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Performance Equivalence of Suboptimally Controlled Nonlinear Systems

Abstract: A procedure is described that shows how a technique used to develop performance bounds for a large class of nonlinear dynamic systems with state-dependent control policies can be extended to determine whether a nonlinear system can be controlled so that it is at least "performance equivalent" to an associated optimally controlled linear system. A procedure for generating one or more control policies to attain this equivalence is also discussed. An example illustrates the fact that more than one control policy may satisfy the equivalence criterion.

Introduction: performance bounds

In order to circumvent many of the practical difficulties associated with the determination of optimal control policies for certain classes of nonlinear dynamic systems, many near-optimal control techniques have been suggested in the control literature (e.g., Refs. 1, 2 and 3). When considering any one of these techniques, one should, ideally, know a priori whether the system is stable under the control policy in question, and whether the resulting system performance can be bounded with meaningful and informative bounds or can be compared with the performance of another system.

There has been a continuing number of technical investigations that have addressed the general area of performance bounds for linear and nonlinear systems. Rekasius[2] investigated performance bounds associated with the suboptimal design of intentionally nonlinear controllers. Durbeck[1] developed upper and lower performance bounds to evaluate an approximation technique for suboptimal control of nonlinear systems. Rissanen[6] investigated the influence of system parameter changes on system performance, and Rissanen and Durbeck[4] generalized certain of their earlier work and derived bounds for systems associated with the so-called Lurie problem.

McClamrock and Aggarwal[7] investigated the existence of upper bounds on the performance index of nonlinear systems, and McClamrock[8] has studied suboptimality and sensitivity in the control and filtering of linear processes.

The purpose of this paper is to show how some of the techniques developed in Refs. 1, 2, 4 and 5 can be extended to determine whether a particular nonlinear system can be controlled so that its resulting performance is at least as good as an associated optimally controlled linear system.

The class of nonlinear time-invariant systems considered here may be described by*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{1a}$$

$$\mathbf{x}(0) = \mathbf{x}_0; \qquad \mathbf{f}(0, 0) = 0. \tag{1b}$$

Equations (1) denote the relationship between the state vector $\mathbf{x}(t)$ and the control vector $\mathbf{u}(t)$. The system is assumed to have an associated performance index (to be minimized)

$$J = \int_0^\infty L(\mathbf{x}, \mathbf{u})\, dt. \tag{2}$$

It is further assumed that the class of control policies $\mathbf{u}$ to be studied may be described in terms of the state vector $\mathbf{x}(t)$ [i.e., $\mathbf{u}(\mathbf{x})$; $\mathbf{u}(0) = 0$]. Therefore, the control policy is said to be a "state-dependent" or feedback control. It is also implied throughout that the scalar function $L(\mathbf{x}, \mathbf{u})$ and the vector function $\mathbf{f}(\mathbf{x}, \mathbf{u})$ are of class $C^{(2)}$, and that $L(\mathbf{x}, \mathbf{u})$ is positive definite.

In the following discussion, system performance bounds associated with arbitrary control policies $\mathbf{u}^*(\mathbf{x})$ are investigated; i.e., bounds on the resulting value of (J) are developed for all initial values of the state vector $\mathbf{x}_0$ in a region R defined in Euclidean space.

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* All vectors are assumed to be of finite dimension.

Define the cost functional (not assumed to be optimal) associated with the system (1) and an arbitrary control policy $\mathbf{u}^*(\mathbf{x})$ as

$$J^*(\mathbf{x}_0) \equiv \int_0^\infty L[\mathbf{x}, \mathbf{u}^*(\mathbf{x})]\, dt. \tag{3}$$

Without direct evaluation of (3) subject to (1) over a region R, the functional $J^*(\mathbf{x}_0)$ is usually very difficult or impossible to obtain except for very special systems. Assume that by any feasible technique (e.g., see Ref. 1), an approximation $V^*(\mathbf{x}_0)$ to the cost surface $J^*(\mathbf{x}_0)$ of class $C^{(2)}$ has been formulated. (In the next section, another means of selecting $V^*(\mathbf{x}_0)$ is suggested). In the discussion that follows, $V^*(\mathbf{x}_0)$ may be considered as any arbitrarily chosen positive definite function of $\mathbf{x}$. The choice is limited practically, however, by demands on such functions as described in the following.

Define $H^*(\mathbf{x})$ such that

$$H^*(\mathbf{x}) \equiv (\partial V^*/\partial \mathbf{x})^T \mathbf{f}[\mathbf{x}, \mathbf{u}^*(\mathbf{x})] + L[\mathbf{x}, \mathbf{u}^*(\mathbf{x})], \tag{4}$$

where T indicates the transposed matrix and $H^*(\mathbf{x})$ is analogous to the Hamiltonian based on the *minimal* cost functional $V^0(\mathbf{x}_0)$ associated with the *optimal* control policy.

Consider a closed region R_M in the space $R^{(n)}$ defined by $\|\mathbf{x}\| \leq M$ and $0 < M < \infty$, and define an arbitrary region $R_\gamma \subseteq R_M$ such that the function $\{H^*(\mathbf{x}) - L[\mathbf{x}, \mathbf{u}^*(\mathbf{x})]\}$ is zero for $\mathbf{x} = \mathbf{u}^* = \mathbf{0}$ and negative elsewhere in R_γ. Further, let $R_\lambda \subseteq R_\gamma$ be the closed region defined by $V^*(\mathbf{x}) \leq \lambda$, $\lambda > 0$. Then, if nonvacuous, R_λ is a region of asymptotic stability for system (1) subject to control policy $\mathbf{u}^*(\mathbf{x})$.

To determine if an arbitrary region $R_{\gamma'} \subseteq R_M$ possesses the properties required of R_γ, it is only necessary to find

$$\max_{\mathbf{x} \in R_{\gamma'}} \{H^*(\mathbf{x}) - L[\mathbf{x}, \mathbf{u}^*(\mathbf{x})]\}, \tag{5}$$

and ascertain whether a unique maximum occurs at $\mathbf{x} = \mathbf{0}$ with value 0. The search over $R_{\gamma'}$ may be done utilizing nonlinear programming techniques.

The above discussion furnishes sufficient background to develop several bounds associated with the system performance parameter J using the arbitrary control policy $\mathbf{u}^*(\mathbf{x})$.

Lemma 1: If with $\mathbf{u}(t) = \mathbf{u}^*[\mathbf{x}(t)]$, $t \geq 0$, the inequality

$$-\rho' L[\mathbf{x}, \mathbf{u}^*(\mathbf{x})] \leq H^*(\mathbf{x}) \leq \rho L[\mathbf{x}, \mathbf{u}^*(\mathbf{x})] \tag{6}$$

is valid for *all* $\mathbf{x} \in R_\lambda$, $\rho' \geq 0$, $1 > \rho \geq 0$, then the resulting system performance indicator $J^*(\mathbf{x}_0)$ may be bounded by

$$V^*(\mathbf{x}_0)/(1 + \rho') \leq J^*(\mathbf{x}_0) \leq V^*(\mathbf{x}_0)/(1 - \rho) \tag{7}$$

for all $\mathbf{x}_0 \in R_\lambda$.

The proof of this lemma was first given by the author in Ref. 1 and generalized in Refs. 4 and 5. Note that the "tightness" of the bounds expressed in (7) depends on the relative magnitudes of $H^*(\mathbf{x})$ and $L[\mathbf{x}, \mathbf{u}^*(\mathbf{x})]$. Several examples of such "performance-bounded" regions $R_\lambda(\rho, \rho')$ are given in Ref. 1 for a specific system.

The limiting case of this lemma is of much interest. Assume that our control policy is given by

$$\mathbf{u}^*(\mathbf{x}) \ni H^*(\mathbf{x}) = 0 \tag{8}$$

for all $\mathbf{x} \in R_\lambda$. If this is possible for $\mathbf{u}^*(\mathbf{x})$ real, the performance of the system can be given a priori as

$$J^*(\mathbf{x}_0) = V^*(\mathbf{x}_0). \tag{9}$$

This result follows directly from (7) where $\rho, \rho' \longrightarrow 0$. Thus, for this particular control policy, which yields a control law that may not be unique, the performance of the system is identically equal to the proposed cost function $V^*(\mathbf{x})$.

A necessary, but not sufficient, condition that there exists a realizable $\mathbf{u}^*(\mathbf{x}) \ni H^*(\mathbf{x}) = 0$ for all $\mathbf{x} \in R_\lambda$ is that $V^*(\mathbf{x}) \geq V^0(\mathbf{x})$ for all $\mathbf{x} \in R_\lambda$, where $V^0(\mathbf{x})$ is the *minimal* cost functional associated with the *optimal* control policy.

Clearly, however, a control policy given by

$$\mathbf{u}^{**}(\mathbf{x}) \ni H^*(\mathbf{x}) \leq 0; \qquad \mathbf{x} \in R_\lambda \tag{10a}$$

is superior to the control policy

$$\mathbf{u}^*(\mathbf{x}) \ni H^*(\mathbf{x}) = 0; \qquad \mathbf{x} \in R_\lambda, \tag{10b}$$

since with control policy (10a),

$$J^*(\mathbf{x}_0) = \left\{ V^*(\mathbf{x}_0) + \int_0^\infty H^*[\mathbf{x}(t)]\, dt \right\} \leq V^*(\mathbf{x}_0). \tag{11}$$

One further comment should be made concerning control policy (10b). A control law satisfying this nonoptimal policy may not be unique, i.e., there may be two or more control laws and, hence, trajectories from the same point $\mathbf{x}_0 \in R_\lambda$ which will yield identical performance. For example, one trajectory may rapidly converge to the desired operating point while expending a great amount of control energy (cost), and another may converge slowly while using only a small amount of control energy. The simple example described below gives several illustrations of these two types of trajectories.

Performance equivalent systems

This section deals with the problem of defining a control policy associated with a nonlinear system such that the resulting controlled system is "performance equivalent" to a particular optimally controlled associated linear system. The exact meaning of the term "performance equivalent" will be made clear in the following.

The nonlinear system with state vector of dimension n described by (1) will again be considered. However, now the system performance index is restricted to the class of a separable quadratic criterion, i.e.,

$$J = \int_0^\infty [||\mathbf{x}(t)||^2_{\mathbf{Q}_1} + ||\mathbf{u}(t)||^2_{\mathbf{Q}_2}]\, dt, \quad (12)$$

where $\mathbf{Q}_1$, $\mathbf{Q}_2$ are assumed to be positive definite. In addition, an associated linear system (also with state dimension n) is defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad (13)$$

with performance criterion (12). The linear system (13) is assumed to be controllable.

Define $V^*_\ell(\mathbf{x}_0)$ such that

$$V^*_\ell(\mathbf{x}_0) = \min_{\mathbf{u}(t)} \left\{ \int_0^\infty [||\mathbf{x}(t)||^2_{\mathbf{Q}_1} + ||\mathbf{u}(t)||^2_{\mathbf{Q}_2}]\, dt \right\}, \quad (14)$$

subject to the linear system relation (13). It is well known that

$$V^*_\ell(\mathbf{x}_0) = ||\mathbf{x}_0||^2_{\mathbf{\Theta}}, \quad (15)$$

where $\mathbf{\Theta}$ is the positive definite solution to the matrix equation

$$\mathbf{Q}_2 + \mathbf{\Theta}^T\mathbf{A} + \mathbf{A}^T\mathbf{\Theta} - \mathbf{\Theta}^T\mathbf{B}\mathbf{Q}_1^{-1}\mathbf{B}^T\mathbf{\Theta} = 0. \quad (16)$$

Lemma 2: If there exists a region R_λ in the space $R^{(n)}$ defined by $||x||^2_{\mathbf{\Theta}} \leq \lambda$, $\mathbf{x} \in R_\lambda$, such that for realizable $\mathbf{u}(\mathbf{x})$,

$$\min_{\mathbf{u}(\mathbf{x})} [||\mathbf{x}||^2_{\mathbf{Q}_1} + ||\mathbf{u}||^2_{\mathbf{Q}_2} + (\partial V^*_\ell/\partial \mathbf{x})^T\mathbf{f}(\mathbf{x}, \mathbf{u})] \leq 0, \quad (17a)$$

and

$$\max_{\mathbf{u}(\mathbf{x})} [||\mathbf{x}||^2_{\mathbf{Q}_1} + ||\mathbf{u}||^2_{\mathbf{Q}_2} + (\partial V^*_\ell/\partial \mathbf{x})^T\mathbf{f}(\mathbf{x}, \mathbf{u})] \geq 0 \quad (17b)$$

for all $\mathbf{x} \in R_\lambda$, then:

(1) There exists at least one realizable control policy $\mathbf{u}^*(\mathbf{x})$ such that the system performance described by (12) for the nonlinear system (1) resulting from $\mathbf{u}^*(\mathbf{x})$ is equal to $||\mathbf{x}_0||^2_{\mathbf{\Theta}}$ for all initial states $\mathbf{x}_0 \in R_\lambda$.

2) A control policy to achieve this performance equivalence is given by

$$\mathbf{u}^*(\mathbf{x}) \ni [||\mathbf{x}||^2_{\mathbf{Q}_1} + ||\mathbf{u}^*||^2_{\mathbf{Q}_2} + (\partial V^*_\ell/\partial \mathbf{x})^T\mathbf{f}(\mathbf{x}, \mathbf{u}^*)] = 0 \quad (18)$$

for all $\mathbf{x} \in R_\lambda$. Note that there may be more than one control algorithm satisfying (18), as illustrated in the example below, and that the control law will typically be nonlinear.

Proof: Since $\mathbf{f}(\mathbf{x}, \mathbf{u})$ is assumed to be of class $C^{(2)}$ the bracketed expression in (17) is a continuous function of $\mathbf{u}$ for all $\mathbf{x} \in R_\lambda$. Thus if conditions (17) are met, there will always exist a $\mathbf{u}(\mathbf{x})$ for all $\mathbf{x} \in R_\lambda$ so that (18) is satisfied. Since $L(\mathbf{x}, \mathbf{u}) \equiv ||\mathbf{x}||^2_{\mathbf{Q}_1} + ||\mathbf{u}||^2_{\mathbf{Q}_2}$ is positive definite, with control policy (18),

$$\dot{V}^*_\ell = (\partial V^*_\ell/\partial \mathbf{x})^T\mathbf{f}(\mathbf{x}, \mathbf{u}^*(\mathbf{x}))$$

is clearly negative definite in R_λ and, therefore, the system is asymptotically stable in R_λ. In addition, by (18), $\dot{V}^*_\ell = -L[\mathbf{x}, \mathbf{u}^*(\mathbf{x})]$ and therefore,

$$J^*_{\mathfrak{N}\ell}(\mathbf{x}_0) = V^*_\ell(\mathbf{x}_0)\, ; \qquad \mathbf{x}_0 \in R_\lambda, \quad (19)$$

where $J^*_{\mathfrak{N}\ell}(\mathbf{x}_0)$ is the nonlinear system performance using control policy (18) and $V^*_\ell(\mathbf{x}_0)$ is the optimal linear system performance.

It is of interest to note that the optimal cost functional $V^*_\ell(\mathbf{x}_0)$ for the linear system may also serve as the approximation $V^*(\mathbf{x}_0)$ to the cost function $J^*(\mathbf{x}_0)$ for the nonlinear system (1) subject to the arbitrary control policy $\mathbf{u}^*(\mathbf{x})$. Thus, as described above, if the inequality (6) can be established for $V^*(\mathbf{x}_0) = V^*_\ell(\mathbf{x}_0)$, $\rho' \geq 0$ and $1 > \rho \geq 0$, the performance $J^*(\mathbf{x}_0)$ may also be bounded by (7). This alternative way of approximating $J^*(\mathbf{x}_0)$ is of importance because of the general difficulty in obtaining a suitable approximation (see Refs. 1, 4 and 7).

A simple example is presented to illustrate the ideas expressed in this section. Let the nonlinear system be represented by

$$\dot{x}_1 = \tanh(x_2)$$
$$\dot{x}_2 = x_1 + x_2 + u, \quad (20)$$

which exhibit saturation. Let the associated linear system be represented by

$$\dot{\mathbf{x}} = \beta(\mathbf{A}\mathbf{x} + \mathbf{b}u), \quad (21)$$

where β is a positive scalar and

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{b}^T = (0, 1). \quad (22)$$

For $\beta = 1.0$, the associated linear system approximates the nonlinear system very well for $|x_2| < 0.8$. For larger values of $|x_2|$ the nonlinear system is near saturation; for $|x_2| > 2.0$, $\dot{x}_1$ becomes essentially invariant, with a value close to ± 1.0. For $\beta < 1.0$, the response of the linear system becomes more sluggish, and tends to roughly approximate the nonlinear system for large values of $|x_2|$.

It is assumed here that the performance criterion common to both systems is†

$$J = \int_0^\infty \left(x_1^2 + u^2\right) dt. \quad (23)$$

† It should be noted that the performance criterion used for this example does not satisfy one of the conditions associated with (12); here $\mathbf{Q}_1$ is not positive definite. $V_\ell(\mathbf{x}_0)$ is still given by (15) but $\mathbf{\Theta}$ can not be obtained by direct solution of the matrix equation (16). Instead, $\mathbf{\Theta}$ may be determined simply from the simultaneous algebraic expressions arising from the optimality conditions for the Hamiltonian functional associated with the system (21). These algebraic expressions would be given by (16) if $\mathbf{Q}_1$ were positive definite.

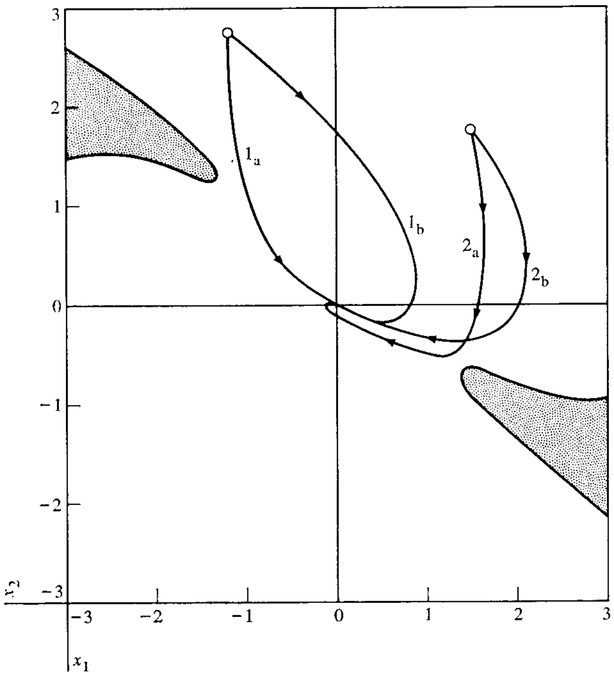

Figure 1 Comparison trajectories with $\beta = 0.8$. The two trajectories starting from each of the two sets of initial conditions shown yield identical performances, respectively. This result will hold for all initial conditions for which the trajectories are contained within the unshaded region.

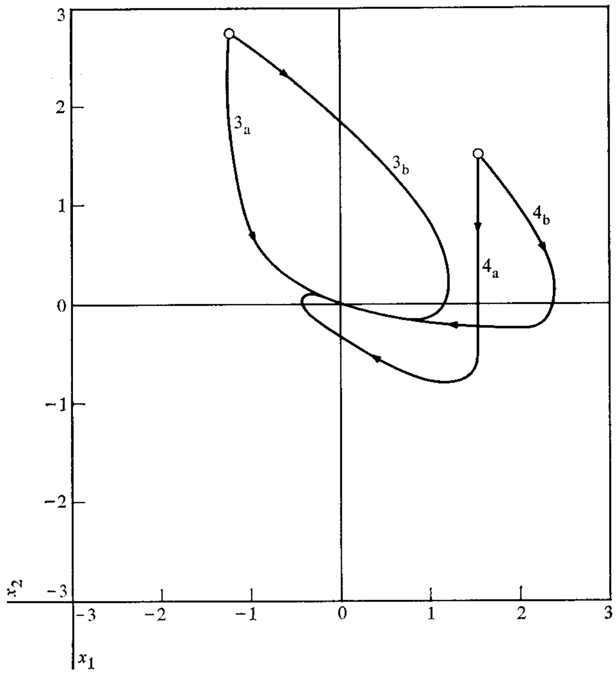

Figure 2 Comparison trajectories with $\beta = 0.4$. Here there is no shaded region and performance equivalence can be attained for all trajectories contained within the area shown.

Then it may be easily shown that Θ is given by

$$\Theta = \{\theta_{ij}\}$$
$$= \frac{1}{\beta}\begin{bmatrix} -1 + \sqrt{6 + 4\sqrt{2}} & 1 + \sqrt{2} \\ 1 + \sqrt{2}, & 1 + \sqrt{3 + 2\sqrt{2}} \end{bmatrix}. \tag{24}$$

The control policy $u(t)$ is selected so that

$$H^*(\mathbf{x}, u) \equiv x_1^2 + u^2 + (\theta_{11}x_1 + \theta_{12}x_2)\tanh(x_2) + (\theta_{12}x_1 + \theta_{22}x_2)(x_1 + x_2 + u) = 0. \tag{25}$$

For $\beta = 0.8$, Fig. 1 shows a region (unshaded) where $u(t)$ may be selected such that (25) is satisfied. Two sets of trajectories with initial conditions $(x_1, x_2) = (-1.218, 2.760)$ and $(1.600, 1.600)$ respectively, are also shown. At each point in the unshaded region there are two control laws that satisfy the policy given by (25) and, hence, two trajectories for each initial condition. Figure 2 is similar, except that policy (25) may be attained over the entire region shown ($\beta = 0.4$). In each case the trajectories identified with the subscript "a" converged more quickly to the origin while initially expending more control energy than those identified with the subscript "b", which converged more slowly. With digital simulation techniques, the performance factor associated with each trajectory pair "a, b" starting at $\mathbf{x}_0$ was found to differ by no more than 0.1% from $V_l^*(\mathbf{x}_0)$, i.e., the *optimal* performance associated with the linear system (21) to (23); the difference must be attributed to numerical integration errors.

Summary

A method has been developed to determine whether a nonlinear system can be controlled so that it is at least performance equivalent to a comparable optimally controlled linear system of the same order. The optimal performance of the linear system then serves as a simple upper bound on the attainable performance of the nonlinear system. It is also shown that the control policy to attain this equivalence may not be unique.

The minimum cost functional for the linear system may also serve as a suitable approximation to the cost functional for the nonlinear system; in this case more informative upper and lower bounds on the nonlinear system performance may be established using Lemma 1.

The difficulty of finding suitable approximations to the cost functional for nonlinear systems increases quickly with increasing system order. The concept of using the performance of a comparative linear system of the same order aids substantially in this regard. The general problem of selecting a comparative linear system which will yield useful bounds has not been specifically addressed here; more work concerning this problem is needed.

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