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Model of Competition in a Two-seller Market

Abstract: In this model of sellers' competition we are concerned with the transition from a single-seller market to a two-seller market, the effects of transition on the first seller, his likely reactions and the thereby changed market situation that awaits the entering competitor. The decision variables considered in this model are the sizes of the two sales forces. We show that the enterer should not view the market as it stands prior to his entry, but as changed by the first seller to accommodate or oppose his entry. Upon the entrance of the competitor, both sellers must increase their sales forces to attain the levels of profit anticipated prior to entry, if indeed these can be attained. Equilibrium strategies are shown to be maximal sales effort on the part of the first seller and either maximal sales effort or abstention from entry on the part of the enterer. A finite series is derived to express the exact expectation of a class of rational functions of a binomially distributed random variable.

Introduction

Sellers' competition has been discussed by several authors. Machlup[1] has considered markets of one, few and many sellers. Sengupta[2] has included the sizes of the sales forces in the determination of the effects of competition. Von Neumann and Morgenstern[3] modeled sellers' competition in a game-theoretic structure, opening the literature to many subsequent contributions. Shubik[4] considered the entry of new competitors into the market. The latter two citations consider price as an equilibrium determining variable.

An equilibrium exists when a change in the strategy of any one of the sellers would worsen his position, given that the other sellers do not vary their respective strategies. Price has traditionally been a variable in the set of strategies considered by economists. However, we believe that there exist market conditions in which all sellers would maintain a price regardless of other circumstances. In this model we analyze both a price-invariant equilibrium and a price-invariant status quo situation.

It is well known that if a single seller enjoys a high profit level, the entrance of a competitor will lead to a decline in prices and profits. In this paper we consider an enterer whose interest in the given business is to share the high profits of the first seller. He wishes to act not as a price competitor, but as a participant in the business of the first seller and therefore commits himself to charge the same price as the first seller who in turn, aware that a price cut might lead to equilibrium at a much lower price level, preserves his price.

We assume that the decision variables of the two companies are the sizes of their sales forces. Under certain assumptions about the selling practices of the companies, we show that profit, profit margin and market share can be expressed as functions of the sizes of the sales forces of both companies and the overlap of their territories. Overlap is shown to be a hypergeometrically distributed random variable. We obtain expressions for expected profit, expected profit margin and expected market share as functions of the sizes of the sales forces and the number of prospects.

Given target values of expected profit, if a solution exists the sizes of the sales forces necessary to achieve the target profits can be determined. Three strategies are shown to be necessary and sufficient to answer any strategy of the opponent. Under certain conditions, if the first seller uses a sufficiently large number of salesmen, he can make entry at any level unprofitable for the enterer. If these conditions are not satisfied, each company improves its own expected profit and degrades its opponent's expected profit by enlarging its own sales force. If the companies wish to maximize expected profit, a Nash equilibrium†[5] exists when the first seller saturates the market and the enterer either saturates the market (if entry is profitable) or abstains from entry (if entry is unprofitable). These two equilibria are mutually exclusive and collectively exhaustive; both are not equilibria simultaneously and the competitors possess the information to select the actual equilibrium.

† If x is the strategy of the first seller, y the strategy of the enterer and $f(x, y)$ and $g(x, y)$ the respective "rewards," then (x^*, y^*) is a Nash equilibrium if $f(x^*, y^*) \geq f(x, y^*)$ and $g(x^*, y^*) \geq g(x^*, y)$ for all feasible choices of x and y.

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An equilibrium can exist in which the enterer saturates the market and the first seller withdraws. This equilibrium is not the one preferred by the first seller and, if he announces his intention of not withdrawing, he can assure himself of attaining the desired equilibrium under an assumption of rational behavior on the part of the enterer.

The upper bound on expected profit margin for either company decreases when the strength of the sales force of the opponent increases, while the actual value of expected profit margin for each company increases with the strength of its own sales force, as long as entry is profitable. Thus the optimal strategies for the maximization of expected profit hold also for maximizing expected profit margin.

Upper and lower bounds on expected market share are obtained which, for each company, increase with the strength of its own sales force, but decrease as that of its opponent increases. Optimal strategies for both are to saturate the market.

In the evaluation of expected profit margin and expected market share, expressions of the form

$$\mathrm{E}\left(\frac{a_1 - b_1 k}{a_2 - b_2 k}\right) \tag{1}$$

occur, where k is a binomially distributed random variable with mean ρN, and E is the expectation with respect to the distribution of k; for discrete k

$$\mathrm{E}f(k) = \mathrm{E}[f(k)] = \sum f(x)\, \mathrm{P}[k = x], \tag{2}$$

where the sum is over all possible values of x and P is the probability measure on the space over which k is defined. If $|b_2/a_2| < 1$, then (1) can be written as an infinite but convergent sum,

$$\mathrm{E}\left(\frac{a_1 - b_1 k}{a_1 - b_2 k}\right) = \frac{a_1}{a_2}\left[1 + \left(1 - \frac{a_2 b_1}{a_1 b_2}\right) \sum_{t=1}^{\infty} \mathrm{E}\left(\frac{b_2 k}{a_2}\right)^t\right]. \tag{3}$$

If the coefficients also satisfy the condition $|b_2/a_2| < 1/N$, we let $\alpha = Nb_2/a_2$, $|\alpha| < 1$, and show in the Appendix that the sum of the expectations in (3) can be evaluated exactly as a finite series:

$$\sum_{t=1}^{\infty} \mathrm{E}\left(\frac{\alpha k}{N}\right)^t = \frac{\alpha \rho N}{N - \alpha} + \frac{\alpha^2 \rho^2 N(N-1)}{(N-\alpha)(N-2\alpha)} + \cdots + \frac{\alpha^N \rho^N N!}{(N-\alpha)\cdots(N-N\alpha)}$$

$$= \sum_{j=1}^{N} \rho^j \binom{\frac{N}{\alpha} - j - 1}{\frac{N}{\alpha} - N - 1} \binom{\frac{N}{\alpha} - 1}{N}^{-1}. \tag{4}$$

Competitive environment

Let the first seller be called the Red company and the enterer the Blue company.

We assume that the selling process is a discrete operation completed in a short time. The Red company sells a product among s prospects; s prospects were available in preceding time periods and s prospects will also be available in succeeding time periods. The transition to sales with a variable number of prospects, dependent on the number of sales in the preceding period and on external market conditions, is straightforward.

Let the sales unit of a company be defined as the number of hours required to sell the product to a prospect. By including the pro-rated hours of selling time wasted approaching people who are not prospects, we can specify that every prospect will purchase one unit of the product with certainty if approached by a salesman. This definition presupposes that the number of salesmen required to sell to s prospects is linear in s. This assumption can be justified if the product is unique and the set of prospects is fairly well identified. We assume sufficient coordination between salesmen so that no two salesmen from the same company approach the same prospect. Prospects are chosen at random from the available set.

Let the Red company possess n sales units. By hypothesis the product is sold to n prospects. Clearly $n \leq s$. Let a be the unit manufacturing cost, c_1 be the selling cost incurred by the sales unit and r be the unit revenue. Then, since each sales unit accounts for the sale of one product, the unit profit is $r - a - c_1$ and the total profit is $n(r - a - c_1)$.

Now the Blue company enters into competition with the Red company. There are m Blue sales units and the associated salesmen, like the Red salesmen, do not interfere with each other. However, there is no communication between Red and Blue salesmen, so it is possible for a prospect to be approached by no salesman, one Red salesman, one Blue salesman, or one Red and one Blue salesman. In the first case no sale will be made, in the second and third cases one sale by Red and by Blue, respectively, and in the fourth case, which we call *overlap*, one sale will be made by Blue with probability p or by Red with probability $q = 1 - p$. We assume that the Red and the Blue products are undifferentiated and priced the same. The preference of the purchaser depends on his confidence in the Red and Blue companies. Red is more favored by virtue of experience and reputation, so $p < q$.

When overlap occurs, the sales effort is greater on both sides because the two salesmen are competing against each other and may have to repeat calls more frequently. Let c_2 be the sales cost incurred by each sales unit in overlap situations; thus $c_2 > c_1$. Here two sales units are expended to obtain one sale. Let k be the number of overlaps among s prospects. From consideration of the equivalent occupancy problem, k is a random variable with hypergeometric distribution

$$P_h(k;\ m, n, s) = \binom{n}{k}\binom{s-n}{m-k}\binom{s}{m}^{-1}. \tag{5}$$

If n cells out of s are occupied at random and if m cells are selected at random without replacement, then $P_h(k; m, n, s)$ is the probability that exactly k occupied cells will be selected. With this distribution, $\mathrm{E}k = mn/s$.

Company objectives

The functions of interest to Red and to Blue are profit, profit margin and market share. The objectives of the companies may be the maximization of the expectation of one or more of these functions, subject to constraints on the others. Frequently, however, a company will not maximize a function because a concomitant of maximization could be rapid growth. This rapid growth might be associated with large debt or with the anticipation of little or no growth in subsequent years and both of these consequences can be undesirable. When the objective of a company is not the maximization of one of these functions, it may be the attainment of a particular value of one of the functions. We do not attempt to prescribe a management philosophy but, where the maximization of a function is desired, we show the optimal strategies and, where the attainment of a target is sought, we show the relevant strategies.

We consider the strategies of both companies with respect to objectives that are, in turn, the expectations of profit, profit margin and market share. Blue's strategy is determined by the choice of a value of m, Red's by the choice of a value of n. If there are s prospects, no more than s units can be sold by either company; hence m and n are bounded from above by s. We define saturation by Blue (or Red) as the choice of m (or n) equal to s. Clearly n is positive and m is positive if Blue chooses to enter the market or zero if Blue abstains from entry.

The following notation is used:

A = number of units sold,
B = cost of manufacturing A units,
C = cost of selling A units,
D = revenue from the sale of A units,
X = profit from the sale of A units,
Y = profit margin from the sale of A units and
Z = market share from the sale of A units.

Variables associated with Blue and Red are subscripted B and R, respectively:

$$\begin{aligned}
A_B &= m - k + pk &&= m - qk,\\
B_B &= aA_B &&= a(m-qk),\\
C_B &= c_1(m-k) + c_2k &&= mc_1 + k(c_2 - c_1),
\end{aligned}$$

$$\begin{aligned}
D_B &= rA_B = r(m-qk),\\
X_B &= D_B - B_B - C_B\\
&= (r-a-c_1)m - (rq - aq + c_2 - c_1)k,\\
Y_B &= \frac{X_B}{(B_B + C_B)}\\
&= \frac{(r-a-c_1)m - (rq - aq + c_2 - c_1)k}{(a+c_1)m - (aq - c_2 + c_1)k} \quad \text{and}\\
Z_B &= \frac{D_B}{D_B + D_R} = \frac{m - qk}{m + n - k}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
A_R &= n - pk,\\
B_R &= a(n - pk),\\
C_R &= nc_1 + k(c_2 - c_1),\\
D_R &= r(n - pk),\\
X_R &= (r-a-c_1)n - (rp - ap + c_2 - c_1)k,\\
Y_R &= \frac{(r-a-c_1)n - (rp - ap + c_2 - c_1)k}{(a+c_1)n - (ap - c_2 + c_1)k} \quad \text{and}\\
Z_R &= \frac{n - pk}{m + n - k}.
\end{aligned}$$

In writing these expressions we have assumed that unit costs are the same for both companies. Because k, the number of sales overlaps among s prospects, is a random variable, X, Y and Z are random functions.

Expected profit

Using the hypergeometric distribution (5), as a consequence of $\mathrm{E}k = mn/s$ we obtain

$$\mathrm{E}X_B = m[r - a - c_1 - (rq - aq + c_2 - c_1)n/s], \tag{6}$$

which is linear in m. We assume $r - a - c_1 > 0$. The expected profit $\mathrm{E}X_B$ is increasing in m if

$$r - a - c_1 > (rq - aq + c_2 - c_1)n/s. \tag{7}$$

Because $r > a$ and $c_2 > c_1$, this inequality holds when

$$n < (r - a - c_1)s/(rq - aq + c_2 - c_1) \equiv \nu. \tag{8}$$

Thus, if $n > \nu$, Blue's entry will not be profitable at any level. If Red wishes to exclude Blue and inequality (8) is satisfied prior to Blue's entry, Red can reverse the inequality by increasing n provided $\nu/s < 1$ or $r-a < c_2/p$.

Similarly the expected profit for Red is

$$\mathrm{E}X_R = n[r - a - c_1 - (rp - ap + c_2 - c_1)m/s], \tag{9}$$

which is increasing in n when

$$r - a - c_1 > (rp - ap + c_2 - c_1)m/s. \tag{10}$$

If inequality (10) can be reversed by Blue, i.e., if Blue can act to exclude Red, then Blue is excludable by Red, for if $r - a - c_1 < (rp - ap + c_2 - c_1)m/s$, then it follows that $r - a < c_2/q < c_2/p$.

Prior to Blue's entry $m = 0$ and Red's expected profit is

$$\mathrm{E}(X_R | \, m = 0) = n(r - a - c_1). \tag{11}$$

If Blue is deterred by increasing n to n^*, $n^* > v > n$, Red's expected profit is increased. Thus Red does not lose by excluding Blue. If Blue is not deterred, accepting an unprofitable entry, Red's expected profit is

$$\mathrm{E}X_R = n^*[r - a - c_1 - (rp - ap + c_2 - c_1)m/s]. \tag{12}$$

The change in expected profit for Red is

$$(n^* - n)(r - a - c_1) - (rp - ap + c_2 - c_1)n^*m/s, \tag{13}$$

which is positive when

$$m < (n^* - n)(r - a - c_1)s/[(rp - ap + c_2 - c_1)n^*]. \tag{14}$$

The right-hand side of (14) is maximized when $n^* = s$ and (13) is negative only if the inequality (14) is reversed. By choosing $n^* = s$, Red can maximize the minimal value of m required to reverse the inequality. Then Blue's entry would cause a diminution of Red's expected profit only if

$$n > s[(r - a)q - c_2]/(r - a - c_1). \tag{15}$$

If Blue is rational, i.e., follows a strategy of non-negative profit, $\mathrm{E}X_R$ is maximized at $n^* = s$, so Red can find no better strategy. If, however, Blue is irrational, choosing a losing strategy to spite Red, $n^* = s$ can be Red's worst strategy although Blue loses more than Red. Blue may pick such a strategy if he has more resources than Red and is willing to suffer initial losses to drive Red out of the market. Such a strategy belongs to a game of economic survival,[4] which we shall not discuss in this context. We affirm here a specific assumption of rationality—if Blue is excludable and Red acts to exclude him, then Blue does not enter.

From the analysis of expected profit alone, it is optimal for Red to saturate the market, i.e., to choose $n = s$. Then if $p > c_2/(r - a)$, Blue should also saturate the market; but if $p < c_2/(r - a)$, Blue should not enter. For $p = c_2/(r - a)$ any non-negative $m \leq s$ yields the same profit. Expected profits derived from the optimal strategies are

$$\mathrm{E}X_B = \begin{cases} s[(r - a)p - c_2] & \text{if } p > c_2/(r - a) \text{ or} \\ 0 & \text{if } p \leq c_2/(r - a) \end{cases} \tag{16}$$

and

$$\mathrm{E}X_R = \begin{cases} s[(r - a)q - c_2] & \text{if } p > c_2/(r - a) \text{ or} \\ s(r - a - c_1) & \text{if } p < c_2/(r - a); \end{cases} \tag{17}$$

for $p = c_2/(r - a)$ Red's expected profit has a value between the extremes.

Red and Blue, in the face of mutual interaction, might want to maintain certain profit levels rather than maximize them. For example, prior to Blue's entry Red's expected profit is $n(r - a - c_1)$. Not taking into consideration any change in n, Blue selects m for entry to yield expected profit $m[r - a - c_1 - (rq - aq + c_2 - c_1)n/s]$. This is possible if there exists a solution m^*, $n^* > 0$ of the equations

$$n^*[r - a - c_1 - (rp - ap + c_2 - c_1)m^*/s] = n(r - a - c_1) \tag{18a}$$

and

$$m^*[r - a - c_1 - (rq - aq + c_2 - c_1)n^*/s] = m[r - a - c_1 - (rq - aq + c_2 - c_1)n/s], \tag{18b}$$

subject to m^*, $n^* \leq s$. If a solution exists, $n^* > n$ from (18a) and therefore $m^* > m$ from (18b), provided the inequality (8) is satisfied for n^*.

If no n^* satisfies (18), the best Red can do is to pick $n^* = s$; then m^* is determined by (18b). The best Blue can do is to choose the lesser of m^* and s. Similarly, if no m^* satisfies (18), the best Blue can do is to pick $m^* = s$; then (18a) fixes the value of n^* and Red should choose the lesser of n^* and s. If, for any n^* (or s) that Red picks, inequality (8) is not satisfied, then Blue should not enter.

Expected profit margin

We showed in the last section that to maximize expected profit Red has a strategy optimal against any rational Blue strategy, namely $n = s$, and Blue has a strategy which is optimal against $n = s$, namely $m = s$ if $p > c_2/(r - a)$ or $m = 0$ if $p < c_2/(r - a)$. In this section we show that the same strategies are also optimal in the case in which the objective is to maximize expected profit margin.

For Blue the expected profit margin is

$$\mathrm{E}Y_B = \mathrm{E}\left[\frac{(r - a - c)m - (rq - aq + c_2 - c_1)k}{(a + c_1)m - (aq - c_2 + c_1)k}\right]. \tag{19}$$

The ratio of the expectation of the numerator to that of the denominator represents an upper bound [because $Y_B(k)$ is a concave function of k] of $\mathrm{E}Y_B$, i.e.,

$$\mathrm{E}Y_B \leq \frac{(r - a - c_1)s - (rq - aq + c_2 - c_1)n}{(a + c_1)s - (aq - c_2 + c_1)n}, \tag{20}$$

which is concave and decreasing in n. Thus Red can bound Blue's expected profit margin.

To evaluate the right-hand side of (19), we first note that the factorial moments of the hypergeometric distribution are

$$Ek = \frac{mn}{s},$$

$$Ek(k-1) = \frac{m(m-1)n(n-1)}{[s(s-1)]},$$

$$\vdots$$

$$Ek(k-1)\cdots(k-j) = \frac{m(m-1)\cdots(m-j)n(n-1)\cdots(n-j)}{[s(s-1)\cdots(s-j)]}.$$

If n and s are large and j is small, the first $j+1$ factorial moments are closely approximated by

$$mn/s,$$
$$m(m-1)n^2/s^2,$$
$$\vdots$$
$$m(m-1)\cdots(m-j)n^{j+1}/s^{j+1},$$

which are the factorial moments of the binomial distribution,

$$P_b(k;\, m, n, s) = \binom{m}{k}\left(\frac{n}{s}\right)^k\left(\frac{s-n}{s}\right)^{m-k}. \tag{21}$$

Hence for sufficiently large n we use the binomial distribution as a close approximation to the hypergeometric distribution.

The expectation EY_B therefore takes the form [see Eqs. (3) and (4)],

$$EY_B = \frac{r-a-c_1}{a+c_1}\left\{1 + \left[1 - \frac{rq - aq + c_2 - c_1}{\alpha(r-a-c_1)}\right]\right.$$
$$\left.\times \sum_{j=1}^{m}\left(\frac{n}{s}\right)^j \binom{\frac{m}{\alpha}-j-1}{\frac{m}{\alpha}-m-1}\binom{\frac{m}{\alpha}-1}{m}^{-1}\right\}, \tag{22}$$

where $\alpha = (aq - c_2 + c_1)/(a + c_1) < 1$. We now show that the magnitude of the series in (22) is decreasing in m for $n/s < 1$ and constant in m for $n/s = 1$.

For $n/s = 1$, k takes the value m with probability one, i.e., $P[k = m] = 1$. Hence $Ek^t = m^t$ for all t and

$$\sum_{t=1}^{\infty} E\left(\frac{\alpha k}{m}\right)^t = \frac{\alpha}{1-\alpha}. \tag{23}$$

The finite series in (22) is also a valid representation, so for $n/s = 1$ we obtain the identity

$$\frac{\alpha}{1-\alpha} = \frac{\alpha m}{m-\alpha} + \frac{\alpha^2 m(m-1)}{(m-\alpha)(m-2\alpha)} + \cdots$$
$$+ \frac{\alpha^m m!}{(m-\alpha)\cdots(m-m\alpha)} \tag{24}$$

for all positive integers m.

Each term of the series in (22), $n/s < 1$, is a term of (24) weighted by $(n/s)^j$, $j = 1, 2, \cdots, m$; $(n/s)^j$ is decreasing in j, so as m increases the series decreases in magnitude. We can write the value of the series as

$$\frac{\alpha}{1-\alpha}\left(\frac{n}{s}\right)^{\beta(m)},$$

where $\beta(m)$ is a real number, $1 \le \beta < m$, and is increasing in m. Thus the expected profit margin for Blue can be expressed as

$$EY_B = \frac{r-a-c_1}{a+c_1}$$
$$\times\left[1 - \frac{r(c_2 - c_1 p)}{(r-a-c_1)(a+c_1)(1-\alpha)}\left(\frac{n}{s}\right)^{\beta(m)}\right]. \tag{25}$$

Clearly (25) is increasing in m.

Similarly we can write

$$EY_R = \frac{r-a-c_1}{a+c_1}$$
$$\times\left[1 - \frac{r(c_2 - c_1 q)}{(r-a-c_1)(a+c_1)(1-\alpha)}\left(\frac{m}{s}\right)^{\gamma(n)}\right], \tag{26}$$

which is increasing in n. If EY_R is positive, $n=s$ is optimal for Red, thus making EY_B, Eq. (25), constant in m and positive only if $p > c_2/(r-a)$. If Blue is rational, $m > 0$ implies $p > c_2/(r-a)$ and also $q > c_2/(r-a)$ because $q > p$. Thus EY_R, Eq. (26), is positive and it follows immediately that the optimal strategies of the preceding section are optimal for this section also.

Expected market share

It is intuitively obvious, regardless of the value of p, that m must be greater than zero when Blue's objective is the maximization of expected market share, even though profit may have to be sacrificed. Blue's expected market share is

$$EZ_B = E\left(\frac{m - qk}{m+n-k}\right)$$
$$= \frac{m}{m+n}\left[1 + \frac{m - q(m+n)}{m}\sum_{t=1}^{\infty} E\left(\frac{k}{m+n}\right)^t\right]. \tag{27}$$

The series in Eq. (27) can be expressed as in Eq. (4) as

$$\sum_{t=1}^{\infty} E\left(\frac{k}{m+n}\right)^t = \binom{m+n-1}{m}^{-1}$$
$$\times \sum_{j=1}^{m}\binom{m+n-1-j}{n-1}\left(\frac{n}{s}\right)^j. \tag{28}$$

The market share function Z_B is concave or convex in k for $q \lessgtr m/(m+n)$, respectively, and is linear in k for $q = m/(m+n)$. Therefore the value of

$$m(s - qn)/(ms + ns - mn) \tag{29}$$

is, respectively, a lower bound, an upper bound or the exact value of EZ_B, Eq. (27), according to the value of q. The bound (29) is decreasing in n; if it is an upper bound of Blue's expected market share, Red can limit it to at most the value $q^2m/[m(q - p) + sp]$ by choosing $n \geq ps/q$. Because (29) is increasing in m, whether it is a lower or an upper bound, the best Blue can do regardless of Red's strategy is to choose $m = s$. Red's best strategy is to choose $n = s$ because its market share is the complement of Blue's. With both m and n equal to s, (29) is seen to be an upper bound with value p. Under this strategy $P[k = s] = 1$ and $EZ_B = p$; the bound is the expectation itself.

Summary

The optimization of expected profit, profit margin and market share leads to a unique strategy for Red and for Blue as long as Blue is not excludable. This strategy is $m = n = s$. If Blue is excludable the optimization of expected profit and profit margin is achieved by $n = s$ and $m = 0$. This strategy also optimizes Red's expected market share. The strategies are equilibrium and Red's strategy is dominating. One can interpret the situation as follows: Prior to Blue's entry Red sells (free from external pressure) at a convenient level, not necessarily saturating the market. With Blue's real or potential entry, Red will increase his sales efforts. If guided solely by the criteria of profit, profit margin or market share, the increased effort will be planned to saturate the market as soon as possible. If Blue is contemplating entry at a certain level, he should expect to enter at a higher level to achieve the same profit. His best return, profitable entry, is also attained by saturating the market. Hence the equilibrium result is strong competition in a saturated market.

Appendix

Theorem

For $0 < |\alpha| < 1$ and k binomially distributed with mean ρN,

$$\sum_{t=1}^{\infty} E\left(\frac{\alpha k}{N}\right)^t = \frac{\alpha \rho N}{N - \alpha} + \frac{\alpha^2 \rho^2 N(N-1)}{(N-\alpha)(N-2\alpha)} + \cdots + \frac{\alpha^N \rho^N N!}{(N-\alpha)\cdots(N-N\alpha)}$$

$$= \sum_{j=1}^{N} \rho^j \begin{bmatrix} \frac{N}{\alpha} - j - 1 \\ \\ \frac{N}{\alpha} - N - 1 \end{bmatrix} \begin{bmatrix} \frac{N}{\alpha} - 1 \\ \\ N \end{bmatrix}^{-1} . \tag{A1}$$

Proof

Let $m(z)$ be the moment generating function and $\hat{P}(z)$ be the probability generating function. Then (see Ref. 6)

$$m(z) = \hat{P}(e^z) \tag{A2}$$

and

$$Ek^t = \left.\frac{d^t m(z)}{dz^t}\right|_{t=0} = \left.\frac{d^t \hat{P}(e^z)}{dz^t}\right|_{t=0} . \tag{A3}$$

Also let

$$m^{(i)} = \frac{d^i m(z)}{dz^i}$$

and

$$\hat{P}^{(i)} = \frac{d^i \hat{P}(e^z)}{d(e^z)^i} ; \tag{A4}$$

thus we generate the sequence of derivatives

$$\begin{aligned} m^{(1)} &= e^z \hat{P}^{(1)}, \\ m^{(2)} &= e^z \hat{P}^{(1)} + e^{2z} \hat{P}^{(2)}, \\ m^{(3)} &= e^z \hat{P}^{(1)} + 3e^{2z} \hat{P}^{(2)} + e^{3z} \hat{P}^{(3)}, \text{ etc.} \end{aligned} \tag{A5}$$

If m_j is the jth moment and $(m)_j$ is the jth factorial moment, then

$$m_j = m^{(j)}|_{z=0}$$

and

$$(m)_j = \hat{P}^{(j)}|_{z=0}. \tag{A6}$$

Evaluating (A5) at $z = 0$ we have

$$\begin{aligned} m_1 &= a_{11}(m)_1, \\ m_2 &= a_{21}(m)_1 + a_{22}(m)_2, \\ &\vdots \\ m_j &= a_{j1}(m)_1 + a_{j2}(m)_2 + \cdots + a_{jj}(m)_j, \end{aligned} \tag{A7}$$

where $a_{j1} = 1$ and

$$a_{ji} = \begin{cases} ia_{j-1,i} + a_{j-1,i-1}, & 1 < i \leq j, \text{ or} \\ 0, & i > j. \end{cases} \tag{A8}$$

Multiplying the jth row of (A7) by $(\alpha/N)^t$ and summing over t, we obtain

$$\sum_{t=1}^{\infty} \left(\frac{\alpha}{N}\right)^t m_t = \sum_{t=1}^{\infty} \left(\frac{\alpha}{N}\right)^t a_{t1}(m)_1 + \sum_{t=2}^{\infty} \left(\frac{\alpha}{N}\right)^t a_{t2}(m)_2 + \cdots + \sum_{t=j}^{\infty} \left(\frac{\alpha}{N}\right)^t a_{tj}(m)_j + \cdots . \tag{A9}$$

We now show by the method of induction that

$$\sum_{t=j}^{\infty} \left(\frac{\alpha}{N}\right)^t a_{tj} = \frac{\alpha^j}{(N-\alpha)(N-2\alpha)\cdots(N-j\alpha)}, \quad j \le N. \tag{A10}$$

Clearly, for $j = 1$

$$\sum_{t=1}^{\infty} \left(\frac{\alpha}{N}\right)^t a_{t1} = \frac{\alpha}{N-\alpha}. \tag{A11}$$

Assume that

$$\sum_{t=j-1}^{\infty} \left(\frac{\alpha}{N}\right)^t a_{t,j-1} = \frac{\alpha^{j-1}}{(N-\alpha)(N-2\alpha)\cdots[N-(j-1)\alpha]}; \tag{A12}$$

then, using relations (8), we find

$$\sum_{t=j}^{\infty} \left(\frac{\alpha}{N}\right)^t a_{tj} = \sum_{t=j}^{\infty} \left(\frac{\alpha}{N}\right)^t (ja_{t-1,j} + a_{t-1,j-1}) = \frac{j\alpha}{N} \sum_{t=j}^{\infty} \left(\frac{\alpha}{N}\right)^t a_{tj} + \frac{\alpha^j}{N(N-\alpha)\cdots[N-(j-1)\alpha]}$$

or

$$\sum_{t=j}^{\infty} \left(\frac{\alpha}{N}\right)^t a_{tj} = \frac{\alpha^j}{(N-\alpha)(N-2\alpha)\cdots(N-j\alpha)}. \tag{A10}$$

It follows from (A7) that

$$\sum_{t=1}^{\infty} \left(\frac{\alpha}{N}\right)^t m_t = \frac{\alpha}{N-\alpha}(m)_1 + \frac{\alpha^2}{(N-\alpha)(N-2\alpha)}(m)_2 + \cdots + \frac{\alpha^N}{(N-\alpha)\cdots(N-N\alpha)}(m)_N + \sum_{j=N+1}^{\infty} \sum_{t=j}^{\infty} \left(\frac{\alpha}{N}\right)^t a_{tj}(m)_j. \tag{A13}$$

Finally,

$$(m)_j = \begin{cases} \rho^j N(N-1)\cdots(N-j+1), & j = 1, \cdots, N, \\ 0, & j = N+1, \cdots, \end{cases} \tag{A14}$$

and the Theorem (A1) is proved.

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