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Thermal Problems of the Pulsed Injection Laser

Abstract: Heat is produced in short periods of time during pulsed operation of an injection laser. The temperature of the laser at the beginning of any pulse due to the heating caused by preceding pulses is calculated for several simple model cases. The results are applied in the description of performance deterioration caused by heating.

1. Introduction

In an earlier publication[1] (hereafter referred to as I) we considered the thermal problems involved in operating an injection laser continuously and in obtaining a single pulse of energy from lasers whose thermal properties are such that they cannot be operated continuously. Similar models and problems were considered independently at about the same time by Engeler and Garfinkel.[2] Somewhat different calculations relating to pulsed energy output were presented by Lasher and Smith.[3] Lasers that cannot be operated continuously are usually operated periodically; they are excited by short pulses of current applied at regular time intervals. The purpose of this paper is to consider the thermal problems associated with pulsed periodic operation of an injection laser.

The problem is as follows. The threshold current of a laser increases rapidly and the pulse energy obtainable at a fixed value of current decreases as the temperature of the laser increases. The temperature of a periodically pulsed injection laser at the beginning of a pulse is raised above that of its environment by the heat generated by preceding pulses. In this paper we present calculations of the temperature of the laser at time zero that results from an infinite series of pulses at preceding times $-ks$ ($k = 1, 2, \cdots$). (Notation and symbols are defined in Table 1.) It is assumed that the pulse duration is small compared with all other times of interest, in particular, the time interval between pulses and the characteristic thermal diffusion time of the structure, so that the heat pulses may be considered as being instantaneous. This problem is sensitive to a greater variety of details of the physical configuration of the laser than was the previous one, so that a general treatment encompassing all relevant details is not possible. A number of limiting cases of various types are treated, however, and provide a guide to the evaluation of more complex configurations.

The power density at a laser junction is high and the heat generated at the junction must flow away through the semiconductor to a much larger interface with a fluid that eventually carries the heat out of whatever apparatus is involved; in so doing the heat current passes through a thermal spreading resistance. In many situations, however, the heat current flows in one dimension through a cylindrical structure before passing through the spreading resistance of the heat sink, which reduces the thermal current density. The first division of the problem is into the case in which the cylindrical or one-dimensional resistance is negligible and the spreading or three-dimensional resistance dominates, and the opposite case in which the spreading resistance is negligible and the one-dimensional resistance dominates. The time-dependent heat conduction problem differs between these two cases.

A second separation of the problem into limiting cases is based on the time scale. As pointed out in I, phenomena that take place on a very short time scale are essentially one-dimensional, even though the three-dimensional features are important in a general treatment encompassing all time scales. The results obtained for the one-dimensional problem can be carried over into the three-dimensional problem for "short" time scales, which means in this context that, for any time t of interest, the thermal diffusion length $(Dt)^{\frac{1}{2}}$, where D is the thermal diffusivity, is much smaller than any linear dimension of the problem.

The short-time-scale approximation is never completely applicable in the problems treated here; since an infinite sequence of pulses is involved, account must be taken of arbitrarily large times. Nevertheless, a significant part of the solution of the thermal problem can be obtained from the very-short-time approximation if the interval between pulses is short. The first approximation to the temperature

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of the laser is obtained in this case simply by allowing the average production of heat to be regarded as a continuous thermal current of constant value flowing from the laser through the thermal resistance to the heat sink. The next approximation, which takes account of the discreteness of the production of heat to first order in the period of the pulses, can be obtained from the short-time approximation. This intuitively apparent assertion is supported by the calculations in sections 2, 3 and 4. The case in which the time between pulses is very large leads one to essentially different results in the one-dimensional and three-dimensional problems. Simplifications in the limit of very large inter-pulse intervals are described in sections 2 through 5.

The models used may be summarized more specifically as follows. The heat is generated in the plane boundary of a semi-infinite solid and is therefore conducted into the solid from only one side. Two cases are considered: (1) The power flows from the junction through a right cylinder with the same cross-section as the junction (the heat flow is one-dimensional). (2) The heat producing region is a finite portion of the surface of a semi-infinite solid. It is assumed in both cases that a quantity of heat per unit area, jVt_1, is generated per pulse uniformly in the area described by the laser junction. (Here V is the junction voltage, essentially the value corresponding to the energy gap of the semiconductor or the energy of the emitted photons.) The generation of heat by the flow of current through the ohmic resistance of the bulk material in electrical series with the junction is also considered.

One-dimensional thermal problems are solved in the next two sections. The three-dimensional problem and the applicability of the results of sections 2 and 3 are discussed in sections 4 through 6. The solution of a particular three-dimensional problem, which supports the points of view of section 4, is presented in section 5. Resistive heating in the three-dimensional problem is discussed briefly in section 6. The results of the thermal theories are applied to the theory of pulsed operation of injection lasers in section 7. Table 1 contains a summary of the notation.

2. One-dimensional heat conduction

The case in which the thermal resistance is determined by flow of heat in one direction through a cylinder is treated in this section. The thermal resistance is finite only if the cylinder has a finite length L. The laser junction at which heat is produced is at one end of the cylinder ($z = 0$) in the present model, and the other end ($z = L$) is maintained at the environmental temperature, $T = 0$. The temperature at any position z at time t after an instantaneous pulse of heat of areal density q appears at the surface is[4,5]

$$T_{\mathrm{I}} = (2q/CL) \sum_{b=1}^{\infty} \cos(b\pi z/2L) \sin(b\pi/2) \times \exp[-D(b^2\pi^2/4L^2)t]. \quad (2.1)$$

Table 1 Summary of notation.

A	Area
b	Summation index; width
C	Specific heat per unit volume
c	Length
D	Thermal diffusivity ($=\kappa/C$)
G_b	Defined by Eq. (2.3)
i	Electric current
i^*	i/i_0
i_0	Threshold current at the ambient temperature
i_t	Threshold current at the actual temperature
J	Current density
j	Current density at the laser junction
j^*	Current density divided by threshold current density
k	Summation index
L	Distance between the laser thermal emission surface and the heat sink surface (one-dimensional case)
L_{D}	Characteristic thermal length; see Eq. (4.11)
m	Summation index
Q	Total heat generated in a single pulse
q	Q/A
R	Electrical resistance
r	Length of position vector
S	Surface region constituting the active area of the laser
s	Time interval between pulses
T	Temperature
T_{I}	Temperature at the beginning of a pulse
T_{IA}	Value of T_{I} below which no stimulated emission occurs [see Eq. (7.3)]
T_{IB}	Value of T_{I} at which stimulated emission ceases before the end of the pulse [see Eq. (7.5)]
T_{IC}	Value of T_{I} at which the light energy is reduced by one-half [see Eq. (7.7)]
T_{IX}	A maximum value of T_{I}, such as T_{IA}, T_{IB} or T_{IC}
T^*	T/T_1 (analogous definition for any temperature)
T_{av}	Temperature increase based on continuous rather than pulsed heat production
T_0	Temperature rise produced by Joule heating after a short current pulse
T_1	Characteristic temperature; see discussion following Eq. (7.1)
t	Time
t_{N}	Characteristic time; see Eq. (7.11)
t_1	Time duration of a current pulse
t_1^*	t_1/t_{N}
V	Voltage
x	Cartesian coordinate in the plane of the laser
z	Cartesian coordinate normal to the plane of the laser
α	$(1/4\pi^{3/2}) \sum_{m=1}^{\infty} m^{-3/2} = 0.11727$
κ	Thermal conductivity
P	Thermal resistance
P′	Thermal resistance that relates the maximum temperature of a region to its heat current
ρ	Radial distance (in the plane of the laser) divided by $(4Dt)^{1/2}$
σ	Electrical conductivity
ω	0.8239; see Eqs. (A8) to (A10)

Here C is the specific heat per unit volume of the solid and D is the thermal diffusivity, i.e., the thermal conductivity κ divided by C. Equation (2.1) is used to find the temperature of the laser at time $t = 0$ after an infinite series of preceding pulses at times $-ks$, $k = 1, 2, 3, \cdots$. Heat is generated at a uniform rate per unit area, q, in each of the pulses. The temperature at $z = 0$ is found by summing terms of the type (2.1) to be

$$T_{\mathrm{I}} = (2q/CL) \sum_{k=1}^{\infty} \sum_{b=1}^{\infty} \exp(-G_b ks) = (2q/CL) \sum_{b=1}^{\infty} [\exp(G_b s) - 1]^{-1}, \tag{2.2}$$

where

$$G_b = \pi^2(2b-1)^2 D/4L^2. \tag{2.3}$$

As $s \rightarrow 0$, Eq. (2.2) becomes

$$T_{\mathrm{av}} = (8qL/\pi^2 \kappa s) \sum_{b=1}^{\infty} (2b-1)^{-2} = (q/s)(L/\kappa). \tag{2.4}$$

Equation (2.4) is the usual formula for the temperature increase produced by the average heat production and flow through the thermal resistance of the cylinder.

To obtain the next term in the expansion of T_{I} as a function of s as $s \rightarrow 0$, we consider that the production of heat can be regarded as an almost continuous process in this limit. In fact, it is a convenient and reasonable approximation to replace those pulses remote in time by rectangular pulses of duration s centered on time $-ks$ that produce the same amount of heat as the instantaneous pulses they replace. If all pulses with an index value $k \geq m$ are so replaced, the heat production earlier than time $-(m - \frac{1}{2})s$ becomes a continuous process and is represented by an integral rather than a sum of a function of the integers.

Thus the first form of Eq. (2.2) becomes

$$T_{\mathrm{I}} = (2q/CL)\left[\sum_{k=1}^{m} \sum_{b=1}^{\infty} \exp(-G_b ks) + \int_{(m+1/2)s}^{\infty} (1/s) \sum_{b=1}^{\infty} \exp(-G_b t)\, dt \right]. \tag{2.5}$$

We show in the appendix by introducing the Euler-Maclaurin summation formula for the sum over b in Eq. (2.5) that T_{I} has the form (to the next order in s)

$$T_{\mathrm{I}} = T_{\mathrm{av}}[1 - \omega(Ds/L^2)^{1/2}], \tag{2.6}$$

where ω is a constant whose value is given in the appendix. By referring to a cylinder or bar of cross section A and introducing its thermal resistance $\mathrm{P} = L/\kappa A$, we put Eq. (2.6) into the form

$$T_{\mathrm{I}} = (q/s)[\mathrm{P}A - \omega\kappa^{-1}(Ds)^{1/2}]. \tag{2.7}$$

Note that the second term in Eq. (2.7) is independent of any geometrical feature, a situation consistent with our previous assertion that this result is also applicable to the three-dimensional problem.

When $s \rightarrow \infty$ the sum in Eq. (2.2) is dominated by the first term ($k = 1$, $b = 1$) and thus T_{I} approaches the value

$$T_{\mathrm{I}} = (2q/CL)\exp(-\pi^2 Ds/4L^2). \tag{2.8}$$

The temperature approaches zero very rapidly with increasing s.

3. Resistive heating in the one-dimensional problem

The cylinder of the preceding section may have an appreciable electrical resistivity. If this is so, then Joule heat will be produced by the flow of electric current and must be taken into account in calculating the temperature of the laser.

At the end of a very short pulse of current of duration t_1 and local current density J in a medium of conductivity σ, the temperature at any point is

$$T_0 = J^2 t_1/\sigma C, \tag{3.1}$$

where J and T_0 are independent of position in the one-dimensional problem we are considering. The temperature due to this pulse at any subsequent time t is[5]

$$T = (4T_0/\pi) \sum_{b=1}^{\infty} [(-1)^{b+1} \exp(-G_b t)]/(2b-1), \tag{3.2}$$

where G_b is defined in Eq. (2.3). The temperature at $t = 0$ after an infinite series of pulses is, in analogy with Eq. (2.2),

$$T_{\mathrm{I}} = (4T_0/\pi) \sum_{b=1}^{\infty} (-1)^{b+1}/[\exp(G_b s) - 1](2b-1). \tag{3.3}$$

As s approaches 0, T_{I} approaches T_{av}, the value the temperature would have if the heat were produced continuously at the average rate. The value of T_{av} for this case is found from Eq. (3.3) to be

$$T_{\mathrm{av}} = T_0 L^2/2sD \tag{3.4}$$

or, in another form,

$$T_{\mathrm{av}} = \mathrm{P}Ri^2 t_1/2s. \tag{3.5}$$

Here the electrical resistance $R = L/A\sigma$ and $i = JA$. Equation (3.5) conforms with Holm's result[6] that the temperature increase in this kind of situation is that which would be produced by the flow of the electrical power through half of the thermal resistance.

The next term in the approximation of T_{I} as s approaches 0 can be obtained easily in this case by straightforward expansion of Eq. (3.3):

$$T_{\mathrm{I}} = T_{\mathrm{av}}[1 - (Ds/L^2)] = (\mathrm{P}Ri^2 t_1/2s) - T_0/2. \tag{3.6}$$

It has been necessary to use the following formulas[7] in obtaining Eqs. (3.4) and (3.6):

$$\sum(-1)^{m+1}(2m-1)^{-1} = \beta(1) = \pi/4$$

and

$$\sum(-1)^{m+1}(2m-1)^{-3} = \beta(3) = \pi^3/32.$$

As in section 2, when s approaches ∞ the first term of Eq. (3.3) dominates and T_I vanishes exponentially with increasing s.

The results of sections 2 and 3 are summarized in Fig. 1. Values of T_I in the one-dimensional problems obtained by numerical summation of the series of Eqs. (2.2) and (3.3) are shown. The approximations given by Eqs. (2.6) and (3.8) are also presented. The independent variable in the figure is the spacing between pulses measured by the quantity sD/L^2. A good approximation to T_I/T_{av} is provided by the simple function

$$T_I/T_{av} = \begin{cases} 1 - \pi^{-1/2}\omega(sD/L^2)^{1/2}, & sD/L^2 \leq 1.47; \\ 0, & sD/L^2 \geq 1.47; \end{cases} \tag{3.7}$$

if the heat is produced at the laser surface; and by

$$T_I/T_{av} = \begin{cases} 1 - (sD/L^2), & sD/L^2 \leq 1; \\ 0, & sD/L^2 \geq 1; \end{cases} \tag{3.8}$$

if the heat is produced uniformly in the laser material.

The results obtained by summing Eqs. (2.2) and (3.3) are also presented in Table 2.

4. Spreading resistance or three-dimensional heat conduction

This section treats the case in which the active region of the laser is a bounded region S on the surface of a semi-infinite solid. The temperature at any point X in a semi-infinite solid (initially at $T = 0$) at time t after an instantaneous pulse of heat of strength Q appears at a point on the surface of the solid is[8]

$$T = (Q/4C)(\pi Dt)^{-3/2} \exp(-r^2/4Dt). \tag{4.1}$$

Here r is the distance from the point of generation to the point X. If heat is generated over a surface region S (the active area of the laser), the temperature of the solid can be found by integrating contributions of the form (4.1) over S, in which case Q is replaced by $q dA$ and q is interpreted as a heat source strength per unit area. If the z coordinate is chosen perpendicular to the boundary plane, then the temperature of the laser (i.e., of the surface region S) may be found by setting $z = 0$. The desired temperature at this junction ($z = 0$) at $t = 0$ after an infinite series of pulses is

$$T_I = (q/4C) \sum_{k=1}^{\infty} (\pi Dks)^{-3/2} \int_S \exp(-r^2/4Dks)\, dA. \tag{4.2}$$

Apparently T is a function of position with respect to the active region S and, in particular, will vary within S. The point of view adopted here is that the deterioration in performance of the laser caused by heating is determined by the maximum temperature in the active region. The maximum value of T occurs at the center of a convex region of high symmetry, such as a rectangle. Thus Eq. (4.2) is henceforth interpreted as a formula for the temperature at the center of the laser and the threshold current of the laser is regarded as a function of this temperature.

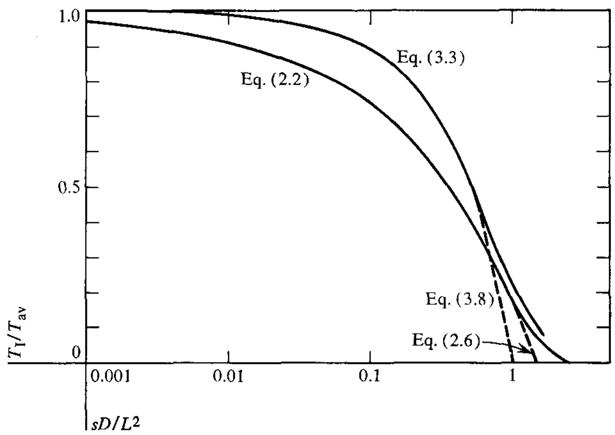


Figure 1 Values of T_I/T_{av} for surface and bulk heating in the one-dimensional laser problem, derived from Eqs. (2.2) and (3.3), as functions of sD/L^2. The approximations given by Eqs. (2.6) and (3.8) are also shown.

Table 2 Values of T_I/T_{av} for the one-dimensional thermal resistance problem, obtained by summing the series of Eqs. (2.2) and (3.3).

sD/L^2	*Eq. (2.2)*	*Eq. (3.3)*
0.01	0.9176	0.9900
0.02	0.8835	0.9800
0.05	0.8158	0.9500
0.1	0.7394	0.8997
0.2	0.6317	0.7962
0.5	0.4109	0.5231
1	0.1853	0.2360
2	0.0290	0.0369

• *Limiting case $s \to 0$*

It is convenient to follow the method of replacing the pulses remote in time by a continuous source of heat as described in section 2 in connection with Eqs. (2.5) and (2.6). Equation (4.2) becomes, in analogy with Eq. (2.5),

$$T_I = (q/4C)\left[\sum_{k=1}^{m} (\pi Dks)^{-3/2} \int_S \exp(-r^2/4Dks)\, dA + (1/s)\int_{(m+1/2)s}^{\infty} (\pi Dt)^{-3/2} \int_S \exp(-r^2/4Dt)\, dA\, dt\right]. \tag{4.3}$$

The time integral in Eq. (4.3) may be regarded as the difference between an integration from 0 to ∞ and an integration from 0 to $(m + \frac{1}{2})s$. The first time integral is easily evaluated and gives a contribution to T_{I} of

$$T_{\mathrm{av}} = (q/2\pi C s D) \int_{\mathrm{S}} (1/r)\, dA, \tag{4.4}$$

which is the steady-state temperature rise at the center of the laser that would be caused by continuous production of heat at a rate q/s. The value T_{av} is obviously closely related to the spreading thermal resistance, although it is not precisely a thermal resistance because this entity is usually defined as the resistance to heat flow from an isotherm, and the laser surface is not an isotherm in the present model. Since it is intuitively expected that the dominant term in the temperature increase will be due to steady-state thermal resistance in the limit $s \to 0$, it is instructive to write T_{av} in terms of a thermal resistance P' and thus obtain

$$T_{\mathrm{av}} = (qA/s)\mathrm{P}'. \tag{4.5}$$

Here A is the area of S and

$$\mathrm{P}' = (1/2\pi\kappa A) \int_{\mathrm{S}} (1/r)\, dA. \tag{4.6}$$

Turning now to the remaining part of Eq. (4.3), we introduce $\rho = r/(4Dt)^{\frac{1}{2}}$ in the space integral over S, thereby transforming S into a surface Σ in the ρ space. Since $t \leq (m + \frac{1}{2})s$ the boundaries of the surface Σ approach ∞ in all directions from any point not on the perimeter of Σ when $s \to 0$. Thus the space integration can be done by allowing the limits of the integral to approach ∞:

$$\int_{\mathrm{S}} \exp(-r^2/4Dt)\, dA = 4\pi Dt. \tag{4.7}$$

In the first part of Eq. (4.3), the summation, t has the value ks. The result (4.7) allows the integration over time from 0 to $(m + \frac{1}{2})s$, which arose in evaluating the second part of Eq. (4.3), to be carried out easily. Thus we find that

$$T_{\mathrm{I}} = T_{\mathrm{av}} + (q/C)(\pi s D)^{-1/2} \times \left[\sum_{k=1}^{m} k^{-1/2} - 2(m + \tfrac{1}{2})^{1/2} \right]. \tag{4.8}$$

The numerical coefficient in the second term of Eq. (4.8) is discussed in the appendix. Thus T_{I} has essentially the form of Eqs. (2.6) and (2.7):

$$T_{\mathrm{I}} = T_{\mathrm{av}} - \omega(q/C)(sD)^{-1/2}. \tag{4.9}$$

Our assertion that the results of the one-dimensional case can be carried over to the three-dimensional case when s is small is confirmed. The resemblance of Eq. (4.9) to (2.6) can be increased by introducing T_{av} from Eq. (4.5),

$$T_{\mathrm{I}} = T_{\mathrm{av}} [1 - \omega(sD/L_{\mathrm{D}})^{1/2}]. \tag{4.10}$$

Here L_{D} is a characteristic length defined by

$$L_{\mathrm{D}} = \kappa A \mathrm{P}'. \tag{4.11}$$

In fact, if Eq. (4.11) is applied to the one-dimensional case one finds that $L_{\mathrm{D}} = L$, a result that is assumed hereafter.

• *Limiting case $s \to \infty$*

One can expect that, if the time delay following a heat pulse is very long, the exact details of the original location of the heat source will have been forgotten and the temperature distribution is describable by Eq. (4.1), wherein Q must be taken as the total heat produced in the pulse, or qA. At the laser itself $r^2 \ll 4Dt$ and the exponential factor can be taken as unity. The temperature T_{I} of the laser is then

$$\begin{aligned} T_{\mathrm{I}} &= (qA/4C)(\pi s D)^{-3/2} \sum_{m=1}^{\infty} m^{-3/2} \\ &= \alpha(qA/C)(sD)^{-3/2}. \end{aligned} \tag{4.12}$$

Here α is the constant

$$(1/4\pi^{3/2}) \sum_{m=1}^{\infty} m^{-3/2}. \tag{4.13}$$

The sum of the $-\frac{3}{2}$ powers of the integers can be found from tables,[9] and the result is $\alpha = 0.11727$.

The qualitative reasoning leading to Eq. (4.12) is confirmed by analysis of Eq. (4.2). The exponential in the integrand of Eq. (4.2) approaches unity for large values of k and the integral approaches A, the area of S. Equation (4.2) then reduces to Eq. (4.12). It is convenient for comparison with earlier results to write Eq. (4.12) in the form

$$T_{\mathrm{I}}/T_{\mathrm{av}} = \alpha(A/L_{\mathrm{D}}^2)(sD/L_{\mathrm{D}}^2)^{-1/2}. \tag{4.14}$$

Equations (4.12) and (4.14) show an essential difference between the one-dimensional and three-dimensional cases. The temperature T_{I} decreases only as an inverse power of s in the latter case, whereas it was found in Eq. (2.7) to decrease exponentially with s in the former case. Thus the thermal effects of the spreading resistance always dominate for large values of s, even though the one-dimensional thermal resistance may be much larger and dominant at small values of s.

5. Heat flow from a rectangular surface region

This section treats the most interesting example of the problem of the preceding section, namely the case in which the laser region S on the surface of a semi-infinite solid is rectangular. Let b be the width of the rectangle and c be its length. The thermal resistance P' of such a rectangle is[10]

$$\mathrm{P}' = (\pi\kappa)^{-1}[b^{-1} \sinh^{-1}(b/c) + c^{-1} \sinh^{-1}(c/b)], \tag{5.1}$$

and the characteristic length L_{D} is

$$L_D = \pi^{-1}[c \sinh^{-1}(b/c) + b \sinh^{-1}(c/b)]. \qquad (5.2)$$

The ratios of L_D to b and c, the dimensions of the rectangle, are shown as functions of c/b in Fig. 2.

Table 3 contains the results obtained by numerical summation of the series of Eq. (4.2) for the case in which S is a rectangle; the integration over S reduces to a product of error functions. Values of T_I/T_{av} are given as functions of sD/L_D^2. Figure 3 shows the same results for the extreme cases $b/c = 1$ and $1/12$. Curves derived from the formulas for the limiting cases, Eq. (2.6) or Eq. (4.10) and Eq. (4.14) [with Eq. (5.2)] are also shown for comparison.

The range of usefulness of the approximations (4.10) and (4.14) for the case in which the thermal resistance is dominated by the spreading resistance of a rectangular surface depends on the value of b/c. The large-s and small-s limits almost overlap if $b/c = 1$, but there is a gap between their ranges of validity for small b/c. Even in the extreme case $b/c = 1/12$, however, the small-s approximation, Eq. (4.10), is valid for values of sD/L_D^2 less than 0.2.

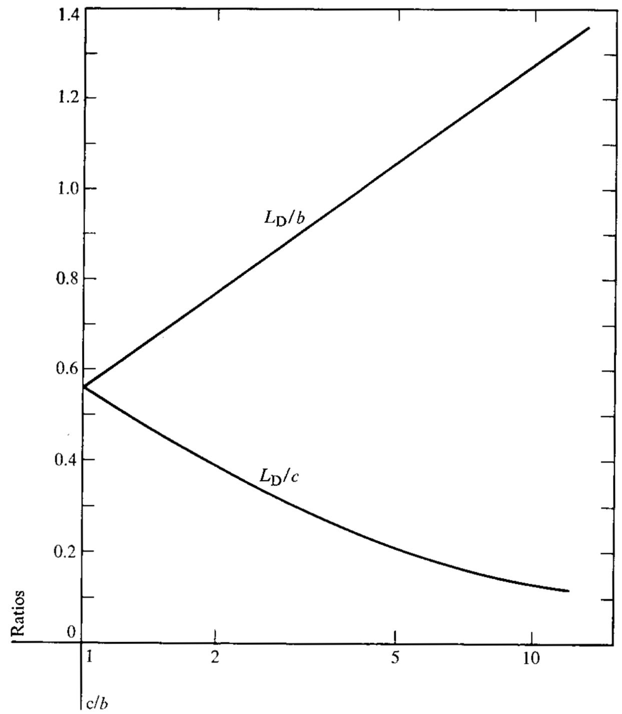


Figure 2 The characteristic thermal length L_D of a rectangular surface region, defined by Eqs. (4.11) and (5.1), as a function of the length-to-width ratio c/b of the rectangle.

Table 3 Values of T_I/T_{av} obtained by summing the series of Eq. (4.2) for a rectangle with sides b and c.

	c/b					
sD/L_D^2	1	2	4	8	12	Eq. (4.10)
0.1	0.739	0.739	0.742	0.749	0.754	0.739
0.2	0.632	0.638	0.653	0.671	0.682	0.632
0.5	0.464	0.482	0.520	0.559	0.579	0.417
1	0.349	0.367	0.415	0.470	0.498	0.176
2	0.255	0.271	0.316	0.379	0.416	
5	0.165	0.176	0.210	0.267	0.308	
10	0.118	0.126	0.152	0.197	0.233	
20	0.084	0.090	0.108	0.143	0.171	

6. Resistive heating in three dimensions

Joule heat is produced by the flow of current away from the laser junction through the bulk material. This heat also contributes to the temperature increase of the laser. In the absence of exact solutions for particular cases of interest, limiting cases somewhat analogous to those described in section 4 are considered. The $s \to 0$ result is taken over from the one-dimensional case, Eq. (3.6):

$$T_I = \mathrm{P}Ri^2t_1/2s - j^2t_1/2\sigma C. \qquad (6.1)$$

The result for the case $s \to \infty$ is again taken from Eq. (4.1). That is, if $(sD)^{\frac{1}{2}}$ is much larger than any dimension of the laser, the details of the original location of the heat source are forgotten. Now Q is taken as the heat produced in the ohmic resistance, $Q = Ri^2t_1$. Thus

$$T_I = \alpha(Ri^2t_1/C)(sD)^{-3/2}. \qquad (6.2)$$

Again denoting the limit of T_I as $s \to 0$ by T_{av}, we find T_{av} to be the first term of Eq. (6.1). Equation (6.2) may then be written, in analogy with Eq. (4.13), as

$$T_I/T_{av} = 2\alpha(A/L_D^2)(sD/L_D^2)^{-1/2}. \qquad (6.3)$$

Furthermore, if one assumes that the electrical resistance problem and the thermal resistance problem are geometrically similar, so that $\mathrm{P}\kappa = R\sigma$, then Eq. (6.1) has the form of the first part of Eq. (3.7):

$$T_I/T_{av} = 1 - (sD/L_D^2). \qquad (6.4)$$

Actually Eqs. (6.1) and (6.4) are only approximations, because the first term in each was obtained by using Holm's theorem, which is applicable to a case in which the contact, or in this case the laser, is both an electrical and a thermal equipotential surface. Values of T_I/T_{av} derived from Eqs. (6.3) and (6.4) are plotted in Fig. 4 for rectangles with $c/b = 1$ and 12. A suggestion derived from inspection of Fig. 3 is incorporated in Fig. 4 in the absence of an exact solution for comparison. Dotted lines joining the two limiting approximations by means of their common tangents are shown.

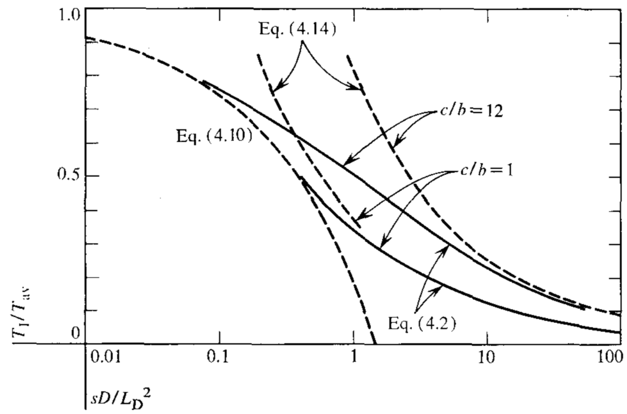
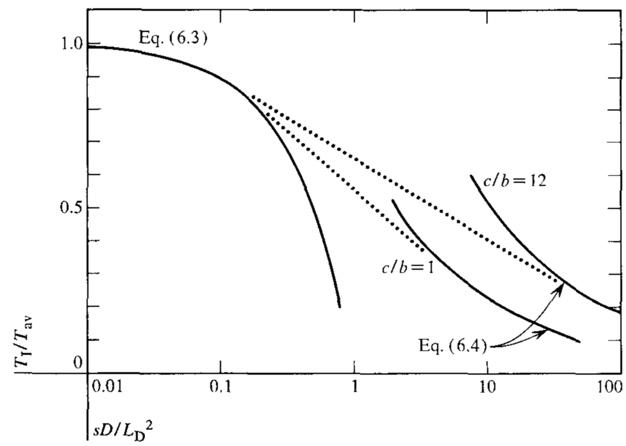


Figure 3 Values of T_I/T_{av} for heat flow from a rectangular surface region into a three-dimensional heat sink, derived from Eq. (4.2) for length-to-width ratios of 1 and 12, as functions of sD/L_D^2. Values derived from the formulas for the limiting cases, Eq. (4.10) and Eq. (4.14) [with Eq. (5.2)], are shown as dashed lines.

Figure 4 Values of T_I/T_{av} for the three-dimensional electrical resistance problem for rectangular surface regions with length-to-width ratios of 1 and 12; the limiting approximations for small and large values of s derived from Eqs. (6.3) and (6.4), respectively, are shown.



7. Application to the injection laser

According to the model used in I the light energy in an injection laser pulse is proportional to the time integral of the excess of the driving current over the threshold current. (The threshold current changes during the pulse because the current heats the laser.) This integral is approximated by the value obtained by simple trapezoidal integration. Figure 2 of I shows that, at least in some cases, this is a good approximation. Thus

$$\int_0^{t_1} (i - i_t)\, dt = \tfrac{1}{2}[2i - i_t(0) - i_t(t_1)]t_1. \tag{7.1}$$

Here t_1 is the time at which light emission ceases, either because the driving current is turned off or because the threshold current rises to the value of the current itself.

In the model used in I the threshold current depends on the temperature of the laser as $\exp(T/T_1)$. The fractional temperature increase during the pulse is $j^* t_1^{*\frac{1}{2}}$. If the temperature of the laser at the beginning of a pulse is T_I, the threshold current at this time is $i_0 \exp(T_I/T_1)$ or, by definition $i_0 \exp T_I^*$. The energy of the pulse as calculated from Eq. (7.1) is then proportional to

$$\{i^* - \tfrac{1}{2}[1 + \exp(j^* t_1^{*1/2})] \exp T_I^*\} t_1, \tag{7.2}$$

where $i^* \equiv i/i_0$. Thus a calculation of T_I^* enables one to determine the degradation of pulse energy due to the heating from other pulses.

Values of T_I at which certain critical stages in the character of the laser output occur may be recognized. As a first example, if the threshold current at temperature T_I is greater than the current itself, stimulated light emission will not occur. Let the lowest temperature for which this is the case be denoted by T_{IA}. Then

$$i = i_0 \exp(T_{IA}/T_1), \tag{7.3}$$

or in a dimensionless notation (see Table 1) this equation may be written as

$$T_{IA}^* = \ln i^*. \tag{7.4}$$

As a second example, laser action will cease before the end of the current pulse if T_I is larger than a temperature T_{IB} defined by

$$i = i_0 \exp(T_{IB}^* + i^* t_1^{*1/2}), \tag{7.5}$$

or

$$T_{IB}^* = \ln i^* - i^* t_1^{*1/2}. \tag{7.6}$$

As a third example, consider that value of T_I at which the light energy of the pulse is one-half of the value for an isolated pulse; that is, a pulse with $T_I = 0$ but with the same i and t_1. From Eq. (7.2) this value defined as T_{IC} is

$$T_{IC}^* = \ln \{\tfrac{1}{2} + i^*/[1 + \exp(i^* t_1^{*1/2})]\}. \tag{7.7}$$

The value of T_{IC}^* given by Eq. (7.7) is based on the assumption that laser action continues through the time t_1, that is, that $T_{IC}^* < T_{IB}^*$.

Let some laser criterion be given that can be expressed by specifying T_{IX}, a maximum permissible value of T_I. Assume that $sD/L_D^2 \ll 1$, so that the approximations of Eq. (2.5) and (3.6) may be used. The temperature at the beginning of a pulse must be determined by adding the contributions from the release of energy at the junction, Eq. (2.5), and from ohmic heating, Eq. (3.6). The following equation for T_I is obtained when this is done:

$$T_I = \frac{\mathrm{P} R i^2 t_1}{2s} - \frac{i^2 t_1}{2\sigma C A^2} + \frac{i V t_1 \mathrm{P}}{s} - \frac{\omega i V t_1 D^{1/2}}{s^{1/2} \kappa A}. \tag{7.8}$$

Equation (7.8) may be regarded as a quadratic equation in $s^{-\frac{1}{2}}$; then an explicit formula for the minimum separation between pulses that is compatible with the operational criterion described by T_{IX} is obtained:

$$\frac{1}{s^{1/2}} = \frac{1}{2 + iRV^{-1}}\left\{\frac{\omega D^{1/2}}{P\kappa A} + \left[\frac{\omega^2 D}{(P\kappa A)^2} + (2 + iRV^{-1})\left(\frac{i}{\sigma C A^2 P V} + \frac{2T_{IX}}{i V t_1 P}\right)\right]^{1/2}\right\}. \quad (7.9)$$

If s is large, T_I is dominated by the three-dimensional or spreading resistance:

$$T_I = \alpha(iV + i^2R)t_1/C(sD)^{3/2}, \quad (7.10)$$

from which an explicit formula for the minimum s value can be obtained. The data in Table 2 can also be used to obtain results of the type represented by Eqs. (7.8) and (7.10).

- *Case of dominant one-dimensional resistance*

Consideration of a specific case illustrates the application of the above considerations. Let a laser junction 50 μm $\times$ 200 μm be at one end of a block of GaAs 60 μm long, with the other end of the GaAs block attached to a semi-infinite copper heat sink. If the laser has a threshold current of 5 A and is operated at a current of 15 A applied in 50-nsec pulses, the power input at the junction is 22.5 W during the pulse. If the conductivity of the GaAs is 600 $(\Omega\text{-cm})^{-1}$, the electrical resistance in series with the junction is 0.1 Ω and another 22.5 W of power are developed in this series resistance. Let the value of T_1, the constant that describes the temperature dependence of the threshold current as in Eq. (7.3), be 90°C. Then the characteristic normalizing time t_N (see I), defined by

$$t_N = \pi\kappa C T_1^2/4j_0^2V^2, \quad (7.11)$$

is 0.96 μsec.

The thermal resistance of the GaAs is 120°C/W and the spreading resistance in the copper [Eq. (5.1)] is 12 °C/W. The thermal time constant of the GaAs, the quantity L^2/D_{GaAs} that appears in Eq. (2.6) and following equations, is 83 μsec. The thermal time constant in the Cu, L_D^2/D_{Cu}, is 25 μsec from Fig. 2. Since these time constants are about 10^3 times the pulse length, it is appropriate to regard the pulse as instantaneous. The following considerations show that the small-s approximation, $sD/L^2 \ll 1$, can be used for the GaAs. If s, the interval between pulses, is 83 μsec so that $sD_{GaAs}/L^2 = 1$, the average laser power and the average resistive power are each 14 mW. The temperature rise from the flow of the average power through the thermal resistance of the copper is 0.3°C. Since the resistive power flows through half of the thermal resistance of the GaAs, the temperature rise due to the thermal resistance of the GaAs is 2.4°C. The total temperature increase caused by the average power flow is 2.7°C, which is small compared with T_1 (90°C). The temperature increase T_I that determines the power output is, of course, considerably smaller still. Thus the interesting range of s, the range in which T_I is large enough to have an appreciable effect on the pulse power, involves only values of sD_{GaAs}/L^2 less than 1 and Eqs. (2.6) and (3.6) can be used.

The situation is somewhat different in the copper. The heat pulses arriving at the copper heat sink are not instantaneous pulses; in fact they are spread out over a time of the order of the thermal time constant of the GaAs, 83 μsec. Since the interesting values of s are less than this value, it is a better approximation to regard the heat current through the spreading resistance of the copper as the average heat current flowing continuously. In any case, the temperature drop in the copper heat sink is small and errors in describing it are of minor significance in the primary calculation.

The approximation just described for calculating T_I, that is, adding the temperature increase caused by the flow of the average power through the spreading resistance of the heat sink to the temperature increase of the GaAs calculated from the theory of sections 2 and 3, leads to a value of T_I appropriate to the calculation of the pulse energy of the model laser. Equation (7.2) was used to calculate a quantity proportional to the light pulse energy. The results are shown in Fig. 5 in which the pulse energy, normalized to the energy of an isolated pulse, is plotted as a function of s, the spacing between pulses.

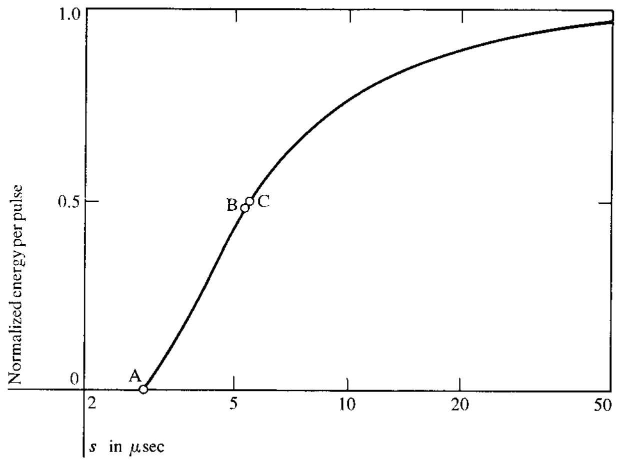


Figure 5 Dependence of the energy output per pulse of the laser described in section 7 on the time s between pulses. The energy is normalized to the energy of an isolated pulse, i.e., the energy value as $s \rightarrow \infty$. The values of s at which T_I attains the values T_{IA}, T_{IB} and T_{IC}, Eqs. (7.4), (7.6) and (7.7), respectively, are indicated.

- *Case of dominant spreading resistance*

As a further illustration, consider a large laser with the p-n junction symmetrically placed in a slab of GaAs 0.3 mm × 0.6 mm × 10 μm ($L = 5$ μm). If the material properties of the GaAs are the same as in the preceding example, the threshold current is 90 A, the thermal resistance from the junction to one side of the GaAs is 0.556 °C/W, and the electrical resistance on one side is 0.463 mΩ. Let the laser be clamped between two large bronze heat sinks with the following properties: $\kappa = 0.5$ W/cm-°C, $C = 3.44$ J/cm^3-°C, $D = 0.145$ cm^2/sec, $\sigma = 73.5$ (mΩ-cm)$^{-1}$. Then L_D in the bronze is 0.023 cm, the thermal resistance is 25.5 °C/W, the electrical resistance is 0.173 mΩ, and the thermal time constant L_D^2/D is 3.63 msec. Since the thermal resistance in the bronze is about 50 times as large as that in the GaAs, neglect of the latter contribution is reasonable. In contrast with the preceding case, the three-dimensional spreading resistance is dominant. Also, the thermal time constant of the GaAs is 0.8 μsec, much less than both the time constant in the bronze and the minimum interesting value of s (the value for which $T_I = T_{IC}$), which is about 7 μsec. Therefore the pulses of heat arriving at the bronze heat sink may be considered to be instantaneous pulses. One-half of the total heat (resistive plus junction) generated in the GaAs is dissipated through each bronze heat sink.

Assume that the laser is operated with 300-A pulses of 50-nsec duration. Then a quantity proportional to the emitted energy can be calculated from Eqs. (7.8) and (7.2). The graph of this quantity as a function of s is very similar to Fig. 5 (since t_1^* has the same value as in the preceding example and i^* is only slightly different), but with the abscissa of Fig. 5 multiplied by a factor of about four.

8. Discussion of approximations

It must be emphasized that the models used involve a number of idealizations. Some of the neglected features would tend to improve the performance and others to degrade it. The treatment of events within the pulse (presented in I) assumed that the current rose discontinuously from zero to a given value during the pulse. In fact, the electronic circuits only provide pulses with finite rise times and some heating of the laser occurs before the current reaches the threshold value and light emission begins. Further, there is a delay between the time at which the current reaches the threshold value and the time that laser action begins, which causes additional heating before the start of light emission. These delays mean that the power output is less than the value calculated from the idealized model and that laser action ceases at a temperature T_I less than the T_{IC} of Eq. (7.7).

Interface thermal resistances have been neglected. These, if present, will cause temperature increases greater than those calculated and degrade performance further.

The temperature dependence of the thermal conductivity was neglected to simplify the analysis. However, the thermal conductivity of semiconductors decreases with increasing temperature, an effect that will also cause temperatures to be somewhat greater than those calculated.

An approximation that tends to make the actual temperatures lower than the calculated ones, and thus to improve the performance of the laser, is the neglect of the energy lost in the form of emitted light. Another probably conservative approximation is the assumption that the threshold current value is determined by the hottest part of the laser; it is conceivable that some cool part of the laser could remain operative even after stimulated emission ceased in the hottest part.

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Appendix

To evaluate the next-higher-order term in the expansion of Eq. (2.5) in powers of s, we define

$$g(u) = \sum_{n=1}^{\infty} \exp(-n^2 u). \tag{A1}$$

Using the Euler-Maclaurin summation formula we obtain[11]

$$g(u) = \int_0^{\infty} \exp(-n^2 u)\, dn = \tfrac{1}{2}(\pi/u)^{1/2}. \tag{A2}$$

The desired sums are

$$\sum_{b=1}^{\infty} \exp(-G_b ks) = g(\pi^2 Dks/4L^2) - g(\pi^2 Dks/L^2)$$

$$= \tfrac{1}{2}L/(\pi Dks)^{1/2} \tag{A3}$$

and

$$\sum_{b=1}^{\infty} \exp(-G_b t) = \tfrac{1}{2}L/(\pi Dt)^{1/2}, \tag{A4}$$

where G_b is defined in Eq. (2.3).

The integral in the second term of Eq. (2.5) is conveniently regarded as the difference between an integral from 0 to ∞ and an integral from 0 to $(m + \frac{1}{2})s$. The first of these integrals can be evaluated exactly as

$$\int_0^{\infty} \sum_{b=1}^{\infty} s^{-1} \exp(-G_b t)\, dt = s^{-1} \sum_{b=1}^{\infty} G_b^{-1} = L^2/2sD. \tag{A5}$$

The result (A5) is equivalent to Eq. (2.5). The remaining part of the integral is easily evaluated with the aid of Eq. (A4):

$$\int_0^{(m+1/2)s} \sum_{b=1}^{\infty} s^{-1} \exp(-G_b t)\, dt = L[(m+\tfrac{1}{2})/\pi s D]^{1/2}. \tag{A6}$$

Substituting Eqs. (A3), (A5) and (A6) into Eq. (2.5) we find

$$\begin{aligned} T_{\mathrm{I}} &= qL/sDC \\ &\quad - (q/CL)(\pi s D)^{-1/2}\left[2(m+\tfrac{1}{2})^{1/2} - \sum_{k=1}^{m} k^{-1/2}\right] \\ &= T_{\mathrm{av}}[1 - \omega_m (sD/L^2)^{1/2}]. \end{aligned} \tag{A7}$$

Here ω_m is defined by

$$\begin{aligned} \omega_m &\equiv \pi^{-1/2}\left[2(m+\tfrac{1}{2})^{1/2} - \sum_{k=1}^{m} k^{-1/2}\right] \\ &= \pi^{-1/2}\Big\{2^{1/2} + \sum_{k=1}^{m} [2(k+\tfrac{1}{2})^{1/2} \\ &\qquad - 2(k-\tfrac{1}{2})^{1/2} - k^{-1/2}]\Big\}. \end{aligned} \tag{A8}$$

Expanding $(k \pm \frac{1}{2})^{\frac{1}{2}}$ by the binomial theorem one obtains

$$\begin{aligned} \omega_m &= \pi^{-1/2}\Big[2^{1/2} + (1/2^5) \sum_{k=1}^{m} k^{-5/2} \\ &\qquad + (7/2^{11}) \sum_{k=1}^{m} k^{-9/2} - \cdots\Big]. \end{aligned} \tag{A9}$$

As the approximation becomes more realistic physically by letting the value of $m \to \infty$, the value of $\omega_m \to \omega$. The sums of the reciprocal powers of the integers are tabulated[9] and ω is found to have the value

$$1.4604/\pi^{1/2} = 0.8239. \tag{A10}$$

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