and

$$\begin{bmatrix}\dot{x}_1{}^{(n+1)}\\ \dot{x}_2{}^{(n+1)}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -6 - 2\beta x_1{}^{(n)} & 0\end{bmatrix}\begin{bmatrix}x_1{}^{(n+1)} - x_1{}^{(n)}\\ x_2{}^{(n+1)} - x_2{}^{(n)}\end{bmatrix} + \begin{bmatrix} x_2{}^{(n)} \\ -6x_1{}^{(n)} - \beta x_1{}^{(n)2} - \cos t\end{bmatrix}, \tag{15a}$$

$$\begin{bmatrix}x_1{}^{(n)}(0)\\ x_2{}^{(n)}(0)\end{bmatrix} + \begin{bmatrix}x_1{}^{(n)}(2\pi)\\ x_2{}^{(n)}(2\pi)\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1{}^{(n+1)}(0) - x_1{}^{(n)}(0)\\ x_2{}^{(n+1)}(0) - x_2{}^{(n)}(0)\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1{}^{(n+1)}(2\pi) - x_1{}^{(n)}(2\pi)\\ x_2{}^{(n+1)}(2\pi) - x_2{}^{(n)}(2\pi)\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}. \tag{15b}$$

for $n = N_1, \cdots$.

Systems (14) and (15) simplify to yield

$$\dot{u}_1{}^{(n)} = u_2{}^{(n)} + \tau_1(x_2{}^{(0)} - \dot{x}_1{}^{(0)}), \tag{16a}$$

$$\dot{u}_2{}^{(n)} = -(6 + 2\beta x_1{}^{(n)})u_1{}^{(n)} - \tau_1(6x_1{}^{(0)} + \beta x_1{}^{(0)2}) + \cos t + x_2{}^{(0)}, \tag{16b}$$

$$u_1{}^{(n)}(0) = u_1{}^{(n)}(2\pi), \tag{16c}$$

$$u_2{}^{(n)}(0) = u_2{}^{(n)}(2\pi), \tag{16d}$$

and

$$\dot{x}_1{}^{(n+1)} = x_2{}^{(n+1)}, \tag{17a}$$

$$\dot{x}_2{}^{(n+1)} = -(6 + 2\beta x_1{}^{(n)})x_1{}^{(n+1)} + \beta x_1{}^{(n)2} - \cos t, \tag{17b}$$

$$x_1{}^{(n+1)}(0) = x_1{}^{(n+1)}(2\pi), \tag{17c}$$

$$x_2{}^{(n+1)}(0) = x_2{}^{(n+1)}(2\pi). \tag{17d}$$

We state the following proposition concerning convergence of the algorithm (16)–(17):

Proposition 3.1: Suppose $0 < \beta \leqslant 0.58$, $r_0 = 0.38$ and $\tau_1 \leqslant 0.28 \times 10^{-8}$. Then the Euler-Newton sequence $\{x_n(\cdot)\}$, based on the initial guess $x_0(\cdot) = 0(\cdot)$, converges to the solution $x^*(\cdot)$ of Eq. (13) in $\bar{S}(0,r_0)$.

Proof: The proposition is proved by a straightforward application of Theorem 3.1 in [1] to Eq. (13). Most of the approximations involving the Green's function for this problem appear in Collatz[8], who has studied this problem in other contexts.

Remark 3.2: The requirement on the step size ($\tau_1 \leq 0.28 \times 10^{-8}$) is necessarily extremely conservative. In practice we again find that convergence of the Euler-Newton sequences are obtained for a much wider range of β, and for τ_1 as large as 1/2 (for certain β). Similar results are expected and obtained for higher-order relaxed Newton method algorithms applied in the TPBVP context.

• *Example 2: A potential problem*
In this example we consider the nonlinear potential problem defined by

$$\begin{aligned}\nabla^2 u(x,y) &= \beta \exp[u^2(x,y)], \quad (x,y) \in G\\ u(x,y) &= 0, \quad (x,y) \in \partial G,\end{aligned} \tag{18}$$

where $G = (0,1) \times (0,1)$ and $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$. Since $u(\partial G) = 0$, we can write (18) in the integral form

$$u(x,y) = \beta \int_0^1 \int_0^1 [K(x,y,\eta,\sigma) \exp(u^2(\eta,\sigma)] \, d\eta \, d\sigma. \tag{19}$$

If we interpret algorithm (7) in the context of (19), we obtain the sequence of linear elliptic (Dirichlet) problems:

Table 1: Values of relaxation constant τ_1 required to obtain convergence for several values of β. (Example 1).

β	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5
τ_1	1	1	1	1	1/2	1/2	1/2	1/3	1/3	1/8

Table 3: Values of relaxation constant τ_2 and number of Newton iterations required to obtain convergence for several values of β. (Example 2).

β	1.0	5.0	6.0	6.5	7.0	7.2	7.4	7.5
τ_2	1	1	1/2	1/2	1/2	1/4	1/4	1/4
Newton iterations	3	4	3	3	4	3	4	4

Table 2: Computed values of x at various time steps for several values of β. (Example 1).

β	Solution values x					
	Time steps t					
	0.0π	0.4π	0.8π	1.2π	1.6π	2.0π
2.0	−0.227325	−0.055409	+0.151716	+0.151716	−0.055401	−0.227325
2.5	−0.234664	−0.054880	+0.149996	+0.149996	−0.054870	−0.234664
3.0	−0.242405	−0.054854	+0.148584	+0.148583	−0.054843	−0.242405
3.5	−0.250707	−0.055396	+0.147480	+0.147470	−0.055383	−0.250707
4.0	−0.259790	−0.056595	+0.146693	+0.146689	−0.056579	−0.259790
4.5	−0.269992	−0.058584	+0.146242	+0.146236	−0.058566	−0.269928
5.0	−0.281867	−0.061575	+0.146160	+0.146152	−0.061553	−0.281867
5.5	−0.296436	−0.065946	+0.146513	+0.146501	−0.065918	−0.296436
6.0	−0.315386	−0.072344	+0.147744	+0.147742	−0.072308	−0.315386
6.5	−0.348514	−0.083514	+0.149329	+0.149295	−0.083929	−0.348514

Table 4: Computed values of $u(x,y)$ for $\beta = 7.5$. (Example 2).

y values	Solution values u						
	x values						
	0.00	0.18	0.36	0.55	0.73	0.91	1.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.18	0.00	−0.2925	−0.4439	−0.4744	−0.3833	−0.1679	0.00
0.36	0.00	−0.2471	−0.5952	−0.7545	−0.5952	−0.2471	0.00
0.55	0.00	−0.4744	−0.7546	−0.8147	−0.6391	−0.2626	0.00
0.73	0.00	−0.3833	−0.5952	−0.6391	−0.5094	−0.2158	0.00
0.91	0.00	−0.1679	−0.2471	−0.2626	−0.2158	−0.9961	0.00
1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

$$\nabla^2 w_n = \beta \exp (u_n{}^2)\, w_n + \tau_1 \, [\exp u_0{}^2 - \nabla^2 u_0],$$

$$w_n\,(\partial G) = 0, \quad \text{for } n = 0,1, \cdots, N_1{-}1, \tag{20}$$

where $w_n = u_{n+1} - u_n$ and

$$\nabla^2 w_n = \beta \exp (u_n{}^2)\, w_n + [\exp u_0{}^2 - \nabla^2 u_0],$$

$$w_n(\partial G) = 0, \quad \text{for } n = N_1, \cdots . \tag{21}$$

We now state the following proposition on convergence of algorithm (20)–(21).

Proposition 3.3: Suppose $0 < \beta < 1.2$, $r_0 = 1$ and $\tau_1 \leqslant 2.8 \times 10^{-5}$ with $u_0(x,y) \equiv 0$. Then the Euler-Newton sequence $\{u_n\}$, defined in (20)–(21), converges to a solution u^* in $\bar{S}(0,1)$.

Proof: The proof follows by an immediate application of the main convergence result in [1] interpreted in the partial-differential equation context. We remark that the Green's function is given by

$$K(x,y,\eta,\sigma) = -\frac{4}{\pi^2} \sum_{j,k=1}^{\infty} \frac{\sin j\pi x \,\sin k\pi y \,\sin j\pi\eta \,\sin k\pi\sigma}{j^2 + k^2}$$

(see, for example Weinberger[9] and the estimates of $\|K\|$ required in the proof of the proposition are easily obtained.

Remark 3,4: Remark 3.2 applies in the above context. In fact, using the results of Falb and Groome[10], it can be shown that global convergence for this problem $[u_0(x,y) = 0]$ can be guaranteed for all $\beta > 0$.

4. Numerical results

In this section we provide the numerical results for the examples discussed in the previous section. We observe that the application of the Euler-Newton algorithm (or the Trapezoidal-Newton algorithm) to nonlinear TPBVP (13) yields a "corresponding sequence" of linear TPBVP's to solve. For the numerical integration of the resultant linear differential equations, a modification of a fourth-order Runge-Kutta method is used. This method requires the values of the right hand sides of the differential equations at points in between the points for which the algorithms yield the value of the solution, in addition to points where the solution is obtained. Since the values are not available at these intermediate points, they are obtained by linear interpolation.

• *Example 1:*

Results of the numerical treatment of the Euler-Newton method for Example 1 appear in Tables 1 and 2. In Table 1 we present the values of the relaxation number τ_1(where $\tau_1 = 1/N_1$) required to obtain convergence [11] of the relaxed Newton sequence for various values of β. In Table 2 we present the actual solutions. We note that, although Proposition 3.1 guaranteed convergence only for very small value of τ_1, the actual computations converged for τ_1 significantly larger.

• *Example 2:*

Results of some particular computations appear in Table 4. The square $[0,1] \times [0,1]$ was partitioned into 121 subsquares and a second-order implicit finite-difference method was used for the numerical approximation. In Table 3 we present the number of Newton iterations required to obtain convergence [12] for various values of β and τ_2. We observe that, although the proposition guaranteed convergence only for $0 < \beta \leqslant 1.41$, the actual computations converged for values of β much greater (see Table 4). We also observe that the relaxed Newton sequences converge for values of β for which the Newton sequences ($\tau_2 = 1$) diverge.

References and notes

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9. H. F. Weinberger, *Partial Differential Equations*, Blaisdell Co., New York, 1965.
10. P. L. Falb and G. R. Groome, private communication.
11. Convergence is here construed to mean that
$$\|x^{(n+1)} - x^n\| = \sum_{i=1}^{2} \{{}_k\max|x_i^{(n+1)}\ (t_k) - x_i^{(n)}\ (t_k)|\} \leqslant 10^{-6},$$
where the t_k are points in the integration routine.
12. Convergence is here construed to mean that
$\|u_{n+1} - u_n\| \leqslant 0.5 \times 10^{-5}$,
where $\|\cdot\|$ is the sup norm in R^{100}

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