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Adhesion and Partial Slip between Normally Loaded Round Surfaces

Abstract: A simple mechanism is sought to account for the frictional energy losses encountered in the mutual contact surface of two bodies pressed normally against each other. The mathematical model of Coulomb friction is assumed valid, and it is shown that slip between the contacting surfaces develops in the outlying regions of contact. This model, used in conjunction with a numerical scheme, leads to a set of nonlinear simultaneous algebraic equations in the surface tractions. Two specific cases are computed: those of a rigid sphere and of a rigid roller each pressed against an elastic medium. The relative size of the slip region is shown to depend on the material properties. Energy dissipation during a half contact cycle is calculated, and the influence of the secondary adhesive tractions on the contact stresses is discussed.

Introduction

In modern friction and wear investigations, extensive use is made of the contact theory of deformable bodies. Contact stress analysis can successfully predict the distribution of tractions in a macroscopic sense. Thus an estimate can be made (from previous measurements) of the performance of the surfaces in the microscopic sense. Contact theory has also been applied directly to the minute surface asperities that can be considered to be statistically distributed over the contact zone [1].

Sliding friction studies have shown that the classical Coulomb theory of friction is applicable as a first-order approximation, although the true process depends on the actual area of contact. Sliding has also been found to be strongly affected by adhesion.

This paper deals with a phenomenon caused by the normal contact of two round, solid bodies that results in some relative sliding motion of adjacent surfaces. When an indenter is pressed against the surface of an elastic body, normal displacements, dependent on the shape of the indenter, are induced at the contact surface. There is also a tendency for relative tangential displacements to take place in the contact region but, due to adhesion, relative motion between adjacent surfaces is always limited. The shear tractions caused by the primary normal pressures, which are responsible for the prevention, or at least reduction, of tangential slip, constitute a secondary stress system.

In the analysis of these problems one of three simple assumptions can be made:

1. The contact is frictionless.
2. Relative sliding cannot be allowed for adjacent points on the surfaces once they enter the contact zone.
3. Sliding in the contact zone is governed by Coulomb's law.

In reality all three models are idealized. The second assumption predicts the existence in the contact zone of a peripheral area of large shear stress in which the ratio of shear to normal stress becomes infinite. Clearly this is impossible. The third assumption, in which limiting the ratio of shear to normal stress results in partial slipping, appears to be more realistic. In addition, this latter model can be used to predict the amount of frictional energy dissipated during the growth of the contact area. The mechanical surface loss, when added to the effects of material hysteresis and stress wave propagation, determines the amount of recoverable work in repeated loading cycles.

Following the classical Hertz theory of a frictionless, normally loaded contact, a successful analysis (based on the second of the above assumptions) of normally loaded, sticky elastic spheres was given by Goodman [2]. Goodman treated the problem on an incremental basis with infinitesimally thin annuli progressively entering the contact region. He assumed that the normal stresses were unaffected by friction. A numerical analysis of this problem was made by the authors [3] without this assumption. That numerical approach provided the foundation of the analysis given in this paper. Partial slip under a flat indenter was analyzed by Conway and Farnham [4]. The effect of fretting (small oscillation in contact) on wear was studied by Wayson [5] and placed into the framework of the IBM engineering wear theory [6]. These latter studies provided grounds for comparison with the present investigation.

The present work uses a numerical method of stress analysis for the discrete normal and shearing tractions in the contact zone. The partial slip problem is not only a mixed boundary value problem of the theory of elasticity; it is also a nonlinear one because the friction coefficient is unknown at the outset, even though it is assumed to be constant. The analysis was made tractable by assuming only one lock-slip boundary within the contact area. This assumption is entirely reasonable in view of Goodman's results [2], which indicate that for rough contact the shear-to-normal pressure ratio increases monotonically with the radius. Of course, Goodman's work is based on an infinite friction coefficient between the materials and, when this condition is not satisfied, the true situation may be modified because several lock-slip boundaries can be formed within the contact region. Indeed, one of the questions answered in this paper concerns the configuration for which the single lock-slip boundary model is applicable. However, further generalizations of the model can be made subsequently.

The specific treatment considered involves a round rigid body indenting a linearly elastic infinite half-space. The indentation of a layer of finite thickness may also be treated by the method in Ref. 3 and the effects of indenter deformation can be included [7]. The method can be applied also to linearly viscoelastic media [8].

This paper presents the formulation of a rigid sphere indenting an elastic half-space, and this treatment is shown to be generally valid for a rigid cylinder indenting an elastic medium. Numerical results are presented for both of these cases.

Nomenclature

a	in.	Contact radius or semicontact length
a_{ij}	in.^{3}/lb	Normal displacement of point i due to unit pressure loading of ring j
b_{ij}		Tangential displacement of point i due to unit pressure loading of ring j
c_{ij}		Tangential displacement of point i due to unit shear loading of ring j
d_{ij}		Normal displacement of point i due to unit shear loading of ring j
E	lb/in.^{2}	Modulus of elasticity of indented medium
$E(\xi)$		Complete elliptic integral of the second kind
h	in.	Thickness of slab indented by cylinder (the slab rests on a rigid frictionless base)
k		Friction coefficient
$K(\xi)$		Complete elliptic integral of the first kind
m/n		Ratio of lock-slip boundary to contact radius
p	lb/in.^{2}	Normal pressure
$P(r/a)$	lb/in.^{3}	Pressure function
q	lb/in.^{2}	Shear traction
$Q(r/a)$	lb/in.^{3}	Shear function
r	in.	Radial coordinate
R	in.	Radius of sphere or cylinder
u	in.	Tangential displacement
$U(r/a)$	in.$^{-1}$	Tangential displacement function
V	lb-in.	Work
w	in.	Normal displacement
$W(r/a)$	in.$^{-1}$	Normal displacement function
ν		Poisson's ratio

Analysis

Rigid sphere indenting an elastic half-space

When considering the axisymmetric contact of a rigid sphere indenting a linearly elastic half-space, the contact zone is a circle of radius a, which is small compared with the radius R of the sphere. For the purpose of analysis, this circle is divided into n concentric rings of equal width $\Delta r = a/n$; we obtain the values of the pressure and shear tractions over these annuli.

The normal pressure distribution is approximated by square steps p_j each of constant height over the pertinent annulus j [Fig. 1(a)]. For shear loading q_j, conical segments are substituted and these rise linearly from the origin [9] as shown in Fig. 1(b). The larger the number of rings, the more accurate is the simulation of the actual stress distribution. In the stress analysis that follows, the dis-

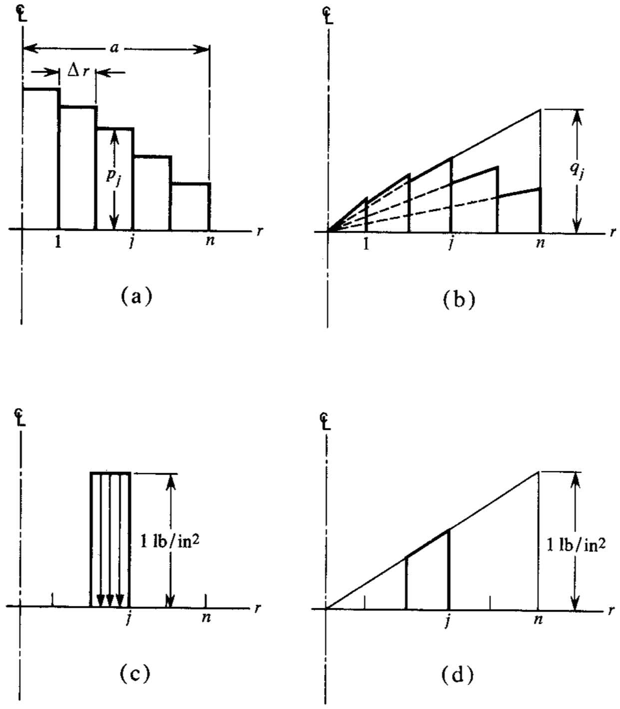


Figure 1 Loading elements used in the analysis: (a) approximation of contact pressure distribution; (b) approximation of shear tractions by piecewise linear segments; (c) pressure-influence loading p_j; (d) shear-influence loading $q_j(r) = r/a$, $(j-1)\Delta r \leq r \leq j\Delta r$.

placement influence coefficients a_{ij}, b_{ij}, c_{ij} and d_{ij}, computed for unit influence loading p_j and q_j [Figs. 1(c) and 1(d)], are used. The computation of normal and tangential displacements due to constant pressures and linearly distributed shear tractions on an infinite half-space is described in the Appendix.

For the normal displacement boundary conditions n equations are written:

$$w_i = w_0 + i^2(\Delta r)^2/2R, \qquad 1 \leq i \leq n. \tag{1}$$

The region of complete adherence is a circular segment devoid of differential tangential motion between indenter and medium and, since the indenter is rigid,

$$\Delta u(r) = 0, \qquad r < a. \tag{2}$$

Slipping is assumed to take place in the remainder of the contact area, and here Coulomb's friction law is assumed valid:

$$q = kp. \tag{3}$$

From experimental observations, slip is expected only in the peripheral regions of the contact area, but the relative extent of the slipping region is unknown. In the analytical approach, the region of adherence is first fixed by assuming that it occupies the interior m annuli out of the total n. The unknown quantities are then computed for each configuration of the relative slip area, which is characterized by the nondimensional ratio m/n.

We now make use of a dimensional argument to prove that the tangential displacements in the adherence region are proportional to the square of the radius. The tractions are linearly proportional to a on the basis of the Hertz theory and thus

$$p(r, a) = aP(r/a); \tag{4}$$

$$q(r, a) = aQ(r/a). \tag{5}$$

The displacement influence coefficients $a_{ij}, \cdots, d_{ij}$ are also linearly proportional to the radial dimension of the annuli. Since the displacements are linear combinations of the tractions and the influence coefficients, they depend on the square of a, so that using the nondimensional radii we can write

$$w(r, a) = a^2 W(r/a); \tag{6}$$

$$u(r, a) = a^2 U(r/a). \tag{7}$$

As the load is applied and a increases, a specific point r first enters the slip region and then, according to the fixed m/n ratio, becomes locked in the adherence region. From this time on, its further tangential displacement is prevented. The total tangential displacement, however, remains proportional to the square of the radius. From this we can conclude that

$$u(r, a) = (r/r_1)^2 u(r_1, a), \qquad 0 < r_1 \leq r \leq am/n. \tag{8}$$

With the displacements written as linear combinations of the discrete tractions, Eq. (1) yields

$$\sum_{j=1}^{n} [(a_{ij} - a_{0j})p_j + (d_{ij} - d_{0j})q_j] = i^2(\Delta r)^2/2R, \qquad 1 \leq i \leq n. \tag{9}$$

The adherence equations (8) yield

$$\sum_{j=1}^{n} [(b_{ij} - i^2 b_{1j})p_j + (c_{ij} - i^2 c_{1j})q_j = 0, \qquad 2 \leq i \leq m, \tag{10}$$

and the Coulomb friction condition Eq. (3) adds

$$q_i = kp_i, \qquad m < i \leq n. \tag{11}$$

The above $(2n-1)$ equations in $(2n+1)$ unknowns (p_i, q_i and k) may be supplemented by the following statements.

It follows from symmetry that the shear is zero at the origin and thus, if n is sufficiently large,

$$q_1 \approx q_2. \tag{12}$$

An additional equation is obtained by writing the adherence condition Eq. (8) for the point $i = \frac{1}{2}$. These simultaneous, nonlinear algebraic equations may be written using vector notation:

$$F_i(\mathbf{x}) = 0, \qquad 1 \leq i \leq 2n+1, \tag{13}$$

where $\mathbf{x}$ is the vector of unknown quantities

$$\mathbf{x} = (p_1, \cdots, p_n; q_1, \cdots, q_n; k). \tag{14}$$

An iterative solution of the system in Eq. (13) by the Newton–Raphson process is expedient, and is outlined briefly below.

A trial solution vector $\mathbf{x}_0$ is taken as the set of pressures for the frictionless indenter with $q_j = 1$ $(1 \leq j \leq n)$ and $k = 0.1$. By expanding the set about the trial vector $\mathbf{x}_0$ and truncating after the linear term, we obtain simultaneous linear equations in the increments $\Delta\mathbf{x}$:

$$F_i(\mathbf{x}_0) + \left.\frac{\partial F_i(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}_0} \Delta\mathbf{x} = 0. \tag{15}$$

An improved set of solutions is available for the next cycle,

$$\mathbf{x} = \mathbf{x}_0 + \Delta\mathbf{x}. \tag{16}$$

Three cycles were found sufficient for satisfactory convergence in most cases.

The friction coefficient k emerges as a single-valued function of the m/n ratio. The horizontal displacement $u(r, a)$ can be computed now for every value of m/n where $0 < m/n < 1$. The frictional energy loss is evaluated as the sum of the products of the shears and the corresponding radial displacements.

At one configuration of the contact radius a, the shear stress distribution is $q(r, a)$. The horizontal displacements $u(r, a)$ can be evaluated by the foregoing numerical analysis. An infinitesimal change da in the contact radius gives rise to the work

$$dV = \int_{am/n}^{a} q(r,a)\, \Delta u(r,a) 2\pi r\, dr, \tag{17}$$

where the radial slip is computed from

$$\Delta u(r,a) = \left[\frac{\partial u}{\partial a} + \frac{\partial u}{\partial (r/a)}\frac{\partial (r/a)}{\partial a}\right] da. \tag{18}$$

Introducing the nondimensional radius $\xi = r/a$, and using Eq. (5) with Eq. (7), we obtain

$$dV = 2\pi a^4\, da \int_{m/n}^{1} \xi Q(\xi)\left[2\,U(\xi) - \xi\frac{dU(\xi)}{d\xi}\right] d\xi, \tag{19}$$

where the integral is independent of the value of a. The derivatives $dU/d\xi$ are computed numerically from the discrete values of U obtained from the stress analysis.

Hence the total frictional work in extending the contact zone from zero to a is

$$V = \frac{2\pi a^5}{5}\int_{m/n}^{1} \xi Q(\xi)\left[2\,U(\xi) - \xi\frac{d\,U(\xi)}{d\xi}\right] d\xi. \tag{20}$$

By carrying out the analysis for various m/n values, the radial slip can be calculated over the contact area with Eq. (18). The slip must be directed opposite to the shear traction; where this is not the situation, the model of partial slip yields spurious results which must be discarded. Thus for each indenter geometry, a consistent model considering a single lock-slip boundary exists only for a certain range of the friction coefficient.

- *Long rigid cylinder indenting an elastic medium in plane strain*

The plane strain displacements due to pressure loading on an elastic half-space can be evaluated only in terms of an unknown constant; thus a thick slab supported on a smooth rigid foundation will be considered. The normal pressure elements are again chosen as steps, and this time the shears are also constant steps [3].

If a/h is small, the dimensional argument for the horizontal displacement in the adherence region is a good approximation and Eq. (10) can be used. Equation (12) is now replaced by $q_1 = \frac{1}{2}q_2$. Otherwise the analysis, using influence coefficients of earlier work [3], proceeds along the previous lines.

Since the elemental area is a straight strip, there is a change indicated in the value of the total work corresponding to Eq. (20):

$$V = \frac{a^4}{4}\int_{m/n}^{1} Q(\xi)\left[2\,U(\xi) - \xi\frac{dU(\xi)}{d\xi}\right] d\xi. \tag{21}$$

Analytical results

The rigid spherical indenter pressed against an elastic half-space was first analyzed assuming $\nu = 0.3$. On the basis of $n = 16$ elements, the limiting frictionless case yielded an accuracy of 0.3 percent in the stresses; even results obtained with $n = 8$ were very close to these.

When values of the friction coefficient k are plotted against the corresponding m/n values (Fig. 2), the resulting curve can be used to find the adherence region for any k. It was found, however, that for values of m/n larger than 0.3 ($k > 0.3$), the slip tendency is in the same direction as the shear traction and thus the assumed model of a single lock-slip boundary appears to break down beyond this limit.

By noting the manner in which the surface stresses vary as m/n increases, several interesting observations can be made (Fig. 3). The shear stress reaches a peak just past the lock-slip boundary, causing a slight downturn in the normal pressure. The shears, in general,

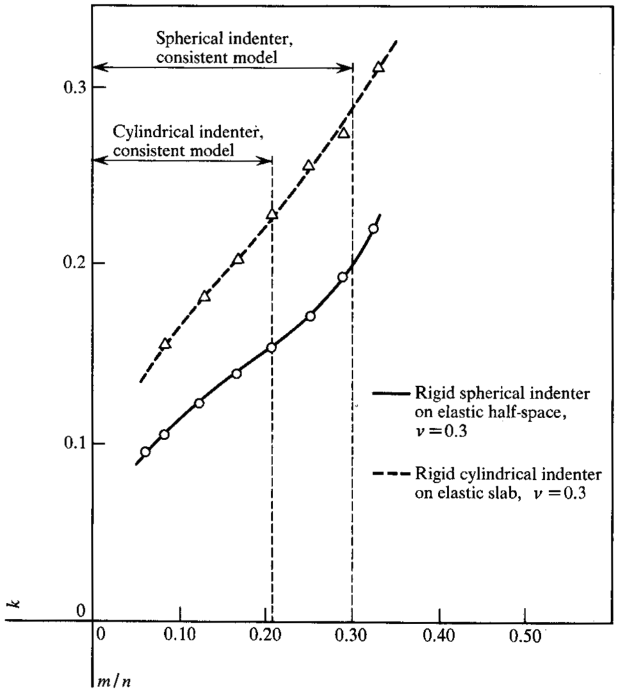
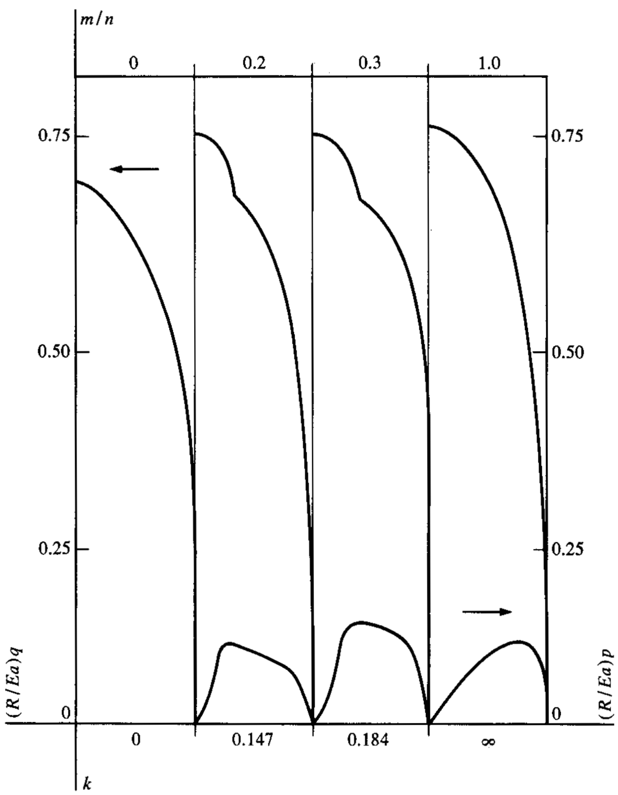


Figure 2 Coefficient of friction k vs m/n.

Figure 3 Contact stress distributions for various values of m/n and k; $\nu = 0.3$.



increase as m/n (and thus k) increases. Pressures are only slightly increased meanwhile and they stay close to the Hertz distribution.

The effect of adhesion on the radial displacement is shown in Fig. 4. The radial displacement for unlimited friction ($k = \infty$) [2] has the opposite curvature from that of the frictionless case. For partial slip the curvature changes sign past the lock-slip boundary.

The frictional energy loss is plotted against m/n in Fig. 5. Note that the curves would start at the origin corresponding to fully lubricated conditions ($k = 0$) and reach a maximum around $m/n = 1/16$. Calculations for the smaller values of m/n, however, involve a large amount of computation. Since Fig. 2 gives the relation between k and m/n, the value of k for a given combination of engineering materials in contact can be entered in that figure, and the relative magnitude of the lock-slip boundary m/n can be obtained. For a full cycle of loading-unloading, the amount of work V indicated in Fig. 5 may be doubled to estimate surface energy losses due to normal loading conditions.

Computations for a rigid cylinder indenting a thick slab were also carried out, using influence coefficients for a slab with Poisson's ratio 0.3 and $a/h = 1/20$. The variation of k vs m/n in this case is shown in Fig. 2. The same general trends in the tractions and the displacements were found as for the spherical indenter case. The range of a consistent model is limited by $m/n = 0.21$, corresponding to $k = 0.23$.

Summary

A numerical method of contact stress analysis has been presented for normally loaded round surfaces that exhibit partial slipping in the contact zone. A unique relation has been shown to exist between the relative size of the adherence region and the coefficient of friction. The model is applicable for moderate values of the friction coefficient, $k \leq 0.3$, for the case of a rigid ball indenting an elastic half-space. In the region of larger values of k, a more complicated model with more than one lock-slip boundary should be considered. The energy dissipation in friction was also computed in the range of moderate k values.

The general method is well suited to handle most geometrical configurations, dissimilar materials in contact, and viscoelastic materials. Thus it is a useful analytical tool for predicting energy dissipation due to surface resistance between repeatedly loaded solids.

Appendix

- *Displacements caused by a constant normal pressure loading over a circular area of radius a on an elastic half-space*

Normal displacements [10]:

$$w = \frac{4(1-\nu^2)pa}{\pi E}\,\mathrm{E}\left(\frac{r^2}{a^2}\right), \qquad r \le a;$$

$$w = \frac{4(1-\nu^2)pr}{\pi E}\left[\mathrm{E}\left(\frac{a^2}{r^2}\right) - \left(1-\frac{a^2}{r^2}\right)\mathrm{K}\left(\frac{a^2}{r^2}\right)\right], \qquad r \ge a.$$

Tangential displacements:

$$u = -\frac{(1-2\nu)(1+\nu)pa}{2E}\,\frac{r}{a}, \qquad r \le a;$$

$$u = -\frac{(1-2\nu)(1+\nu)pa}{2E}\,\frac{a}{r}, \qquad r \ge a.$$

- *Displacements caused by linear surface shear [$q(r) = q_0 r$ over a circle of radius a] on an elastic half-space*

The following formulas result from writing integrals of Bessel functions in terms of hypergeometric functions which, in turn, are related to complete elliptic integrals [11].

Normal displacements:

$$w = \frac{(1-2\nu)(1+\nu)q_0 a^2}{2E}\left(1-\frac{r^2}{a^2}\right), \qquad r \le a;$$

$$w = 0, \qquad r > a.$$

Tangential displacements:

$$u = \frac{4(1-\nu^2)q_0 a^2}{3\pi E}\,\frac{a}{r}\left[\left(\frac{2r^2}{a^2}-1\right)\mathrm{E}\left(\frac{r^2}{a^2}\right) - \left(\frac{r^2}{a^2}-1\right)\mathrm{K}\left(\frac{r^2}{a^2}\right)\right], \qquad 0 < r \le a;$$

$$u = \frac{4(1-\nu^2)q_0 a^2}{3\pi E}\left[\left(\frac{2r^2}{a^2}-1\right)\mathrm{E}\left(\frac{a^2}{r^2}\right) - 2\left(\frac{r^2}{a^2}-1\right)\mathrm{K}\left(\frac{a^2}{r^2}\right)\right], \qquad r \ge a.$$

The influence coefficients can be evaluated from the above formulas with the aid of superposition. Corresponding displacements for a slab under normal and shear loading are given in [12].

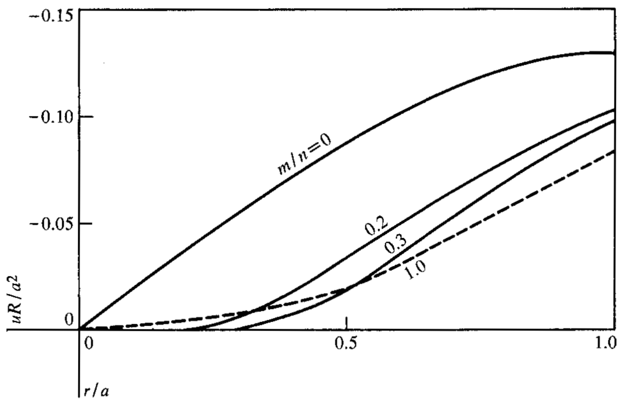
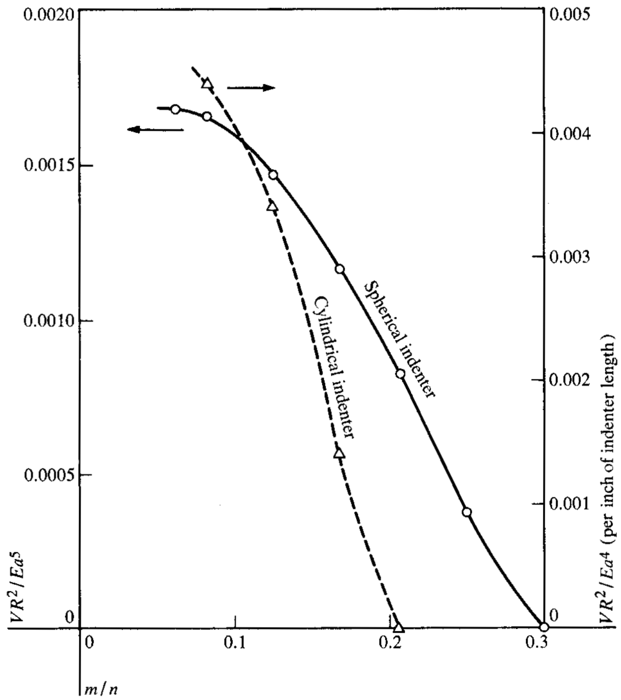


Figure 4 Horizontal displacements for given values of m/n; $\nu = 0.3$.

Figure 5 Frictional energy dissipation in one half-cycle of loading vs m/n.



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