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Maximal Biflow in an Undirected Network

Abstract: In this network flow problem we deal with two distinct commodities, each commodity being identified by a pair of source and sink nodes. The problem consists of maximizing the total flow (biflow) of the two commodities. It is solved by an inductive algorithm which starts with a maximal multiterminal flow from the set of sources to the set of sinks in the network, yields the value of the maximal biflow and terminates with the construction of the maximal biflow itself. Computational experience shows that this algorithm can also be used in the three-commodity flow problem to obtain a good lower bound for the value of a maximal three-commodity flow.

Introduction

The important problem of maximizing flow from a source to a sink in a network was first solved by a simple node labeling procedure in 1956.[1] This procedure, together with the "max-flow min-cut" theorem,[1] has initiated a number of sophisticated methods for the solution of network flow problems. In 1960 Gomory and Hu[2] solved the multiterminal network flow problem.

In many problems, however, it is necessary to deal with several distinct commodities, each commodity being identified by a pair of source and sink nodes. The problem consists of maximizing the total flow of all commodities. It is known that this multicommodity maximal flow problem belongs to a class of large linear programs of special structure.[3] Jewell[4] and Ford and Fulkerson[1] gave early solutions. More recently, Tomlin[5] and Sakarovitch[6] gave solutions based on special considerations in linear programming, for instance, the decomposition principle of Dantzig and Wolfe.[7] In 1963 Hu[8] directly solved a two-commodity maximal flow problem using the characteristic features of networks. Then in 1966 Rothschild and Whinston[9] gave an alternative approach to the same problem.

In this paper we describe a special case of a general approach to network flow problems. This approach consists of solving the problem for a given global network using the maximal solution of the same problem for a reduced network; the maximal feasible solution for the reduced network is considered as an initial unfeasible solution for the global network. The algorithm starts with this solution and progressively reduces the maximal value previously found until the final solution is a feasible maximal solution for the global network.

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Definitions and theorems

Let $G = (X, U)$ be a finite, undirected, connected network. There is no loop in G; X is the set of nodes x of G; U is the set of edges u of G. A non-negative integer $c(u)$, called the capacity of u, is associated with each edge of G. Let x_1, x_2, y_1 and y_2 be four distinct nodes of G with x_1 and x_2 not related by an edge of G. The problem is to find a maximal biflow from x_1 to y_1 and from x_2 to y_2 subject to the capacity constraints.

For convenience of notation we introduce arbitrary directions on the edges of G, corresponding to a node-edge incidence matrix e in which

$e(x, u) = 0$ if node x is not an extremity of edge u,
$e(x, u) = 1$ if node x is the initial extremity of edge u or
$e(x, u) = -1$ if node x is the terminal extremity of edge u.

In particular we take for all u, $u \in U$,

$$e(x_1, u) = 0 \text{ or } 1 \quad \text{and} \quad e(x_2, u) = 0 \text{ or } 1.$$

The graph is denoted by $G = (X, U, e)$.

- *Multiterminal flow*

A multiterminal flow φ from x_1 and x_2 to y_1 and y_2 in G is a function

$$\varphi : U \rightarrow R$$

such that

$$\sum_{u \in U} \varphi(u)e(x, u) = 0, \qquad x \in X - \{x_1, x_2, y_1, y_2\};$$

$$\sum_{u \in U} \varphi(u)e(x, u) = v_1 \text{ (or } v_2), \qquad x = x_1 \text{ (or } x_2);$$

$$\sum_{u \in U} \varphi(u)e(x, u) = v_1' \text{ (or } v_2'), \qquad x = y_1 \text{ (or } y_2); \quad \text{and}$$

$$v = v_1 + v_2 = -(v_1' + v_2').$$

Feasible flow
A flow φ is said to be feasible if

$$|\varphi(u)| \leq c(u), \qquad \forall u \in U.$$

Maximal flow
A flow φ is said to be maximal if its value v is a maximum.

Cut
Let X_{12} and Y_{12} be a partition of the set X with $x_1 \in X_{12}$, $x_2 \in X_{12}$, $y_1 \in Y_{12}$ and $y_2 \in Y_{12}$; then the set of edges (X_{12}, Y_{12}) with one extremity in X_{12} and the other in Y_{12} is a cut separating x_1, x_2 and y_1, y_2. The capacity of the cut (X_{12}, Y_{12}) is

$$c(X_{12}, Y_{12}) = \sum_{u \in (X_{12}, Y_{12})} c(u).$$

Theorem 1 Max-flow min-cut[1]
The value of a maximal flow from x_1, x_2 to y_1, y_2 in G is equal to the capacity of a minimal cut separating x_1, x_2 and y_1, y_2.

Theorem 2 Integrality and non-negativity
There exists a maximal feasible flow φ from x_1, x_2 to y_1, y_2 in G with, for all u, $u \in U$,

$\varphi(u)$ an integer,

$$e(x_1, u)\varphi(u) \geq 0 \quad \text{and} \quad e(x_2, u)\varphi(u) \geq 0.$$

Chain
A sequence of nodes $x_0, \cdots, x_i, \cdots, x_n$ and of edges $u_0, \cdots, u_i, \cdots, u_{n-1}$ with

$$e(x_i, u_i)e(x_{i+1}, u_i) = -1$$

is called a chain relating x_0 and x_n. If chain E relates nodes x_0 and x_k and if chain E' relates x_k and x_n, we denote by $E \circ E'$ the chain relating x_0 and x_n.

Forward and backward edges
An edge u_i of the chain is said to be a forward edge (backward edge) if

$$e(x_i, u_i) = 1(-1).$$

The set of forward edges of a chain E is denoted by E^+; the set of backward egdes of E is denoted by E^-.

Cycle
If x_0 and x_n are the same node of G, a chain relating x_0 and x_n is called a cycle of G and is denoted by K.

Cycle flow
Let φ be a flow from x_1, x_2 to y_1, y_2 in G and let K be a cycle of G. The set

$$\varphi(K) = \{\varphi(u) \mid u \in K\}$$

is called a cycle flow contained in φ if

$\varphi(u) \gtrless 0$ for all u, $u \in K^+$, and

$\varphi(u) \lessgtr 0$ for all u, $u \in K^-$.

Theorem 3 Cycle flow
Let φ' be any feasible flow of value v, v an integer, from x_1, x_2 to y_1, y_2 in G; then there exists a feasible flow φ of value v from x_1, x_2 to y_1, y_2 in G that satisfies the integrality and non-negativity theorem and does not contain a cycle flow.

Theorem 4 Chain flow
Let φ be a feasible flow from x_1, x_2 to y_1, y_2 in G satisfying Theorem 3 and let u_i be an edge of G with extremities x_i and x_{i+1} such that $e(x_i, u_i)\varphi(u_i) > 0$; then there exist

a positive number ϵ, $\epsilon \leq |\varphi(u_i)|$,
a chain E_1 relating x_1 (or x_2) and x_i with
$\varphi(u) > 0$ if $u \in E_1^+$ or with $\varphi(u) < 0$ if $u \in E_1^-$, and
a chain E_2 relating x_{i+1} and y_1 (or y_2) with
$\varphi(u) > 0$ if $u \in E_2^+$ or with $\varphi(u) < 0$ if $u \in E_2^-$

such that $E_1 \circ u_i \circ E_2$ is a chain E relating x_1 (or x_2) and y_1 (or y_2) and such that the flow defined as follows is a feasible flow:

$$\begin{aligned} \varphi'(u) &= \varphi(u) - \epsilon \quad \text{if} \quad u \in E^+, \\ \varphi'(u) &= \varphi(u) + \epsilon \quad \text{if} \quad u \in E^- \quad \text{or} \\ \varphi'(u) &= \varphi(u) \qquad \text{otherwise.} \end{aligned}$$

We say that the chain E passes by edge u_i and we call this operation an ϵ-diminution of flow along the chain E.

Bunch
A set of chains relating two sets of nodes X_1 and X_2 and passing by edge u_i is called a bunch relating X_1 and X_2 and passing by u_i.

Theorem 5 Bunch flow-diminution
Let φ be a feasible integral flow from x_1, x_2 to y_1, y_2 in G satisfying the cycle-flow theorem and let u_i be an edge of G with extremities x_i and x_{i+1} such that $e(x_i, u_i)\varphi(u_i) > 0$; then there exists a bunch B relating x_1, x_2 and y_1, y_2 and passing by u_i such that the flow φ' obtained after a finite sequence of ϵ-diminutions along chains of B is feasible and integral with $\varphi'(u_i) = 0$.

Bunch flow-diminution procedure

We use a node scanning and labeling procedure similar to the classical method developed by Ford and Fulkerson.[1]

- *Biflow*

A biflow φ_1, φ_2 from x_1 to y_1 and from x_2 to y_2 in G is a pair of functions

$$\varphi_1 : U \rightarrow R \quad \text{and} \quad \varphi_2 : U \rightarrow R$$

such that

$$\sum_{u \in U} \varphi_1(u)e(x, u) = 0, \qquad x \in X - \{x_1, y_1\};$$

$$\sum_{u \in U} \varphi_1(u)e(x_1, u) = v_1;$$

$$\sum_{u \in U} \varphi_1(u)e(y_1, u) = -v_1;$$

$$\sum_{u \in U} \varphi_2(u)e(x, u) = 0, \qquad x \in X - \{x_2, y_2\};$$

$$\sum_{u \in U} \varphi_2(u)e(x_2, u) = v_2; \qquad \text{and}$$

$$\sum_{u \in U} \varphi_2(u)e(y_2, u) = -v_2.$$

Feasible biflow

A biflow φ_1, φ_2 is said to be feasible if for all u, $u \in U$,

$$|\varphi_1(u)| + |\varphi_2(u)| \leq c(u).$$

Maximal biflow

A biflow φ_1, φ_2 is said to be maximal if its value $v = v_1 + v_2$ is a maximum.

- *Biflow and multiterminal flow*

Let φ_1, φ_2 be a feasible biflow of value $v = v_1 + v_2$ from x_1 to y_1 and from x_2 to y_2 in G; then the function φ defined by

$$\varphi(u) = \varphi_1(u) + \varphi_2(u)$$

is a feasible multiterminal flow of value v from x_1, x_2 to y_1, y_2 in G.

Theorem 6

The value of a maximal feasible biflow from x_1 to y_1 and from x_2 to y_2 in G is not greater than the value of a maximal feasible multiterminal flow from x_1, x_2 to y_1, y_2 in G.

Corollary 7

The value of a maximal feasible biflow from x_1 to y_1 and from x_2 to y_2 in G is not greater than the value of a minimal cut separating x_1, x_2 and y_1, y_2 in G.

Theorem 8 Interchange

If φ_1, φ_2 is a feasible biflow from x_1 to y_1 and from x_2 to y_2 in G and if K is a cycle of G with

$$\varphi_1(u) > 0 \quad \text{if} \quad u \in K^+ \quad \text{or}$$

$$\varphi_1(u) < 0 \quad \text{if} \quad u \in K^-,$$

then there exists an $\epsilon > 0$ such that the biflow φ_1', φ_2' is feasible, i.e.,

$$\varphi_1'(u) = \varphi_1(u) - \epsilon \text{ and } \varphi_2'(u) = \varphi_2(u) - \epsilon \text{ if } u \in K^+,$$

$$\varphi_1'(u) = \varphi_1(u) + \epsilon \text{ and } \varphi_2'(u) = \varphi_2(u) + \epsilon \text{ if } u \in K^-$$

or

$$\varphi_1'(u) = \varphi_1(u) \qquad \text{and } \varphi_2'(u) = \varphi_2(u) \qquad \text{if } u \notin K.$$

Theorem 9

Let φ_1, φ_2 be a feasible biflow from x_1 to y_1 and x_2 to y_2 in G; the chain-flow and bunch flow-diminution theorems are applicable to either flow φ_1 or flow φ_2 with the other flow remaining fixed. The bunch flow-diminution procedure is unchanged.

Construction of a maximal biflow

We start with a maximal feasible multiterminal flow from x_1, x_2 to y_1, y_2 in G and progressively construct a maximal feasible biflow φ_1, φ_2 from x_1 to y_1 and from x_2 to y_2. The algorithm proceeds in three phases:

1. Construct an initial integral biflow.
2. Find the value of a maximal feasible biflow.
3. Determine the maximal feasible biflow itself.

- *Phase 1*

Consider a maximal feasible multiterminal flow φ from x_1, x_2 to y_1, y_2 in G of value $v = v_1 + v_2 = -(v_1' + v_2')$. By application of the chain-flow theorem and the integrality and non-negativity theorem, it is always possible to find two feasible integral multiterminal flows φ_1' from x_1 and x_2 to y_1 and φ_2' from x_1 and x_2 to y_2, where φ_1' has value $-v_1$ and φ_2' has value $-v_2$, such that

$$\varphi = \varphi_1' + \varphi_2'$$

and, for all u, $u \in U$,

$$e(x_1, u)\varphi_1'(u) \geq 0,$$

$$e(x_1, u)\varphi_2'(u) \geq 0,$$

$$e(x_2, u)\varphi_1'(u) \geq 0 \quad \text{and}$$

$$e(x_2, u)\varphi_2'(u) \geq 0.$$

We illustrate this phase with the network G of Fig. 1 in which nodes are represented by circles, edges by lines and capacities by numbers assigned to the lines; directions of the edge flows are represented by arrows and their values by numbers assigned to the arrows.

Then in the network $G^* = (X, U^*, e)$ (see Fig. 1d) obtained from G by the addition of one supplementary

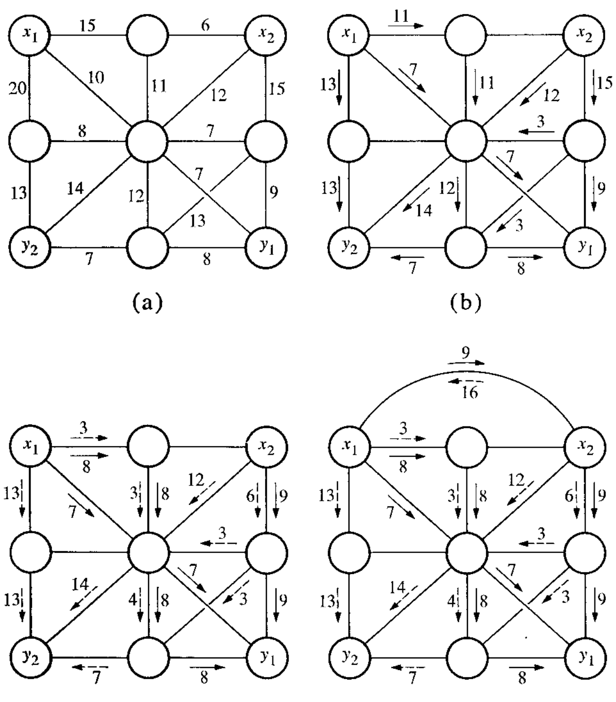

Figure 1 Network G—an example of Phase 1: (a) nodes, edges and capacities; (b) maximal flow from x_1, x_2 to y_1, y_2; (c) maximal flows from x_1, x_2 to y_1 (φ_1', solid arrow) and from x_1, x_2 to y_2 (φ_2', dashed arrow); and (d) initial biflow (end of Phase 1).

edge u^* relating nodes x_1 and x_2, with $c(u^*) = 0$ and $e(x_1, u^*) = 1$, the pair of functions φ_1, φ_2 defined by

$$\varphi_1(u) = \varphi_1'(u), \qquad u \in U;$$

$$\varphi_2(u) = \varphi_2'(u), \qquad u \in U;$$

$$\varphi_1(u^*) = \sum_{u \in U} \varphi_1'(u) e(x_2, u); \quad \text{and}$$

$$\varphi_2(u^*) = -\sum_{u \in U} \varphi_2'(u) e(x_1, u)$$

is a biflow of value $v = v_1 + v_2$ from x_1 to y_1 and from x_2 to y_2 in G^* which satisfies the condition

$$|\varphi_1(u)| + |\varphi_2(u)| \leq c(u), \qquad u \in U,$$

where $\varphi_1(u^*)$ and $-\varphi_2(u^*)$ are non-negative integers.

- *Phases 2 and 3*

In Phase 2 the algorithm simultaneously reduces the values of $\varphi_1(u^*)$ and $-\varphi_2(u^*)$ by the same integer or half-integer ϵ, $\epsilon > 0$, and yields either a maximal feasible biflow from x_1 to y_1 and from x_2 to y_2 in G or a minimal cut separating x_1, y_2 and x_2, y_1 in G. In the latter case Phase 2 also yields the value of a maximal biflow.

In Phase 3 the algorithm constructs the maximal biflow itself. This will be proved in the following section. Before describing the algorithm, we introduce the notions of unsaturated chain and of maximal modification of the biflow.

Saturated chain

A chain E relating x_1 and x_2 in G is said to be saturated if there is either a forward edge u of the chain such that

$$\varphi_1(u) \geq 0, \quad \varphi_2(u) \leq 0 \quad \text{and}$$

$$|\varphi_1(u)| + |\varphi_2(u)| = c(u)$$

or a backward edge u such that

$$\varphi_1(u) \leq 0, \quad \varphi_2(u) \geq 0 \quad \text{and}$$

$$|\varphi_1(u)| + |\varphi_2(u)| = c(u).$$

Otherwise, chain E is said to be unsaturated. A node scanning and labeling routine finds such an unsaturated chain if one exists.

Modification of the biflow along an unsaturated chain

If E is an unsaturated chain relating x_1 and x_2 in G, there exists a positive number ϵ such that, starting with an initial biflow φ_1, φ_2 of value v, the pair of functions φ_1', φ_2' obtained as

$$\varphi_1'(u) = \varphi_1(u) + \epsilon, \quad \varphi_2'(u) = \varphi_2(u) - \epsilon \quad \text{if } u \in E^+,$$

$$\varphi_1'(u) = \varphi_1(u) - \epsilon, \quad \varphi_2'(u) = \varphi_2(u) + \epsilon \quad \text{if } u \in E^-$$

or

$$\varphi_1'(u) = \varphi_1(u), \qquad \varphi_2'(u) = \varphi_2(u) \qquad \text{if } u \notin E,$$

with

$$\varphi_1'(u^*) = \varphi_1(u^*) - \epsilon, \quad \varphi_2'(u^*) = \varphi_2(u^*) + \epsilon$$

is a biflow of value v from x_1 to y_1 and from x_2 to y_2 in G, and

$$|\varphi_1'(u^*)| + |\varphi_2'(u^*)| = |\varphi_1(u^*)| + |\varphi_2(u^*)| - 2\epsilon.$$

Maximal modification of the biflow

With the contraint that $\varphi_1'(u^*)$ remains non-negative and $\varphi_2'(u^*)$ remains non-positive, the maximal value of ϵ is given by

$$2\epsilon = \min \{\min_{u \in E^+} [c(u) - \varphi_1(u) + \varphi_2(u)], \min_{u \in E^-} [c(u) + \varphi_1(u) - \varphi_2(u)], 2\varphi_1(u^*), -2\varphi_2(u^*)\}.$$

Theorem 10

The biflow obtained after any maximal modification is integral or half-integral.

Flow of type 2 can be reduced by $2\Delta_0 - \Delta_1$ along B_3^* and one obtains $\varphi_2(u^*) = 0$.

Further, Theorem 5 indicates that

$$\Delta - \Delta_1 > 0 \Rightarrow \text{there exists a bunch } B_2^* \text{ relating } x_2 \text{ and } y_2 \text{ in } \bar{G}^* \text{ passing by } \bar{u}.$$

Flow of type 2 can be reduced by $\Delta - \Delta_1$ along B_2^* and one obtains $\varphi_2(\bar{u}) = 0$.

The two bunches $B_3 = B_3^* - u^*$ and $B_2 = B_2^* - \bar{u}$ form a bunch B relating x_1 and y_1 in $\bar{G}^*$; B is unsaturated and it is possible to increase the flow of type 1 along B by the value

$$v' = \min (2\Delta_0 - \Delta_1, \Delta - \Delta_1).$$

The final biflow is a feasible biflow from x_1 to y_1 and from x_2 to y_2 in $\bar{G}^*$, obtained after a reduction by $[\Delta_1 + (\Delta - \Delta_1) + (2\Delta_0 - \Delta_1)]$ and an augmentation by v' of the initial value $v + \Delta$, and

$$\varphi_1(\bar{u}) = \varphi_2(\bar{u}) = \varphi_1(u^*) = \varphi_2(u^*) = 0.$$

This biflow is then a maximal feasible biflow from x_1 to y_1 and from x_2 to y_2 in G of value

$$\begin{array}{lll} v & \text{if} & \Delta \geq 2\Delta_0 \quad \text{or} \\ v + \Delta - 2\Delta_0 & \text{if} & \Delta < 2\Delta_0. \end{array}$$

Example

Refer to the network in Fig. 1d; after a sequence of maximal modifications along unsaturated chains we obtain the network in Fig. 4a. A maximal biflow from x_1 to y_1 and from x_2 to y_2 is shown in Fig. 4b, where a minimal cut separating x_1, y_2 and y_1, x_2 has been indicated.

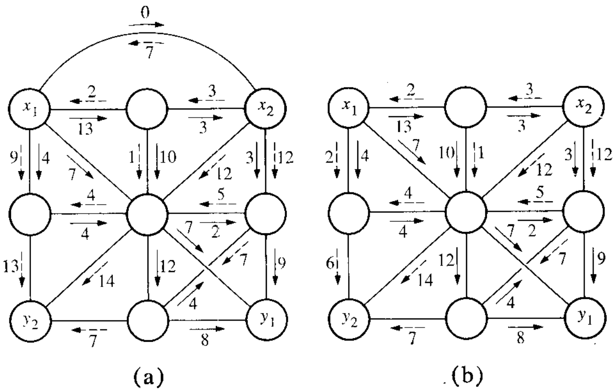

Figure 4 Network G: (a) final unfeasible biflow (end of Phase 2) and (b) maximal biflow and minimal cut (end of Phase 3).

Discussion and extension

The most interesting feature of the algorithm described is that it takes only a short time to do Phase 2 and it is therefore easy to compute the value of a maximal biflow in a given network. This may be sufficient in a number of problems where one is not really interested in the construction of the maximal feasible biflow itself.

This algorithm, with a classical, maximal-flow, search procedure, can be used to determine the initial multiterminal flows from x_1, x_2 to y_1 and from x_1, x_2 to y_2 in G.

The algorithm can also be used to compute easily a lower bound of the value of a multicommodity flow between a set of sources and a set of sinks in a given network. In particular, we have obtained results in the case of a three-commodity flow problem. To find a maximal three-commodity flow from x_1 to y_1, x_2 to y_2 and x_3 to y_3, the procedure is as follows:

1. Group x_1, x_2 and x_3 and find a maximal flow φ from x_1, x_2, x_3 to y_1, y_2, y_3.
2. Separate x_1, x_2 and x_3 and find a maximal flow φ_{12}, φ_3 from x_1, x_2 to y_1, y_2 and from x_3 to y_3.
3. Fix φ_3 and separate x_1 and x_2; find a maximal biflow φ_1, φ_2 from x_1 to y_1 and from x_2 to y_2 with the capacities now being $c - \varphi_3$.
4. Fix φ_1 and resume the procedure with φ_3 and φ_2, etc.

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