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Stochastic Model for Manufacturing Cost Estimating

Abstract: The unit manufacturing cost (i.e., its estimator) for a given manufacturing program with stochastic demand and operation yield is assumed to be a random variable. For a simple series production line the probability distribution of the unit manufacturing cost has been derived by either the transform method, which uses Mellin and Laplace transforms, or the method of moments, which uses either the Gram-Charlier series approximation or the Pearson system of frequency curves. The estimates and 90%-confidence intervals for the base manufacturing cost are computed for two device-component products. The model cost estimates are very close to the actual values and the confidence intervals are sufficiently narrow to be useful in applying contingencies to the predictions.

Introduction

The economics of production and inventory control for a variety of production processes is critically influenced by the uncertainty of demand for goods as well as by the uncertain nature of the yield of operations at various stations or stages of the production process. Before undertaking a manufacturing program it is essential to have an estimate of the total program cost and the corresponding unit cost. In many situations it is possible to express the uncertainties associated with demand quantities, yield of operations and cost components such as labor and material in terms of suitable probability distributions. We show that both program cost and unit cost are specific functions of demand quantities, yield of operations and the various cost components. Thus both program cost and unit cost, regarded as functions of random variables, have probability distributions that permit estimation at any desired confidence level. In the planning stage it is possible to evaluate the effect of alternative methods of production with different costs of operation and different yield distributions on both the expected values and the confidence limits associated with the program.

Essential to the construction of the stochastic model for manufacturing cost estimating is the ability to derive the probability distribution of any rational algebraic function of random variables. If these random variables are assumed to be mutually independent, the probability distribution of a rational algebraic function of these random variables can be derived by the "transform" method, which utilizes Mellin and Laplace transforms. Two theorems necessary for this purpose concerning the relation between Laplace and Mellin transforms have been proved under fairly general assumptions about the probability distributions of the independent random variables. However, if the random variables are correlated and if the number of such variables is large, the direct method of "moments" can be used to obtain the probability distributions of rational algebraic functions of random variables. This latter method has also been investigated and general expressions for moments have been obtained. Once the moments are available, either the Gram-Charlier series approximation or standard statistical tables for the Pearson system of frequency distributions can be used to obtain the probability distributions. These methods are outlined briefly and applied to the analysis of data.

In an environment of planned production, it is necessary to incorporate "lead time" for various operations into the expression for manufacturing cost. Appropriate modification has been made for the case in which the total lead time for all operations is less than one period for cost estimation. In addition, in the actual production environment several products usually share the cost of an operation. Modification of our cost estimating procedure has been explored for such situations.

Manufacturing data have been analyzed using this model and comparison of results with actual cost indicates that the probabilistic model yields estimates with confidence intervals sufficiently narrow to be meaningful to the decision maker. The results of the analysis are included in the final section of this paper.

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Cost estimating model

The model applies to manufacturing processes involving a series of operations. We assume that the product passes through n operations or gates (where gates are several operations combined) and that it goes through each operation only once. Because of the yield factor the number of input units is greater than or equal to the number of output units in each operation. An additional assumption is that the yield value is zero with zero probability; there is no product reworking at any of the gates. In many process-oriented manufacturing environments, this is indeed the case. The modification of the model to incorporate reworking only increases the complexity of the manufacturing cost function without changing the analysis qualitatively.

In the development of this model, the operation yield is assumed to have either a uniform or a beta probability distribution. Whereas the uniform probability density implies minimum prior knowledge, the beta distribution assumes specific prior knowledge and Bayesian estimation of percent yield.[1] However, the applicability of this model is not limited to these distributions; the fixed and variable cost elements associated with each operation or gate in the manufacturing cycle are allowed to be subject to uncertainties characterized by any suitable probability distribution.

The total manufacturing program cost in dollars per unit of production has two major components, the base manufacturing cost (BMC) and the support cost. Figure 1 indicates schematically a more detailed decomposition of manufacturing cost. Our model is concerned primarily with the estimation of BMC. We deal with the estimation of BMC (per unit of final product) for the whole program period. If the program period is partitioned into several non-overlapping periods and we estimate BMC for each of these periods, we can estimate BMC for the whole program period. The principal components of BMC are

1. direct labor cost,
2. direct material cost and
3. burden cost that consists of base burden (standard overhead for the whole plant), indirect labor cost, indirect material cost, depreciation, department expenses, occupancy cost and maintenance and service costs, etc., associated with every operation.

These costs are expressed in terms of cost per unit for the ith operation as L_i, M_i and B_i, respectively.

The following notation is used in developing the mathematical model:

B = program BMC per acceptable finished unit,
$B(t)$ = BMC per unit for time period t,
n = number of operations (or gates),
r_n = final production required (number of acceptable units from the last operation) and
Y_i = yield of the ith operation, $0 < Y_i \leq 1$, $i = 1, 2, \cdots, n$.

The number of acceptable units entering the ith operation is seen to be

$$r_n\left(\prod_{j=i}^{n} Y_j\right)^{-1}.$$

The burden cost (per unit) is written as

$$B_i = e_i + r_n^{-1}\left(\prod_{j=i}^{n} Y_j\right)f_i,$$

where e_i is the part of the burden cost that is proportional to the number of units starting the ith operation and f_i is the fixed burden cost for the operation.

Under the assumption that no reworking is done at any of the operations, B [or $B(t)$ when appropriate] is given by

$$B = \sum_{i=1}^{n}\left[(L_i + M_i + B_i)\left(\prod_{j=i}^{n} Y_j\right)^{-1}\right]$$
$$\equiv W_n + \sum_{i=1}^{n}\left[c_i\left(\prod_{j=i}^{n} Y_j\right)^{-1}\right], \qquad (1)$$

where

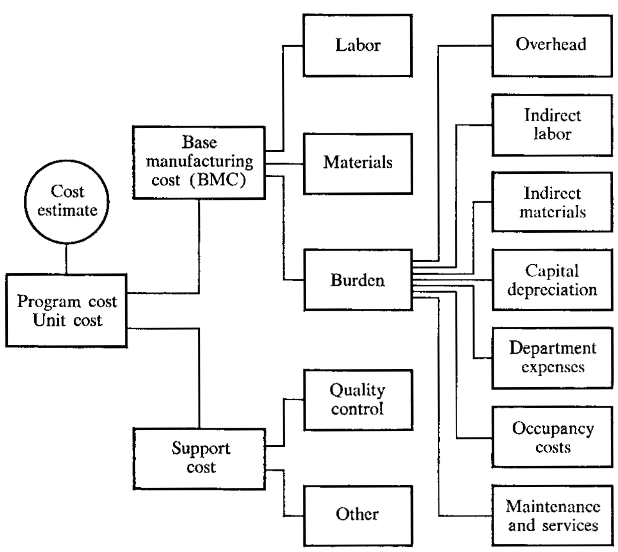
$$c_i = L_i + M_i + e_i,$$

$$W_n = r_n^{-1}\sum_{i=1}^{n} f_i$$

and r_n, Y_i, f_i and c_i are assumed to be random variables.

The expression for BMC is a rational algebraic function of random variables. Specifically, the function involves sums, products and quotients of random variables. Under the assumption that these random variables are independent, we use transform methods to obtain the probability

Figure 1 Manufacturing program cost elements.



distribution and moments of B. When the number of operations is very large and the random variables are not independent, it is preferable to obtain the moments of B directly to derive the probability distribution.

• *Transform method*

The manufacturing cost function can be reduced to arithmetic operations on independent random variables. The basic results for these operations can be stated as follows:

1. The Laplace transform of the density function of the sum of statistically independent random variables is the product of the Laplace transforms of the individual density functions of the random variables. Let the random variable Z be defined as the sum of n statistically independent random variables X_i, $Z = X_1 + X_2 + \cdots + X_n$, and let $L_Z(s)$, $L_{X_1}(s)$, $\cdots$, $L_{X_n}(s)$ be the Laplace transforms of the density functions of these random variables. Then

$$L_Z(s) = L_{X_1}(s)L_{X_2}(s) \cdots L_{X_n}(s). \tag{2}$$

2. The Mellin transform of the density function of the product of statistically independent random variables is the product of the Mellin transforms of the individual density functions of the random variables. Let the random variable P be defined as the product of n statistically independent random variables Y_i, $P = Y_1 Y_2 \cdots Y_n$ and let $M_P(\alpha)$, $M_{Y_1}(\alpha)$, $\cdots$, $M_{Y_n}(\alpha)$ be the Mellin transforms of the density functions of these random variables. Then

$$M_P(\alpha) = M_{Y_1}(\alpha)M_{Y_2}(\alpha) \cdots M_{Y_n}(\alpha). \tag{3}$$

3. The Mellin transform of the density function of the ratio of two statistically independent random variables is given by the relation

$$M_R(\alpha) = M_X(\alpha)M_Y(2 - \alpha), \tag{4}$$

where $R = X/Y$ and $M_R(\alpha)$, $M_X(\alpha)$ and $M_Y(\alpha)$ are the Mellin transforms of the density functions of R, X and Y, respectively.

To deal with rational algebraic expressions involving the sum, the product and the ratio of random variables, it is necessary to obtain the conversion of the Laplace transform to the Mellin transform and vice-versa. Two theorems[2] establishing the relation between the Laplace and the Mellin transforms are the following:

Theorem 1

If $L(s)$ is the Laplace transform of $f(t)$ such that all the singularities and branch points of $L(s)$ lie in the left half of the complex s plane, the Mellin transform of $f(t)$ is

$$M(\alpha) = (2\pi j)^{-1}\Gamma(\alpha) \int_{c-j\infty}^{c+j\infty} L(s)(-s)^{-\alpha}\, ds, \qquad \operatorname{Re}\alpha > 0. \tag{5}$$

In the special case in which the singularities of $L(s)$ are poles in the left half-plane, the Mellin transform of $f(t)$ is

$$M(\alpha) = \Gamma(\alpha)\sum [\text{residues of } L(s)(-s)^{-\alpha} \text{ at poles of } L(s)], \qquad \operatorname{Re}\alpha > 0.$$

Theorem 2

If the Mellin transform of $f(t)$ exists, the Laplace transform of $f(t)$ exists and is

$$L(s) = (2\pi j)^{-1} \int_{c-j\infty}^{c+j\infty} M(\alpha)\Gamma(1 - \alpha)s^{\alpha-1}\, d\alpha. \tag{6}$$

The manufacturing cost for the n-stage process has been defined as $W_n + Z_n$, Eq. (1), where now

$$Z_n = \sum_{i=1}^{n} \left[c_i \left(\prod_{j=i}^{n} Y_j \right)^{-1} \right].$$

A simple recurrence relation for Z_n is

$$Z_n = (Z_{n-1} + c_n)/Y_n, \quad \text{where} \quad Z_0 = 0. \tag{7}$$

It is assumed that W_n and Z_n are statistically independent. A general approach to the development of the probability law for the manufacturing cost proceeds as follows:

First, $Z_1 = c_1/Y_1$. Let $M_{Z_1}(\alpha)$, $M_{c_1}(\alpha)$ and $M_{Y_1}(\alpha)$ be the Mellin transforms of the density functions of these three random variables. Then

$$M_{Z_1}(\alpha) = M_{c_1}(\alpha)\, M_{Y_1}(2 - \alpha). \tag{8}$$

Next, $Z_2 = (Z_1 + c_2)/Y_2$. The Laplace transform of the density function of $Z_1 + c_2$ is $L_{Z_1+c_2}(s) = L_{Z_1}(s)L_{c_2}(s)$, where $L_{Z_1}(s)$ is obtained using Eq. (6). The Mellin transform of the density function of $Z_1 + c_2$ can now be obtained by Eq. (5) and hence the Mellin transform of the density function of Z_2 is

$$M_{Z_2}(\alpha) = M_{Z_1+c_2}(\alpha)M_{Y_2}(2 - \alpha). \tag{9}$$

A straightforward continuation of this procedure is used to develop the Mellin transform of the density function of Z_n. Once $M_{Z_n}(\alpha)$ is computed, $L_{Z_n}(s)$ can be obtained using Eq. (5). Thus the Laplace transform of the density function of the manufacturing cost is $L_{W_n}(s)L_{Z_n}(s)$. This procedure is best suited to numerical computation; it can be illustrated analytically, however, for small values of n and as an example the expression for $L_{W_2}(s)L_{Z_2}(s)$ is derived in the Appendix.

• *Moments method*

Under fairly general regularity conditions the probability density function $f(x)$ of a random variable can be approximated by a Gram-Charlier series of Type A,[3]

$$f(x) = \alpha(x) \sum_{i=0}^{\infty} a_i H_i(x), \tag{10}$$

where $\alpha(x) = (2\pi)^{-1/2} \exp(-\frac{1}{2}x^2)$, the $H_i(x)$ are Hermite polynomials and the a_i are polynomials in the moments about the mean of the random variable.

We let the sth moment about the mean of BMC be

$$U_s = \mathrm{E}\{[B - \mathrm{E}(B)]^s\}, \tag{11}$$

where E is the expectation operator, and

$$U_s^0 \equiv \mathrm{E}(B^s) \tag{12}$$

is the sth moment of B about the origin. Then

$$U_s^0 = \sum_{r=0}^{s} \binom{s}{r} \mathrm{E}(W_n^{s-r} Z_n^r). \tag{13}$$

If we assume that W_n and Z_n are mutually independent, Eq. (13) becomes

$$U_s^0 = \sum_{r=0}^{s} \binom{s}{r} \mathrm{E}(W_n^{s-r}) \mathrm{E}(Z_n^r). \tag{14}$$

For any probability distribution with finite moments, the moments about the mean and origin, Eqs. (11) and (12) respectively, are related according to

$$U_s = \sum_{j=0}^{s} (-1)^j \binom{s}{j} U_{s-j} U_j^0. \tag{15}$$

A standard table of multinomials such as the one in Ref. 4 can be used to compute U_s^0. Knowing the first four moments, we can use the Pearson system of frequency curves to obtain the probability distribution of BMC. The estimation of the required constants is described in Ref. 5. In evaluating the probabilities we can use various tables[6–8] or the programs of Bargmann and Ghosh.[9] We have taken this approach in addition to that of series approximation in the analysis of manufacturing data.

If the manufacturing program has a multiperiod structure, B is regarded as $B(t)$ and BMC for the complete cycle is obtained by summing over the time periods, i.e.,

$$B_T = \left[\sum_{t=1}^{T} r_n(t) \right]^{-1} \sum_{t=1}^{T} r_n(t) B(t). \tag{16}$$

The probability distribution of B_T can be obtained in the same way as that of $B(t)$.

- *Modifications*

Lead time

The mathematical expression for BMC needs minor modification if we take into account "lead time" for various operations. Lead time can be important because the yield of the operations, as well as the fixed and variable cost elements, is predicted for each of the production periods (e.g., quarterly, semi-annually, etc.) and there can be significant changes in yield from one time period to the next; the changes in fixed and variable cost components for each operation are less dynamic in nature. By lead time we mean the number of days that an operation on a batch of units started in some previous time period extends into the present time period. We obtain expressions for BMC in any time period for the case in which the combined lead time on all operations is less than one time period. Derivations are essentially similar, but more complex, when the total lead time exceeds one time period. We assume that the daily production rate or the capacity for any operation remains unchanged between successive time periods. The main modification in the BMC formula results from the need to relate production quantities to the corresponding yields and cost components.

The lead times (in days) for the n operations are denoted by $h_1, h_2, \cdots, h_n$. Suppose we are concerned with a particular production quantity $r_n(t)$ at the nth or final operation in the tth period. Let $r_n(t-1)$ and $r_n(t+1)$ denote the requirements for the preceding and succeeding periods, respectively, and let $d(t-1)$, $d(t)$ and $d(t+1)$ be the number of production days in the corresponding time periods. By assumption,

$$h_1 + h_2 + \cdots + h_n \leq \min [d(t-1), d(t), d(t+1)].$$

The total output $r_n(t)$ of the nth operation in the tth period consists of a part

$$r_n(t)[h_n/d(t)] \equiv r_n(t)[1 - p_n(t)] \tag{17}$$

that has to pass through all the operations in the $(t-1)$th period and a part

$$r_n(t)[1 - h_n/d(t)] = r_n(t)p_n(t)$$

that passes through the nth operation in the tth period. To yield this latter amount, the input to the nth operation must be $r_n(t)p_n(t)/Y_n(t)$.

If we define

$$p_{n-1}(t) \equiv 1 - h_{n-1}[d(t) - h_n]^{-1},$$

then, analogously, the part $r_n p_n(1 - p_{n-1})/Y_n$ (where the time-period argument has been suppressed) of the output of the $(n-1)$th operation passes through all the first $n-1$ operations of the $(t-1)$th period and the part $r_n p_n p_{n-1}/Y_n$ passes through the $(n-1)$th operation in the tth period. This association of the yield in different time periods with the quantities produced can be extended back to the first operation. For the general case we set

$$p_{i-1}(t) = 1 - h_{i-1}\left[d(t) - \sum_{j=i}^{n} h_j \right]^{-1},$$
$$i = 1, 2, \cdots, n;\ h_0 \equiv 0. \tag{18}$$

It remains now to associate operation costs with such quantities. Recall that c_i and f_i represent the unit cost and fixed cost of the ith operation. Then, based on the nested decomposition of production quantities passing through n, $n-1$, $n-2$, $\cdots$ operations and affecting the nth-stage output in period t, one can show that BMC (per unit) is

$$\begin{aligned} B = {} & [1 - p_n(t)]\left[\sum_{i=1}^{n} f_i(t-1)\right] \\ & \times \{r_n(t)[1 - p_n(t)] + r_n(t-1)p_n(t-1)\}^{-1} \\ & + p_n(t)\left[\sum_{i=1}^{n} f_i(t)\right] \\ & \times \{r_n(t)p_n(t) + r_n(t+1)[1 - p_n(t+1)]\}^{-1} \\ & + [1 - p_n(t)] \sum_{i=1}^{n} \left\{c_i(t-1)\left[\prod_{j=i}^{n} Y_j(t-1)\right]^{-1}\right\} \\ & + p_n(t) \sum_{i=1}^{n-1} \left\{\prod_{j=i+1}^{n} [1 - p_j(t)] \right. \\ & \left. \times \sum_{m=1}^{i} c_m(t-1)\left[\prod_{l=m}^{i} Y_l(t-1)\right]^{-1}\right\} \\ & + \sum_{i=1}^{n} \left\{c_i(t) \prod_{j=i}^{n} p_j(t)\left[\prod_{l=i}^{n} Y_l(t)\right]^{-1}\right\}. \end{aligned} \tag{19}$$

The method of moments can be used to obtain the probability distribution of B.

Cost sharing

The basic problem is to assign the fixed cost of a shared operation to the individual products. We illustrate the method for two products P and Q. Let n, r_n and Y_i ($i = 1, 2, \cdots, n$) be, respectively, the number of operations, the quantity requirement and the yield of P. Similarly, let m, r_m and Z_j ($j = 1, 2, \cdots, m$) be the corresponding items for Q. Consider a particular shared operation which is the i'th operation for P and the j'th operation for Q. Let f denote the fixed burden associated with this operation. The total input for this operation is

$$r(i', j') = r_n\left(\prod_{i=i'}^{n} Y_i\right)^{-1} + r_m\left(\prod_{j=j'}^{m} Z_j\right)^{-1}.$$

If the fixed burden f is prorated on the basis of input quantity, the share of burden for P is

$$\left[r_n r(i', j')^{-1}\left(\prod_{i=i'}^{n} Y_i\right)^{-1}\right]f,$$

with a similar expression for Q. The modification of the BMC formula is straightforward.

Data analysis

Manufacturing data from the IBM Components Division were analyzed using our stochastic model. Production records for two device components involving twelve and twenty-four operations, respectively, were used; for the device having twelve operations the manufacturing data relate to the period April through December, 1966, and for the other component the costs and yields refer to one batch of production in 1968. In the manufacturing environment, inventories are maintained at various operations and the data have been adjusted to allow for removal of production to stock as well as additions from stock to the production line.

- *Twelve operations*

Working back from the quantity of finished units out of the twelfth operation, we generated the theoretical input quantities for all the earlier operations using the observed average yield for each operation. The actual range (l_i, u_i) of the yield Y_i was known from the data and the yield was assumed to vary uniformly in this range. To compensate for minor differences between the theoretical and actual input quantities at each stage, the unit cost c_i for each operation was allowed to have a uniform probability distribution, but was restricted to be within $\pm 5\%$ of the observed cost. The inventory-adjusted data for a single-period analysis and for a three-period analysis are given in Table 1.

The expected value or estimator $\mathrm{E}(B)$ of the base manufacturing cost is given in Table 2 as calculated with our model using the moments method. Both the Pearson frequency curves (based on four moments) and the Gram-Charlier series (for six moments) were used to evaluate the probability distribution. The first moment, which is $\mathrm{E}(B)$, is the same in all cases and compares favorably with the actual value of B. The 90%-confidence interval is seen to be narrowed both by the Gram-Charlier determination, because the six-moment series carries more information about the distribution function, and by the three-period analysis, because the input data could be represented more realistically. The "best" value of $\mathrm{E}(B)$, computed using u_i for the yield and $0.95\, c_i^{\mathrm{obs}}$ for the cost, is 44.621; the analogous "worst" value is 60.705.

- *Twenty-four operations*

The stochastic model was also applied to the analysis of data for one batch of production of a different device component. The purpose of this study was to show the effect on the cost estimate of changes in the probability distribution of the yield and, as such, only the unit cost at each operation was considered. As in the first manufacturing program, the unit cost c_i was allowed to have a uniform probability distribution between 95% and 105% of the average observed c_i. Two probability distributions for B were obtained. In Case 1 each operation yield Y_i was assumed to have uniform probability density in the range $l_i \leq Y_i \leq u_i$. These data are given in Table 3. In Case 2 the yields for operations 3, 13, 16, 23 and 24 were assumed to have beta probability distributions.

To derive the required beta distributions, another (observed) value m_i of the most likely yield of the ith

Table 1 Inventory-adjusted data* for a twelve-operation manufacturing program.

Operation	Single period			Three periods					
	April to December $r_{12} = 1.087 \times 10^5$			April to June $r_{12} = 0.663 \times 10^5$		July to September $r_{12} = 0.202 \times 10^5$		October to December $r_{12} = 0.222 \times 10^5$	
	Range of yield l_i, u_i	*Average unit cost* c_i	*Fixed cost* $f_i \times 10^{-5}$	*Range of yield* l_i, u_i	*Average unit cost* c_i	*Range of yield* l_i, u_i	*Average unit cost* c_i	*Range of yield* l_i, u_i	*Average unit cost* c_i
1	0.507, 0.913	0.742566	2.117	0.710, 0.766	0.689102	0.850, 0.913	0.744763	0.507, 0.662	0.923448
2	0.603, 0.811	0.532138	1.681	0.850, 0.881	0.498439	0.603, 0.878	0.605208	0.603, 0.878	0.605208
3	0.944, 0.986	0.737519	1.009	0.944, 0.983	0.656418	0.975, 0.983	0.643722	0.977, 0.986	1.097143
4	0.917, 0.968	1.082945	2.005	0.722, 0.954	1.804975	0.950, 0.968	1.136260	0.917, 0.955	1.590702
5	0.983, 0.994	0.940457	3.317	0.983, 0.987	0.835003	0.984, 0.986	1.028571	0.987, 0.994	1.073186
6	0.940, 0.990	1.238370	3.532	0.940, 0.968	1.587255	0.975, 0.980	1.163969	0.982, 0.990	1.122476
7	0.912, 0.980	1.059261	1.057	0.912, 0.969	1.075663	0.969, 0.971	0.979653	0.962, 0.980	1.080350
8	0.957, 0.990	0.573231	0.675	0.969, 0.990	0.619823	0.957, 0.987	0.605208	0.972, 0.989	0.501208
9	0.986, 0.999	0.448993	1.134	0.986, 0.999	0.470742	0.998, 0.999	0.643722	0.993, 0.997	0.408050
10	0.989, 0.999	0.798848	1.695	0.989, 0.996	0.639688	0.995, 0.997	1.136260	0.997, 0.999	0.885313
11	0.601, 0.841	5.130086	8.933	0.601, 0.776	5.692240	0.727, 0.841	1.028571	0.653, 0.717	5.023981
12	0.941, 0.997	2.579089	0.391	0.943, 0.997	1.694659	0.957, 0.967	1.163969	0.941, 0.947	5.506383

* The cost data have been modified by an arbitrary scale factor.

Table 2 Summary of computed results for the twelve-operation data in Table 1.*

	E(B)		90%-*confidence interval*	
	Single period	*Three periods*	*Single period*	*Three periods*
Pearson curves	50.687	50.687	47.165, 54.836	49.704, 52.520
Gram-Charlier series	50.687		48.141, 53.541	

* The observed value of B is 50.510.

operation was obtained in addition to l_i and u_i. Following general practice, we used $\frac{1}{6}(l_i + 4m_i + u_i)$ and $\frac{1}{6}(u_i - l_i)$, respectively, as estimates of the means and variances of the beta distributions. The related parameters a_i and b_i of the beta distributions are given in the footnote of Table 3.

The first four central moments of the computed probability distribution of the base manufacturing cost for each case are listed in Table 4 (obtained with the Pearson curves). The values for the two cases are significantly different and indicate the sensitivity of the probability distribution of B to the assumptions about yield and operation cost uncertainties.

Table 3 Inventory-adjusted data for one production batch of a twenty-four operation manufacturing program.*

Operation	*Average unit cost* c_i	*Range of yield* l_i, u_i
1	458.06	0.957, 0.997
2	27.58	0.957, 0.997
3†	26.78	0.930, 0.980
4	9.50	0.955, 0.995
5	6.62	0.959, 0.999
6	10.32	0.955, 0.995
7	6.80	0.959, 0.999
8	9.56	0.952, 0.992
9	6.98	0.955, 0.995
10	9.65	0.956, 0.996
11	4.57	0.954, 0.994
12	49.32	0.955, 0.995
13†	14.75	0.940, 0.980
14	2.35	0.999, 1.000
15	35.42	0.957, 0.997
16†	25.72	0.930, 0.980
17	16.89	0.910, 0.960
18	76.73	0.947, 0.997
19	29.75	0.935, 0.985
20	63.93	0.950, 0.990
21	11.11	0.970, 0.990
22	43.42	0.970, 0.990
23†	17.01	0.155, 0.295
24†	0.00	0.610, 0.790

* The final quantity $r_{24} = 0.812 \times 10^5$.
† In Case 2 of the analysis the probability distribution of the yield was changed from a uniform to a beta distribution; the parameters (a_i, b_i) of the beta distributions for operations 3, 13, 16, 23 and 24 are (4.1900, 0.2000), (3.2300, 0.1170), (4.1900, 0.2000), (1.4600, 5.0300) and (4.2000, 1.8000), respectively.

Table 4 Summary of computed results for the twenty-four operation data in Table 3.*

Central moment	*Case 1* Uniform probability densities	*Case 2* Uniform and beta probability densities
1 [$\equiv E(B)$]	1359.34	1298.54
2	4.80036×10^5	1.20089×10^5
3	5.42264×10^8	0.94723×10^8
4	6.17326×10^{12}	0.15564×10^{12}

* The observed value of B is 1268.63.

Appendix

The probability density function of the manufacturing cost (when there are two operations) is obtained explicitly under the following assumptions:

1. Let c_i have a uniform probability distribution; the density function is

$$f(c_i) = (\lambda_i - \mu_i)^{-1}, \qquad \mu_i \le c_i \le \lambda_i.$$

2. Let Y_i have a beta probability distribution; the density function is

$$f(Y_i) = Y_i^{a_i - 1}(1 - Y_i)^{b_i - 1}[B(a_i, b_i)]^{-1}, \qquad 0 \le Y_i \le 1;$$

also $a_i, b_i > 0$ and $B(a_i, b_i) = \Gamma(a_i)\Gamma(b_i)[\Gamma(a_i + b_i)]^{-1}$.

3. Let W_2 have a uniform probability distribution; the density function is

$$f(W_2) = (\gamma - \theta)^{-1}, \qquad \theta \le W_2 \le \gamma.$$

4. Let c_i, Y_i and W_2 be statistically independent.

The base manufacturing cost of a two-stage process is

$$B = W_2 + Z_2 = W_2 + (Z_1 + c_2)/Y_2,$$

where $Z_1 = c_1/Y_1$. The Mellin transforms of c_1, Y_1 and Z_1 are

$$M_{c_1}(\alpha) = (\lambda_1^\alpha - \mu_1^\alpha)[\alpha(\lambda_1 - \mu_1)]^{-1},$$

$$M_{Y_1}(\alpha) = B(a_1 + \alpha + 1, b_1)[B(a_1, b_1)]^{-1}$$

and

$$M_{Z_1}(\alpha) = M_{c_1}(\alpha) M_{Y_1}(2 - \alpha) = A_1 \frac{\lambda_1^\alpha - \mu_1^\alpha}{\alpha} \frac{\Gamma(a_1 - \alpha + 1)}{\Gamma(a_1 + b_1 - \alpha + 1)}, \qquad \text{(A1)}$$

where $A_1 \equiv \Gamma(b_1)[(\lambda_1 - \mu_1)B(a_1, b_1)]^{-1}$.

The Laplace transform of the density function of Z_1 can be obtained using Eq. (6),

$$L_{Z_1}(s) = (2\pi j)^{-1} \int_{c-j\infty}^{c+j\infty} M_{Z_1}(\alpha)\Gamma(1 - \alpha)s^{\alpha-1}\, d\alpha. \qquad \text{(A2)}$$

The integrand has simple poles at $\alpha = n$, $n = 1, 2, 3, \cdots$, and $\alpha = a_1 + n + 1$, $n = 0, 1, 2, \cdots$; hence the integral is

$$\begin{aligned} L_{Z_1}(s) &= \sum [\text{residues of } M_{Z_1}(\alpha)\Gamma(1 - \alpha)s^{\alpha-1}] \\ &= A_1\left[\sum_{n=1}^{\infty} \frac{\Gamma(a_1 - n + 1)}{\Gamma(a_1 + b_1 - n + 1)} \frac{\lambda_1^n - \mu_1^n}{n} \frac{(-s)^{n-1}}{(n-1)!}\right. \\ &\quad \left. + \sum_{n=0}^{\infty} \frac{\Gamma(-a_1 - n)}{\Gamma(b_1 - n)} \frac{\lambda_1^{a_1+n+1} - \mu_1^{a_1+n+1}}{a_1 + n + 1} \frac{s^{n+a_1}(-1)^n}{n!}\right] \\ &= A_1\left\{\frac{\Gamma(-a_1 - 1)}{\Gamma(b_1)} s^{a_1}[\lambda_1^{a_1+1}F(1 - b_1, a_1 + 2, -\lambda_1 s)\right. \\ &\quad - \mu_1^{a_1+1}F(1 - b_1, a_1 + 2, -\mu_1 s)] \\ &\quad + \frac{\Gamma(a_1 + 1)}{\Gamma(a_1 + b_1 + 1)} \frac{1}{s}[F(a_1 + b_1, a_1, -\lambda_1 s) \\ &\quad \left. - F(a_1 + b_1, a_1, -\mu_1 s)]\right\}, \end{aligned} \qquad \text{(A3)}$$

where $F(p, q, z)$ is a Kummer's function.

The Laplace transform of the density function of c_2 is

$$L_{c_2}(s) = (e^{-s\mu_2} - e^{-s\lambda_2})[s(\lambda_2 - \mu_2)]^{-1}.$$

Then

$$L_{Z_1+c_2}(s) = L_{Z_1}(s)L_{c_2}(s)$$

and, using Eq. (5), we obtain

$$\begin{aligned} M_{Z_1+c_2}(\alpha) &= (2\pi j)^{-1}\Gamma(\alpha) \int_{c-j\infty}^{c+j\infty} L_{Z_1+c_2}(s)(-s)^{-\alpha}\, ds \\ &= A_1[\Gamma(1 - \alpha)(\lambda_2 - \mu_2)]^{-1} \\ &\quad \times \{\Gamma(a_1 - \alpha)[\phi(a_1 - \alpha, \lambda_2) \\ &\quad - \phi(a_1 - \alpha, \mu_2)] \\ &\quad + \Gamma(-1 - \alpha)[\psi(-1 - \alpha, \lambda_2) \\ &\quad - \psi(-1 - \alpha, \mu_2)]\}, \end{aligned} \qquad \text{(A4)}$$

where

$$\begin{aligned} \phi(a_1 - \alpha, \mu_2) &= [\Gamma(-a_1 - 1)/\Gamma(b_1)] \\ &\quad \times \{\mu_2^{\alpha - a_1}[\lambda_1^{a_1+1}{}_2F_1(1 - b_1, a_1 - \alpha, a_1 + 2, -\lambda_1/\mu_2) \\ &\quad - \mu_1^{a_1+1}{}_2F_1(1 - b_1, a_1 - \alpha, a_1 + 2, -\mu_1/\mu_2)]\}, \end{aligned}$$

$$\begin{aligned} \phi(a_1 - \alpha, \lambda_2) &= [\Gamma(-a_1 - 1)/\Gamma(b_1)] \\ &\quad \times \{\lambda_2^{\alpha - a_1}[\lambda_1^{a_1+1}{}_2F_1(1 - b_1, a_1 - \alpha, a_1 + 2, -\lambda_1/\lambda_2) \\ &\quad - \mu_1^{a_1+1}{}_2F_1(1 - b_1, a_1 - \alpha, a_1 + 2, -\mu_1/\mu_2)]\}, \end{aligned}$$

$$\psi(-1-\alpha,\mu_2) = [\Gamma(a_1+1)/\Gamma(a_1+b_1+1)] \times \{\mu_2^{\alpha+1}[{}_2F_1(a_1+b_1,-1-\alpha,a_1,-\lambda_1/\mu_2) - {}_2F_1(a_1+b_1,-1-\alpha,a_1,-\mu_1/\mu_2)]\},$$

$$\psi(-1-\alpha,\lambda_2) = [\Gamma(a_1+1)/\Gamma(a_1+b_1+1)] \times \{\lambda_2^{\alpha+1}[{}_2F_1(a_1+b_1,-1-\alpha,a_1,-\lambda_1/\lambda_2) - {}_2F_1(a_1+b_1,-1-\alpha,a_1,-\mu_1/\lambda_2)]\}$$

and ${}_2F_1(p,q,r,x)$ is a hypergeometric function.

The other Mellin transforms needed are

$$M_{Y_2}(\alpha) = B(b_2, a_2+\alpha-1)[B(a_2,b_2)]^{-1} \quad \text{and}$$

$$M_{Z_2}(\alpha) = M_{Z_1+c_2}(\alpha)B(b_2,a_2-\alpha+1)[B(a_2,b_2)]^{-1} = \frac{A_2\Gamma(a_2-\alpha+1)}{\Gamma(1-\alpha)\Gamma(a_2+b_2-\alpha+1)} \times \{\Gamma(a_1-\alpha)[\phi(a_1-\alpha,\lambda_2)-\phi(a_1-\alpha,\mu_2)] - \Gamma(-1-\alpha)[\psi(-1-\alpha,\lambda_2) - \psi(-1-\alpha,\mu_2)]\}, \tag{A5}$$

where

$$A_2 = A_1\Gamma(b_2)[(\lambda_2-\mu_2)B(a_2,b_2)]^{-1}.$$

The Laplace transform of the density function of Z_2 is

$$\bar{L}_{Z_2}(s) = (2\pi j)^{-1}\int_{c-j\infty}^{c+j\infty} M_{Z_2}(\alpha)\Gamma(1-\alpha)s^{\alpha-1}\,d\alpha.$$

The integrand has simple poles at $\alpha = a_1+n$, $\alpha = a_2+n+1$ and $\alpha = n$, $n = 1, 2, 3 \cdots$, in the right half-plane. Hence

$$L_{Z_2}(s) = \sum[\text{residues of } M_{Z_2}(\alpha)\Gamma(1-\alpha)s^{\alpha-1}]$$
$$= A_2\sum_{n=1}^{\infty}\left\langle\frac{(-1)^n}{n!}\frac{s^{a_2+n}}{\Gamma(b_2-n)}\{\Gamma(a_1-a_2-n-1) \times[\phi(a_1-a_2-n-1,\lambda_2) - \phi(a_1-a_2-n-1,\mu_2)] - \Gamma(-2-a_2-n)[\psi(-2-a_2-n,\lambda_2) - \psi(-2-a_2-n,\mu_2)]\}\right.$$
$$+\frac{s^{n+a_1-1}\Gamma(a_2-a_1-n+1)}{\Gamma(a_2+b_2-a_1-n)} \times\left\{\frac{(-1)^n}{n!}[\phi(-n,\lambda_2)-\phi(-n,\mu_2)] - \Gamma(-1-a_1-n)[\psi(-a_1-1-n,\lambda_2) - \psi(-a_1-1-n,\mu_2)]\right\}$$
$$+\frac{s^{n-1}\Gamma(a_2-n+1)}{\Gamma(a_2+b_2-n+1)} \times\left\{\Gamma(a_1-n)[\phi(a_1-n,\lambda_2)-\phi(a_1-n,\mu_2)] - \frac{(-1)^{n-1}}{(n-1)!}[\psi(-1-n,\lambda_2) \left. - \psi(-1-n,\mu_2)]\right\}\right\rangle. \tag{A6}$$

Correspondingly, although easier to obtain, the Laplace transform of the density function of W_2 is

$$L_{W_2}(s) = (e^{-s\theta} - e^{-s\gamma})[s(\gamma-\theta)]^{-1}. \tag{A7}$$

Finally, the Laplace transform of the density function of the base manufacturing cost B is given by

$$L_B(s) = L_{W_2}(s)L_{Z_2}(s) \tag{A8}$$

and the probability density itself can be obtained by inverting the transform $L_B(s)$.

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