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Iterated Consensus Method for Multiple-output Functions

Abstract: The iterated consensus method for obtaining prime implicants of Boolean functions has several advantages with respect to the conventional tabular method. However, when one attempts to apply the iterated consensus method to multiple-output functions using the rules set forth in the existing literature, it is possible that some of the prime implicants will not be produced. This communication presents an algorithm which assures that all prime implicants of multiple-output functions will be found.

The conventional tabular method of obtaining prime implicants has several disadvantages: the method requires obtaining the canonical expansion to start with; a large number of terms may be generated and have to be handled in the process; and a large number of matches may have to be made.

The iterated consensus method of obtaining prime implicants overcomes these disadvantages. Briefly stated, the iterated consensus method for a single-output function operates as follows (assuming that the states of the Boolean function $f(x_1, \cdots, x_n)$ are listed in tabular form): all pairs of rows—original rows and rows that may be added to the table—are systematically compared for subsumption and consensus.

One row *subsumes* another row if it has a 1 in every column in which the other row has a 1, and if it has a 0 in every column in which the other row has a 0. The subsuming row is also said to be *absorbed* or *included* by the other row. Any row that subsumes another is eliminated. In Example 1, the second row subsumes the first and is eliminated. In Example 2, either row is eliminated.

Example 1

x_1	x_2	x_3	x_4
1	0	–	–
1	0	–	0

Example 2

x_1	x_2	x_3	x_4
1	0	–	–
1	0	–	–

Two rows generate a *consensus* row if, in one column only, one row has a 1 and the other a 0. The consensus row has a dash in that column. The consensus row also has a dash in any column in which both of the two original rows have dashes. The consensus row has a 1 in any column in which either of the two original rows has a 1, and it has a 0 in any column in which either of the two original rows has a 0. If a consensus row does not subsume any row in the table, it is added to the table. In Example 3, the first two rows generate the last (consensus) row. The consensus row is added to the table only if it does not subsume a row already in the table.

Example 3

x_1	x_2	x_3	x_4	x_5	x_6
1	0	–	0	0	–
1	–	–	0	1	1
1	0	–	0	–	1

When this process terminates, the rows of the table comprise all of the prime implicants of the function.

Application of the iterated consensus method to multiple-output functions, $f_i(x_1, \cdots, x_n)$, $i = 1, \cdots, m$, requires some additional rules. Unfortunately, the rules that exist in the current literature do not always produce all the prime implicants.[1-4]

The method used for multiple-output functions has been described as a direct extension of the method for single-output functions. A *tag*, composed of dashes and 0's, specifies the output functions with which each input term, or *identifier*, is associated. A dash in an output column indicates that the output is associated with the input term in the corresponding row; a 0 indicates that the output is not associated with the input term. (The symbols may differ from reference to reference, but the concept is the same.) In addition to those given for single-output functions, the following rule has been devised,

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quoted here from Ref. 2: "If two rows have identical identifier portions and differ in their tag positions, a new row is formed having the same identifier and dashes wherever either of the original rows have dashes in their tags. The original rows are then removed from the table." In Example 4, the first two rows generate and are replaced by the last row.

Example 4

Identifier			*Tag*			
x_1	x_2	x_3	f_1	f_2	f_3	f_4
1	–	0	–	–	0	0
1	–	0	0	–	0	–
1	–	0	–	–	0	–

Mage[4] notes in a recent paper that all prime implicants may not be produced by the preceding rules. He points out an exception that occurs when the identifier of one row is subsumed by the identifier of another row, but where the tag of the subsuming row has a 0 in those columns in which the subsumed row has a dash.

In Example 5, the identifier of the first row is subsumed by the identifier of the second row, and the tag of the second row has a 0 in the f_1 column while the first row has a dash in this same column.

Example 5

Identifier				*Tag*	
x_1	x_2	x_3	x_4	f_1	f_2
0	0	0	–	–	0
0	0	0	0	0	–
–	0	0	1	–	0

The preceding rules do not produce the required prime implicant 0000– –.

Mage[4] offers an additional rule to resolve this type of problem: "If the identifier portion of one row may be absorbed by the identifier portion of a second row, the tag elements associated with the absorbed identifier must be altered to reflect its application to those functions to which the absorbing row applies." According to this rule, the second row in Example 5 would become 0000– – and the other two rows would remain unchanged.

Even with these two additional rules, however, the iterated consensus method still cannot be satisfactorily extended to multiple-output functions. In Example 6, no two rows have identical identifiers nor can any identifier be absorbed by another. Yet, two required prime implicants, 010– – and 001– –, are missing from the table.

Example 6

Identifier			*Tag*	
x_1	x_2	x_3	f_1	f_2
0	–	0	–	0
0	1	–	0	–
0	0	–	–	0
0	–	1	0	–
–	0	0	–	0
–	1	1	0	–

We now present one rule that permits successful extension of the iterated consensus method to multiple-output functions: Two rows generate an *intersection* (or *product*) row if there is no column in which one row has a 1 and the other a 0. The identifier of the intersection row, like that of a consensus row, has a dash in any column in which both of the original rows have dashes; it has a 1 in any column in which either of the two original rows has a 1; and it has a 0 in any column in which either of the two original rows has a 0. The tag of the intersection row, however, has a dash in any column in which either of the two original rows has a dash, and it has a 0 otherwise. If an intersection row does not subsume any row in the table, it is added to the table.

In Example 6 the first two rows generate the intersection row 010– –, and the third and fourth rows generate the intersection row 001– –. These two intersection rows are added to the table. (All other generated intersection rows subsume existing rows.)

It should be noted that a necessary condition for a consensus row to be of value is that the two original rows must have at least one tag column with dashes in both rows; otherwise the consensus row tag will consist of all 0's, implying that the identifier applies to no output. For an intersection row to be of value, it is necessary that the two original rows have at least one tag column with a dash in one row and a 0 in the other; otherwise the intersection row will subsume the original rows.

To summarize, the iterated consensus method applied to multiple-output functions operates as follows. All pairs of rows are systematically compared for subsumption, intersection and consensus. Any row that subsumes another is eliminated.

Two rows generate an intersection row if there is no column in which one row has a 1 and the other a 0. Two rows generate a consensus row if, in one column only, one row has a 1 and the other a 0. Each column of a generated row is defined from the two original rows as follows, with one exception.

Original rows			*Generated row*
0	1	→	–
–	–	→	–
1	1	→	1
–	1	→	1
0	0	→	0
–	0	→	0

The one exception is the generation of the tag of an intersection row:

–	0	→	–

The authors have found this algorithm for the iterated consensus method for multiple-output functions easily programmable in APL\360.

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