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An Analysis of the Effectiveness of Hybrid Transmission Schemes

Abstract: A comparison is made of the performance of pure retransmission, forward error correction and hybrid (error detecting/correcting) schemes for data transmission in a noisy (probability of error, $P > 10^{-4}$) binary symmetric channel. The performance calculations are based on the use of BCH codes for error detection and correction up to the full correction capability of the code. It is shown that a probability of undetected error of less than 10^{-9} error/bit, can be achieved by correcting only a few errors while retaining a reasonable throughput and a very low retransmission rate. The best codes in the class considered are specified and the complexity of instrumentation is estimated. Finally, various combinations of possible systems employing half duplex and reverse channel operation are used in a comparison of the transmission schemes. For line error rate worse than 10^{-4} error/bit, a hybrid system operating with a reverse channel is superior to the other possibilities.

Introduction

In most of the applications of data transmission for computer to computer or computer to terminal communication, a very high reliability is needed; i.e., the probability of information-bit error must be less than 10^{-9}. Depending upon the channel error rate, two basic approaches, both using a feedback channel, allow the achievement of this high reliability. The first approach uses error detection combined with retransmission on request. This method has been used for many years with different error detection schemes (e.g., horizontal redundancy checking, horizontal and vertical redundancy checking, and the use of a cyclic code) and with different retransmission procedures.[1,2] Retransmission is attractive because it requires only a minimum of hardware, but it is efficient only for channel error rates less than 10^{-4} as in low-speed voice channels.

The second approach is the hybrid transmission scheme, which uses retransmission combined with partial error correction to reduce the number of retransmission requests. This method requires a larger amount of hardware for performing the error correction; it will be shown that it becomes attractive when the channel error rate becomes large (10^{-4} to 10^{-1}), as happens, for example, on a voice channel when the number of levels is increased in order to transmit at higher rates.[3,4] In that case, the error correcting scheme has to be carefully chosen in order to have a code rate high enough to take advantage of the improvement in the data rate.

Three approaches are possible for designing an error correction scheme. They all suppose a good knowledge of the burst error characteristic of the channel. The first two approaches consist of using a random error correcting code after proper randomization of the errors (by interleaving, for example). The first method uses block codes, the second, convolutional codes and the third, burst error correcting codes.

We will limit the paper to a detailed study of the first approach, i.e., the use of block codes for error correction on channels with high error rates, and we assume that an efficient randomizing device has been designed. Answers will be given to the four basic questions:

1) What is the maximum transmission rate compatible with a maximum probability 10^{-9} of undetected information-bit error?
2) What is the influence of partial error correction on the retransmission rate?
3) What is the influence of the channel error rate on the probability of undetected error and on the transmission rate?
4) What is the influence of the channel error rate on the probability of undetected error when using forward error correction (no retransmission)?

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Basic assumptions

The technique used for evaluating transmission schemes in this paper consists of grouping the possible error patterns into four subsets according to their effect on the decoding: no error, correct decoding, decoding failure (retransmit) and undetected decoding error. To evaluate the various probabilities, it is assumed that

1) the bit errors are caused by a memoryless, binary symmetric channel (BSC), and
2) the standard array of the codes has a homogeneous distribution of n-tuple Hamming weights among the rows in which such weights can occur.

Classification of the error patterns

A linear, systematic, binary block code can be specified by its parity check matrix $\mathbf{H}$:

$$\mathbf{H} = \begin{bmatrix} \mathbf{P} \\ \mathbf{I}_{n-k} \end{bmatrix}, \tag{1}$$

where $\mathbf{P}$ is an arbitrary $k \times (n-k)$ matrix defining the code and where $\mathbf{I}_{n-k}$ is the identity matrix of order $(n-k)$. When an n-tuple is received, represented by an n-place row vector $\mathbf{r}$, the first step of the decoding is to determine the syndrome vector $\mathbf{s}$:

$$\mathbf{s} = \mathbf{r}\mathbf{H}.$$

The vector $\mathbf{s}$ is an $(n-k)$-place row vector, so that there are 2^{n-k} possible syndromes, one of which is the all-0 syndrome; the corresponding n-tuples are the 2^k code words. Given an n-tuple that is not a code word, all the n-tuples corresponding to the same syndrome are obtained by adding a codeword to the n-tuple. In that manner, given a linear code of block length n, the set of the 2^n possible n-tuples can be broken down into 2^{n-k} disjoint subsets corresponding to the 2^{n-k} possible syndromes. The set can be represented as a rectangular array, the *standard array*, with 2^{n-k} rows and 2^k columns. Each row contains all the n-tuples corresponding to one particular syndrome. The first row contains the 2^k code words. The leftmost column contains the "coset leader," i.e., the minimum weight n-tuple of the row. If the code can correct up to t errors, all the n-tuples with weight less than or equal to t are coset leaders.

Taking another point of view, one may consider the standard array as representing all the possible error patterns. Then it may be divided into four subsets (Fig. 1):

1) Subset I contains the all-0 n-tuple only. It corresponds to an *error-free transmission.*
2) Subset II contains all the coset leaders of weight less than or equal to the maximum error correcting capability θ used for the code: $0 \leq \theta \leq t$; i.e., the code could correct up to t errors but is used to correct only up to θ errors.

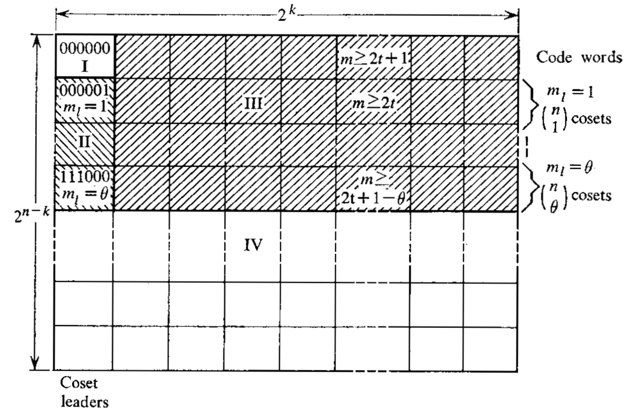


Figure 1 Error pattern array.

The only error patterns corrected by the code are those of subset II. Each time such an error pattern occurs there is a *correct decoding.*
3) Subset III contains all the n-tuples (except the coset leaders) whose coset leaders are in Subsets I and II. Each time such an error pattern occurs there is a *decoding error,* for the decoding algorithm takes as an error pattern corresponding to a given syndrome the most likely n-tuple of the coset, i.e., the coset leader (minimum weight). When a decoding error occurs, the number of undetected errors in the block after decoding may be equal to the number of errors introduced during transmission increased by the weight of the coset leader; i.e., the minimum number of undetected errors per block is, in most cases, $2t+1$ independently of θ.
4) Subset IV contains all the other cosets. Each time an error pattern of Subset IV occurs, there is a *decoding failure*; we will assume that in that case retransmission is requested. For a "close-packed" or "perfect" code, Subset IV is empty when $\theta = t$.

Weight distribution

Since the code is supposed capable of correcting up to t errors, the minimum weight of the code words is $2t+1$. In the cosets in which the coset leader has weight m_l, the minimum weight of the n-tuple is at least $2t+1-m_l$, for each n-tuple of the coset is the sum of the coset leader and of the code words.

We make the following hypothesis concerning the weight distribution of the n-tuples: *The n-tuples of weight m are uniformly distributed in the standard array within the rows in which they can exist.* In the particular case of interest here, i.e., making an estimate of the probability of decoding error, this hypothesis is a good approximation for two reasons:

1) As is well known,[5] given a certain bit position in the code word, the bit in this position is a ONE for 2^{k-1} of

the code words and a ZERO for the other 2^{k-1}. This also applies within each coset. Thus, the average weight of all cosets is the same, arguing for a horizontal and vertical homogeneity of the weight distribution. As a consequence of this hypothesis, the weight distribution of the code words is a binomial distribution, since the total number of n-tuples of weight m is $\binom{n}{m}$. In fact, it has been verified that the actual distribution for BCH codes is very close to the binomial distribution.[6]*

2) The number of coset leaders of weight m_t is $\binom{n}{m_t}$; i.e., the number increases rapidly when the number of errors that can be corrected increases and at the same time the minimum weight $(2t + 1 - m_t)$ of the n-tuple in the corresponding coset decreases. Since the most likely error patterns are those with lower weight,† the most important contribution to the probability of decoding error is that of the cosets containing high-weight coset leaders. Due to the relatively large number of these cosets, the hypothesis of uniform weight distribution becomes more accurate.

- *Error characteristics after decoding*

The decoding after one transmission of a block of n bits can be characterized by the following probabilities:

P_O = probability of no error (error pattern of Subset I)
P_C = probability of correct decoding (Subset II)
P_U = probability of undetected error (Subset III)
P_D = probability of decoding failure (Subset IV); this equals the probability of requesting a retransmission in a hybrid system (both error correction and retransmission).

These probabilities can be expressed as a function of the probability $P(m, n)$ of having exactly m errors in a block of n bits. For a memoryless, binary symmetric channel with error rate P:

$$P(m, n) = \binom{n}{m} P^m (1 - P)^{n-m}. \tag{2}$$

Hence, for a $(n, k; t, \theta)$ block code‡, (see Appendix A),

$$P_O = P(0, n), \tag{3}$$

$$P_C = \sum_{m=1}^{\theta} P(m, n), \tag{4}$$

$$P_U = \sum_{m=2t+1-\theta}^{2t+1} P(m, n) \frac{\sum_{u=2t+1-m}^{\theta} \binom{n}{u}}{2^{n-k} - \sum_{q=0}^{2t-m} \binom{n}{q}} \tag{5}$$

* Compare with the weight enumerators given in Ref. 7.
† Since, to reach $P_i < 10^{-9}$, $t > nP$ must hold.
‡ The notation $(n, k; t, \theta)$ specifies the block length n; the number k of information bits (where $n - k$ is the number of check bits); the maximum number t of errors that can be corrected; and the effective number θ of errors that will be corrected by the decoder.

$$P_D \approx 1 - P_O - P_C - P_U. \tag{6}$$

In a hybrid system, the interesting parameter is the probability of undetected block error P_{UB} including retransmissions:

$$P_{UB} = P_U + P_U P_D + P_U P_D^2 + \cdots$$
$$= \frac{P_U}{1 - P_D}. \tag{7}$$

Knowing the probability of undetected block error P_{UB}, we can compute the probability of undetected bit error P_b, and the probability of undetected information-bit error P_i from the following (see Appendix B):

$$P_b \approx \frac{2t + 1}{n} P_{UB}, \tag{8}$$

$$P_i \approx \frac{(2t + 1)k}{n^2} P_{UB}. \tag{9}$$

Finally, the throughput rate of a hybrid transmission scheme is given by

$$R \equiv \frac{\text{number of information bits transmitted in a block}}{\text{total number of bits transmitted before accepting the block}}$$
$$= \frac{k}{n + n(P_D) + n(P_D)^2 + \cdots}$$
$$= \frac{k}{n}(1 - P_D), \tag{10}$$

where an ideal retransmission scheme is assumed; i.e., there is assumed to be no retransmission delay (line delay, modem turn-around time). As will be discussed in the last section, this rate can be reached with a full duplex low-speed return channel if confirmation and retransmission requests are to be transmitted after each block. When only a half duplex channel is available, it can be attained by transmitting the confirmation and retransmission requests for a large number of blocks at a time. When this logic cannot be implemented, n has to be replaced by $n + d + s$ in relation (10), where d is the retransmission delay and s the length of the return message:

$$R' = \frac{k}{n + d + s}(1 - P_D).$$

Furthermore, it is assumed that transmission in the feedback channel is error free. This is a reasonable hypothesis since the feedback information rate is generally very low.

Results

The relations in the preceding section permit a precise and practical evaluation of a hybrid retransmission scheme (error correction plus transmission) that uses block codes. Values of these relations have been computed for actual codes. The numerical application covers two particular cyclic codes: the primitive BCH codes,[8,9] which are the most powerful random-error correcting codes known, and the Golay code,[10] a nonprimitive BCH code, which is one of the rare "perfect" binary codes known. Both codes are quite close to the Hamming-Plotkin-Elias bound;[11] i.e., they almost reach the maximum efficiency that can be attained with block codes for the block lengths considered here. Computations cover the range of block length from 15 to 511 bits and the range of BSC error rate from 10^{-1} to 10^{-4}.

- *Maximum transmission rate with BCH codes*

It is interesting to compare the normalized capacity C of the memoryless BSC channel,

$$C = 1 + P \log_2 P + (1 - P) \log_2 (1 - P) \quad \text{bits/channel use,} \tag{11}$$

with the maximum transmission rate that can be reached with BCH codes. (In doing this, we neglect the complexity of the decoding.) In Fig. 2, a probability of undetected information-bit error, $P_i \leq 10^{-9}$, is assumed. For a given P, the maximum rate increases slowly with the block length n, but remains far from the channel capacity; indeed, the higher the error rate, the further from capacity it remains. Each point on the diagram represents the coding schemes giving the largest throughput rate for a given block length. The first number associated with each point gives the number θ of errors effectively corrected; the second number t gives the maximum error correcting capability of the code. It is interesting to note that in most of the cases in which $P > 10^{-4}$ the higher rates are obtained with a hybrid transmission scheme. For $P \leq 10^{-4}$, error detection with retransmission is the best scheme.

Practically, the complexity of the decoder[3,7,12] limits the number of different values of θ to three or four; Table 1 shows the corresponding maximum throughput rate of a hybrid transmission scheme for several values of channel probability of error P, such that the probability P_i of undetected information-bit error is less than or equal to 10^{-9}.

- *Probability of retransmission*

In a hybrid transmission scheme the role of error correction is to improve the transmission rate by reducing the number of retransmission requests. Figures 3(a)-(d) show how the probability of retransmission varies as a function of channel error rate where the number θ of errors corrected is a parameter. For a given block length and a given number of corrected errors, the curves obtained are roughly independent of the code, for in the expression

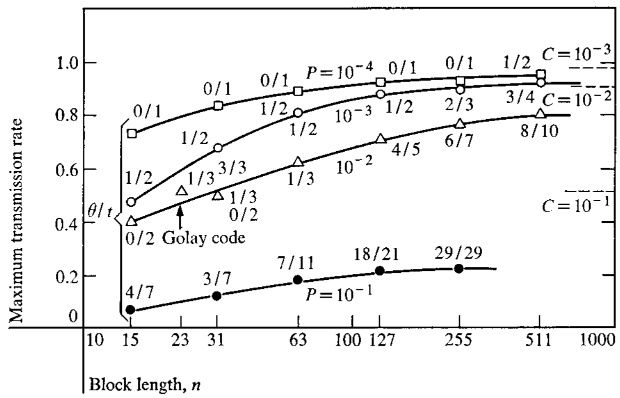
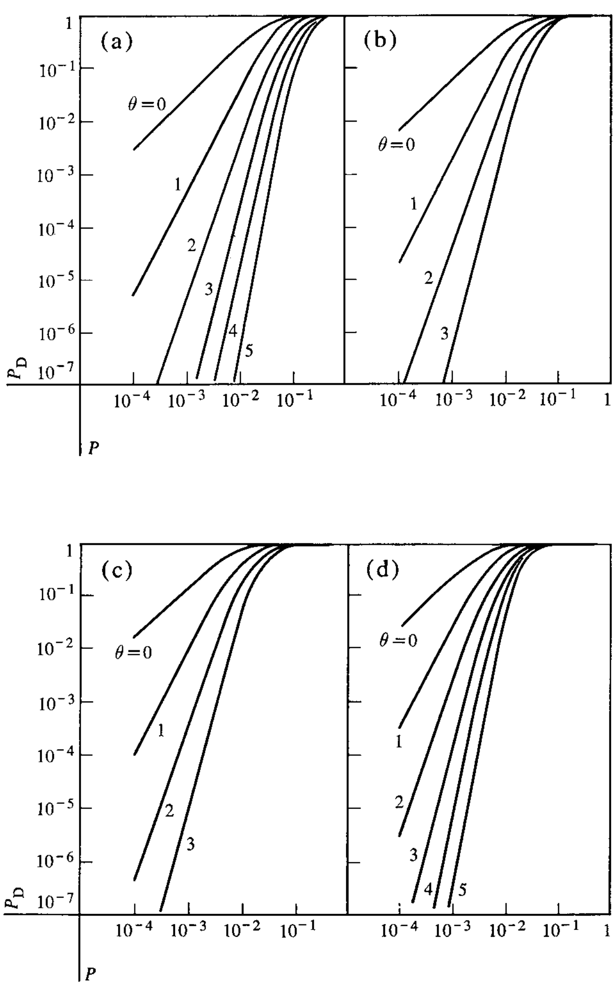


Figure 2 Maximum transmission rate vs block length for codes with a feedback channel, where probability of information-bit error after decoding $P_i \leq 10^{-9}$. Curves correspond to channel error probability $P = 10^{-1}$, 10^{-2}, 10^{-3} and 10^{-4}.

Figure 3 Probability of retransmission P_D vs error probability P of the memoryless binary symmetric channel for several block lengths n. (a) $n = 31$; (b) $n = 63$; (c) $n = 127$; and (d) $n = 255$.



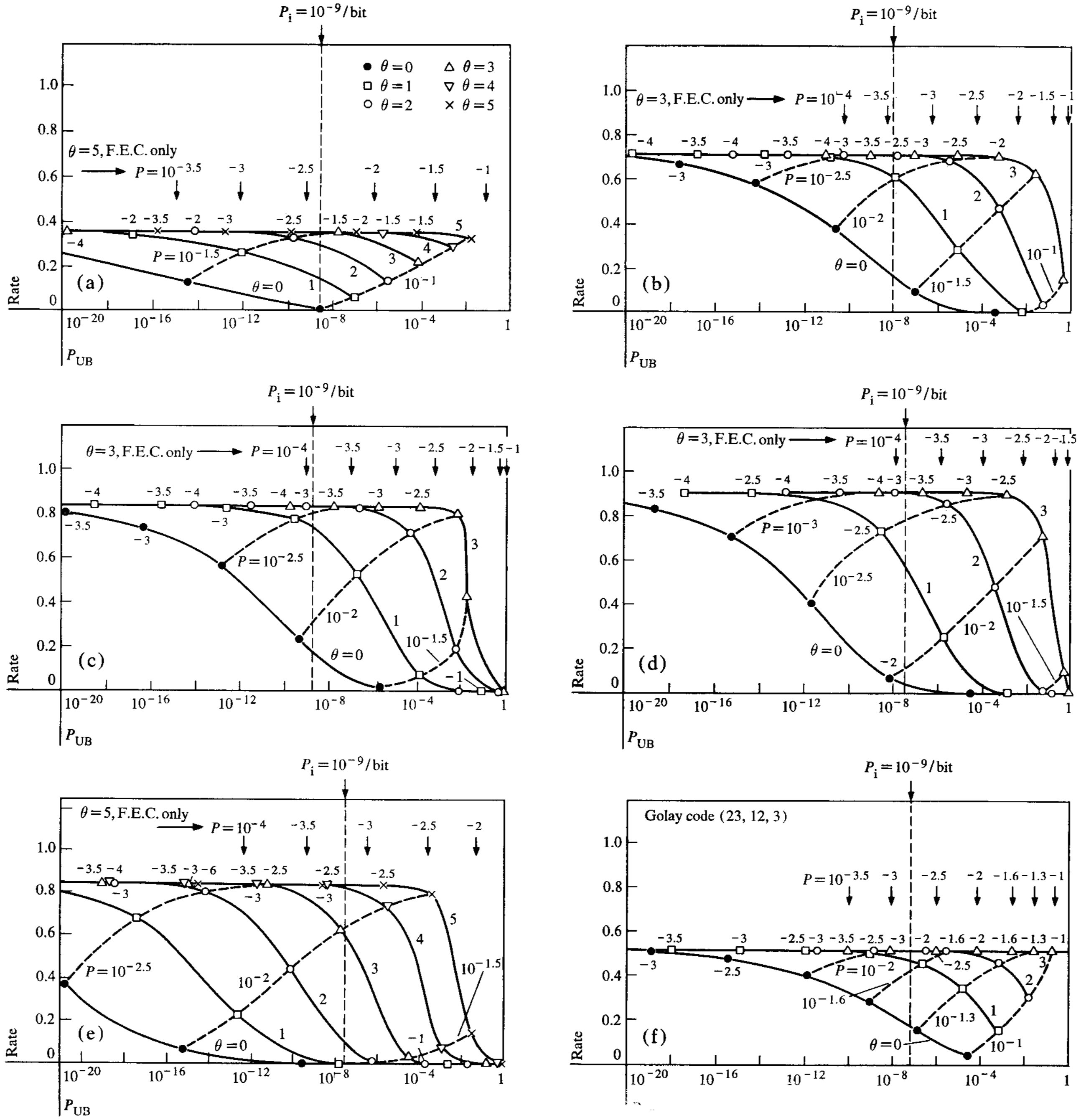


Figure 4 Performance of several linear block codes. (a) (31, 11; 5) BCH code; (b) (63, 45; 3) BCH code; (c) (127, 106; 3) BCH code; (d) (255, 231; 3) BCH code; (e) (255, 215; 5) BCH code; (f) (23, 12; 3) Golay code. The solid curves correspond to a given number θ of errors effectively corrected and the dashed curves correspond to a given channel probability of error P. The vertical arrows give the probability of undetected block error P_{UB} for the corresponding value of P in the absence of retransmission. The legend showing the value of θ for the various shapes of data points is given in (a), but is the same for all other parts of the figure. The negative numbers on the figure are $\log_{10}$ of P.

of P_D the only term depending on the code is P_U, which is generally much smaller than $1 - P_O$ and P_C. Taking, for example, the (63, 45; 3, 1) code for $P = 10^{-2}$, we see in Fig. 3(a) that 13 percent of the blocks contain errors after correction of one error; in fact, only 6 out of 18 check bits are used for the one-error correction. The other 12 are used for error detection to achieve, with retransmission, a probability of undetected error of 10^{-9}.

These curves can be used for another purpose: to give the order of magnitude of the probability of undetected

Table 1 Maximum throughput rate for hybrid transmission.

P	Rate	Typical code $(n, k; t, \theta)$
10^{-1}	0.13	(31, 6; 7, 3)
10^{-2}	0.62	(63, 45; 3, 1)
10^{-3}	0.92	(511, 475; 4, 3)
10^{-4}	0.96	(511, 493; 2, 1)

error in a forward-error-correction (FEC) transmission scheme using no retransmission. In that case the maximum error correcting capability of the code has to be used. For example, with the (63, 45; 3, 3) code in Table 1, the error rate at $P = 10^{-2}$ without retransmission is $P_i \approx 7.9 \times 10^{-4}$.

- *Performance of a code as a function of the channel error rate*

When a hybrid transmission scheme is used on a channel with large error rate, P, the transmission rate depends drastically on the channel error rate. Figures 4(a)-(f) show how the rate and the block probability of undetected error vary with P for different codes capable of correcting from 0 (pure retransmission) up to t errors. Taking, for example, the (63, 45; 3, 1) code, we can see in Fig. 4(b) that, for single-error correction, the rate drops very rapidly as P goes from 10^{-2} to $10^{-1.5}$. For P less than $10^{-2.5}$, the transmission rate becomes close to k/n and is thus independent of the channel error rate because the percentage of retransmission requests can be neglected. The vertical dashed line in the middle of the figures indicates an information bit probability of undetected error $P_i = 10^{-9}$. Finally, a series of vertical arrows indicates the block error rate when the full error correcting capability of the code ($\theta = t$) is used without retransmission (i.e., when forward error correction is used). For codes correcting $\theta = t = 3$ or more errors, retransmission does not improve the block error probability very much (only one order of magnitude). Obviously, no improvement is observed with the Golay code [Fig. 4(f)], since Subset IV is empty.

The above performance evaluation of block random error correcting codes indicates that, for the range of channel probability of error greater than 10^{-4} (as in a high-data-rate modem on a voice-grade line), a hybrid transmission scheme, which incorporates both some error correction and error detection with retransmission, is superior to both pure error detection with retransmission and forward error correction.

An important practical result is that only a few errors need to be corrected to reach very high reliability while still retaining a reasonable throughput. That is, even for a large block length, only a small portion of the error correcting capability of the code needs to be used. In fact, the error correction is used to bring the channel reliability up to the point where the throughput rate is not degraded by successive retransmissions.

Discussion

To compare the performance of the hybrid transmission scheme with that of the two other schemes, pure retransmission and forward error correction, a numerical example is given. The following system parameters are assumed:

- transmission bit rate = 4800 bit/sec,
- average transmission delay + turn around = 300 msec or 1440 bits,
- binary symmetric channel error rate = 10^{-3}. *

The practical implementation of retransmission depends 1) on the nature of the feedback channel available (half duplex, reverse channel), and 2) on the retransmission algorithm. Three typical cases of retransmission (A, B and C) are considered as indicated in Table 2. In Case A each block of data is acknowledged separately before retransmission or transmission of the next block. In Case B the blocks are grouped into superblocks; i.e., after transmission of a superblock, the transmitter is notified which blocks are in error. Two hybrid schemes are considered: Case D, with half duplex and superblocks and Case E, with a reverse channel. Finally, in Case F, forward error correction is assumed and the same code is used as in the hybrid schemes, Cases D and E.

Table 2 gives a general comparison of the six different schemes. In each case but Case F, the designed probability of information-bit errors is $P_i \leq 10^{-9}$. The codes chosen are those having the highest information-bit rate R consistent with an error rate better than 10^{-9}.

For the constraints of this example, the system of Case E (hybrid + reverse channel) is superior to the others. Comparison of E with C (retransmission + reverse channel) shows a rate improvement of 15 percent with a drastic reduction in the retransmission rate. Comparison of E with F (forward error correction) shows a reliability improvement of four orders of magnitude though it is necessary to consider the cost of a reverse channel versus that of a more complex decoder.

The results are based on the assumption that the channel errors are randomly distributed. This is not normally a good assumption for a telephone channel unless, as previously mentioned, sufficient error randomization has been introduced. The randomizer memory size (4000 to 8000 bits) is comparable to the bit content of the superblocks; thus, only one memory need be used to accomplish both randomization and block grouping. Consequently, approximately the same amount of storage is needed in each of

* This rate represents the third quartile of the data given in Ref. 4.

Table 2 Comparison of transmission schemes (channel bit rate = 4800 bits/sec, channel error rate $P = 10^{-3}$, turn-around time = 300 msec or 1440 bits).

	Retransmission			*Hybrid*		*Forward error correction*
	Case A	*Case B*	*Case C*	*Case D*	*Case E*	*Case F*
	Half duplex	*Half duplex with superblocks*	*Reverse channel with superblocks*	*Half duplex with superblocks*	*Reverse channel with superblocks*	
Code	(127, 113; 2, 0)	(127, 113; 2, 0)	(127, 113; 2, 0)	(255, 231; 3, 2)	(255, 231; 3, 2)	(255, 231; 3)
P_i	5×10^{-13}	5×10^{-13}	5×10^{-13}	3.4×10^{-10}	3.4×10^{-10}	3.5×10^{-6}
R bit/sec	305	2800	3750	3210	4350	4350
Retransmission rate	12%	12%	12%	0.2%	0.2%	0
Blocks/superblock	1	32	1 or 32	16	1 or 16	1
Transistors/encoders	300	300	300	400	400	400
Transistors/decoders	300	300	300	2800	2800	$\gg$2800

the six transmission schemes, with memory being used most efficiently in Cases B and D, and eventually in C and D. Finally, an evaluation of the coder and decoder complexity is given by an estimate of the number of transistors needed for implementing the corresponding shift registers and control logic (assuming the use of four-phase MOS shift registers with six transistors per stage). Except for the decoder of the forward error correction scheme (Case F), the complexity of the encoder and decoder is within the practical and economical range of the present large-scale integration technology.

Conclusions

The preceding study has shown the advantages of the hybrid transmission scheme versus the pure retransmission and the forward error correction schemes when a high reliability is needed for data transmission over a noisy channel. Nevertheless, this study was limited to linear block codes under the assumption of a memoryless binary symmetric channel. A comparative study using convolutional codes under the same hypothesis would be interesting. The validity of the results for a real channel depends essentially on whether or not interleaving can randomize the channel as was assumed. This may not always be possible. For a given real channel, burst error correcting codes can be made to outperform random error correcting codes, but this is true only if the real channel cooperates to cluster the errors into pure bursts with the correct guard spaces. A comparative study of burst error correcting codes using different models of real channels would be worthwhile.

Acknowledgment

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Appendix A: Probability of undetected error

The exact expression of P_U is:

$$P_U = \sum_{m=2t+1-\theta}^{n} P(m, n) \times \text{Prob (error pattern weight} = m \text{ in Subset III).} \quad \text{(A-1)}$$

Since the channel is supposed memoryless, we will neglect $P(m, n)$ for $m > 2t + 1 > nP$ in comparison with $P(2t + 1 - \theta, n)$; i.e.,

$$P(m, n) \approx 0 \quad \text{for} \quad m > 2t + 1. \quad \text{(A-2)}$$

We will consider only the error patterns of weight $m \leq 2t + 1$. Such an error pattern of weight m cannot be located in the cosets whose coset leader has weight $2t - m$ or less, i.e., in the $\sum_{q=0}^{2t-m} \binom{n}{q}$ first cosets. The hypothesis of uniform distribution of the weight in the standard array gives the following approximation for the probability of an error pattern of weight m in Subset III:

$$P(m \mid \text{III}) \approx \frac{\sum_{u=2t+1-m}^{\theta} \binom{n}{u}}{2^{n-k} - \sum_{q=0}^{2t-m} \binom{n}{q}}, \quad \text{(A-3)}$$

where the numerator (denominator) represents the number of cosets that contain weight-m error patterns in Subset III (the standard array of error patterns).

Appendix B: Relation between the probability of undetected block error P_{UB} and the probabilities of undetected bit error P_b and P_i

Taking a given bit, say bit l, of an n-bit block, we state that the probability of this bit not being in error, given that there are exactly m errors in the block, is

$P\{\text{bit } l \text{ not in error} \mid m \text{ errors}\}$

$$= \binom{n-1}{m} \Big/ \binom{n}{m} = \frac{n-m}{n}. \tag{B-1}$$

The probability of bit l being not in error and of having exactly m errors is

$P\{\text{bit } l \text{ not in error}, m \text{ errors}\}$

$$= \left(\frac{n-m}{n}\right) P'(m, n), \tag{B-2}$$

where $P'(m, n)$ is the probability of having exactly m errors in a block after decoding. Since the number of errors in a block may vary from $(2t + 1)$ in the worst case (independently of θ as noticed) to n:

$P\{\text{bit } l \text{ not in error} \mid \geq 2t + 1 \text{ errors}\}$

$$\approx \frac{\sum_{m=2t+1}^{n-1} \left(\frac{n-m}{n}\right) P'(m, n)}{\sum_{m=2t+1}^{n-1} P'(m, n)}. \tag{B-3}$$

If we assume

$$P'(m + 1, n) \ll P'(m, n), \qquad m > 2t + 1, \tag{B-4}$$

(B-3) reduces to (B-1); hence,

$$P\{\text{bit } l \text{ in error} \mid \text{block error}\} \approx \frac{2t+1}{n} \tag{B-5}$$

or

$$P_b = P(\text{bit } l \text{ in error}) \approx \left(\frac{2t+1}{n}\right) P_{UB} \tag{B-6}$$

and, more precisely, the probability of undetected information-bit error P_i is

$$P_i \approx \frac{(2t+1)k}{n^2} P_{UB}, \tag{B-7}$$

under the assumption (B-4).

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