J. F. Janak

Thermal Expansion in a Constrained Elastic Cylinder

Abstract: The stress developed in an elastic cylinder of finite length undergoing thermal expansion with one end clamped is expressed in terms of a series expansion of a biharmonic function, appropriate derivatives of which give the displacements and stresses within the cylinder. The coefficients in this series are determined by a least-squares fit to the boundary conditions at the ends of the cylinder and values of the stress on various surfaces are found as functions of the height-to-radius ratio. All components of the stress tensor become infinite at the circumference on the clamped end. A tabulation is included of quantities of interest in any cylindrical problem in which the curved surface is a free surface.

Introduction

There has been considerable interest in the problem of stress induced by thermal shrinkage or expansion in an elastic body with one or more surfaces constrained. Such problems arise, for example, in the encapsulation of electronic circuit components. The calculation of these stresses in a circular cylinder is the subject of this paper. The cylindrical symmetry reduces the problem to a two-dimensional one, offering the possibility of a tractable solution in terms of a series of biharmonic functions. In previous work studies of rectangular plates have been carried out variationally[1] and studies of long bars (constrained on a long edge) have been made using series expansions.[2] The cylindrical geometry, although permitting solution of a three-dimensional problem in two-dimensional terms, leads to considerable mathematical complexity in the sense that the terms in the series for various stresses are not orthogonal and the coefficients must be found by solving a set of linear equations. Certain mathematical constants involved in the solution are independent of the dimensions of the cylinder and can be given as functions of the elastic moduli. A considerably more extensive tabulation of these quantities than has heretofore appeared[3] is given here.

Theory

The problem is to determine the displacements and stresses in a cylinder of radius R and height h that undergoes thermal expansion with one end ($z = h$) clamped to a rigid, nonexpanding plate. The other end ($z = 0$) and the curved surface or circumference ($r = R$) of the cylinder are free surfaces, i.e., no forces are applied.

Let u be the displacement of a point in the cylinder in the radial direction and w the displacement in the z direction. If we assume that all quantities are independent of the angular coordinate θ, the strains are

$$\epsilon_{rr} = \frac{\partial u}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{u}{r}, \quad \epsilon_{zz} = \frac{\partial w}{\partial z},$$
$$\epsilon_{rz} = \epsilon_{zr} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right), \quad \epsilon_{r\theta} = \epsilon_{z\theta} = 0 \tag{1}$$

and the dilation is

$$\Delta = \epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz} = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}. \tag{2}$$

The cylinder temperature is assumed to be raised by a constant amount ΔT above the temperature at which no strain exists. If λ and μ are the Lamé constants at the elastically isotropic cylinder, in terms of which Young's modulus E and Poisson's ratio ν are expressed as

$$E = \mu(3\lambda + 2\mu)/(\lambda + \mu) \text{ and } \nu = \tfrac{1}{2}\lambda/(\lambda + \mu), \tag{3}$$

the stress-strain relations are[4]

$$T_{rr} = 2\mu\epsilon_{rr} + \lambda\Delta - (3\lambda + 2\mu)\alpha\Delta T,$$
$$T_{\theta\theta} = 2\mu\epsilon_{\theta\theta} + \lambda\Delta - (3\lambda + 2\mu)\alpha\Delta T,$$
$$T_{zz} = 2\mu\epsilon_{zz} + \lambda\Delta - (3\lambda + 2\mu)\alpha\Delta T, \tag{4}$$
$$T_{rz} = 2\mu\epsilon_{rz} \text{ and}$$
$$T_{r\theta} = T_{z\theta} = 0,$$

The author is located at the IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598.

where α is the linear coefficient of thermal expansion; there is a constant strain $\alpha\Delta T$ in each principal direction.

The equations of equilibrium are

$$\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} + \frac{\partial T_{rz}}{\partial z} = 0 \tag{5a}$$

and

$$\frac{\partial T_{rz}}{\partial r} + \frac{T_{rz}}{r} + \frac{\partial T_{zz}}{\partial z} = 0. \tag{5b}$$

These are subject to the following boundary conditions: At $z = h$ (on the plate) the cylinder is constrained to have zero displacement, so that

$$u(r, h) = w(r, h) = 0; \tag{6a}$$

at $r = R$ and at $z = 0$ no forces are applied, so that

$$T_{rr}(R, z) = T_{rz}(R, z) = 0 \text{ and} \tag{6b}$$

$$T_{zz}(r, 0) = T_{rz}(r, 0) = 0. \tag{6c}$$

Love[4] has shown that problems of cylindrical symmetry can be solved in terms of a single biharmonic function. For example, let

$$\nabla^4\Phi \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)^2 \Phi = 0 \tag{7}$$

and define the following expressions (which are equivalent, but not identical, to those given by Love):

$$u = -\frac{1}{2\mu}\frac{\partial^2\Phi}{\partial r \partial z},$$
$$w = \frac{1}{2\mu}\left(\frac{\lambda + 2\mu}{\lambda + \mu}\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right). \tag{8}$$

The stresses obtained in terms of Φ from Eqs. (4) using Eqs. (1) and (2) automatically satisfy the equilibrium conditions, Eqs. (5), and the present problem is reduced to constructing the function Φ that satisfies the boundary conditions given in Eqs. (6).

We begin this construction by expanding Φ in a series of solutions of the biharmonic equation (7), each term of which automatically satisfies the boundary conditions on the circumference of the cylinder. The coefficients in this series are adjusted so that the remaining boundary conditions are satisfied when the number of terms in the series becomes infinite; inasmuch as the terms in the series are nonorthogonal, this adjustment is made using a least-squares procedure. The coefficients of certain trivial solutions of the biharmonic equation are chosen so that the boundary conditions are satisfied in an average sense and it turns out that the trivial solutions included in Φ serve to remove the constant thermal stresses and to add a constant to the axial displacement w.

We define the reduced quantities

$$x = r/R,$$
$$\zeta = z/R,$$
$$l = h/R, \tag{9}$$
$$K^2 = \frac{\lambda + 2\mu}{\lambda + \mu} = 2(1 - \nu)$$

and write

$$\Phi = a_1\zeta^2 + a_2\zeta^3 + b_1x^2 + b_2x^2\zeta + \eta(x, \zeta). \tag{10}$$

The first four terms are the trivial solutions of the biharmonic equation that are regular at $x = 0$ (the constant term and the term proportional to ζ have been omitted since they do not contribute to any displacements or stresses) and η represents the nontrivial part of Φ, which is still to be constructed. In terms of the coefficients a_1, a_2, b_1 and b_2 the displacements are

$$u = -\frac{b_2}{\mu R^2}x + \text{terms in } \eta,$$

$$w = \frac{1}{R^2}\left\{\frac{a_1}{\lambda + \mu} + \frac{2(\lambda + 2\mu)b_1}{\mu(\lambda + \mu)} + \left[\frac{3a_2}{\lambda + \mu} + \frac{2(\lambda + 2\mu)b_2}{\mu(\lambda + \mu)}\right]\zeta\right\} + \text{terms in } \eta,$$

where the "terms in η" are obtained from Eqs. (8) with Φ replaced by η.

- *Evaluation of coefficients*

The stresses all involve derivatives of u and w, so it is clear that the a_1 and b_1 terms in Φ contribute constants to w and appear nowhere else. Without loss of generality we can choose $b_1 = 0$ and use a_1 to fix the constant term in w. The terms in a_2 and b_2 produce a displacement field like that due to unconstrained thermal expansion, i.e., constant strains with no shear; they add constants to the principal stresses and do not affect the shear stress. The boundary conditions require that T_{rr} vanish on the circumference and that T_{zz} vanish on the free end ($z = 0$). It follows, then, that

$$\int_0^1 xT_{zz}(x, 0)\,dx = \int_0^l T_{rr}(1, \zeta)\,d\zeta = 0. \tag{11}$$

We shall choose η in such a way that its contribution to $T_{rr}(x, \zeta)$ automatically vanishes when $x = 1$. It will then turn out that η makes no contribution to the average of T_{zz} over the free end; thus Eqs. (11) will contain no terms in η. To satisfy Eqs. (11), a_2 and b_2 must be chosen to remove the thermal stress term $-(3\lambda + 2\mu) \times \alpha\Delta T$ from T_{rr} and T_{zz}. In this way $T_{rr}(1, \zeta)$ is made identically zero for all ζ, while $T_{zz}(x, 0)$ presumably becomes zero pointwise as the number of terms in the series for η becomes infinite. The required values of a_2 and b_2 are

$$a_2 = \tfrac{1}{3}R^3(3\lambda + 5\mu)\alpha\Delta T; \qquad b_2 = -R^3\mu\alpha\Delta T. \tag{12}$$

The displacements and stresses now become[4]

$$\begin{aligned}
u &= R\alpha\Delta T x - \frac{1}{2\mu}\frac{\partial^2\eta}{\partial r\partial z}, \\
w &= R\alpha\Delta T(\zeta - l + B) \\
&\quad + \frac{1}{2\mu}\left(\frac{\lambda + 2\mu}{\lambda + \mu}\nabla^2\eta - \frac{\partial^2\eta}{\partial z^2}\right), \\
T_{rr} &= \frac{\partial}{\partial z}\left[\frac{\lambda}{2(\lambda + \mu)}\nabla^2\eta - \frac{\partial^2\eta}{\partial r^2}\right], \\
T_{\theta\theta} &= \frac{\partial}{\partial z}\left[\frac{\lambda}{2(\lambda + \mu)}\nabla^2\eta - \frac{1}{r}\frac{\partial\eta}{\partial r}\right], \\
T_{zz} &= \frac{\partial}{\partial z}\left[\frac{3\lambda + 4\mu}{2(\lambda + \mu)}\nabla^2\eta - \frac{\partial^2\eta}{\partial z^2}\right], \\
T_{rz} &= \frac{\partial}{\partial r}\left[\frac{\lambda + 2\mu}{2(\lambda + \mu)}\nabla^2\eta - \frac{\partial^2\eta}{\partial z^2}\right],
\end{aligned} \tag{13}$$

where we have rewritten a_1 so that the constant contribution to w is $R\alpha\Delta T(B-l)$. The constant B is chosen so that the boundary condition on w is satisfied on the average:

$$\int_0^1 xw(x, l)\,dx = 0. \tag{14}$$

- *Construction of* $\eta(x, \zeta)$

We write η in the form

$$\begin{aligned}
\eta(x, \zeta) &= 2\mu R^3\alpha\Delta T \\
&\quad \times[\sum_s xJ_1(\beta_s x)(A_s \sinh\beta_s\zeta + C_s\cosh\beta_s\zeta) \\
&\quad + \sum_s J_0(\beta_s x)(B_s\sinh\beta_s\zeta + D_s\cosh\beta_s\zeta)],
\end{aligned} \tag{15}$$

where J_0 and J_1 are Bessel functions and A_s, B_s, C_s, D_s and β_s are constants to be determined. The quantities $J_0(\beta x) \times\exp(\pm\beta\zeta)$ and $xJ_1(\beta x)\exp(\pm\beta\zeta)$ are well known solutions[4] of the biharmonic equation (7).

The boundary conditions (6b) on the circumference of of the cylinder take the form

$$\sum_s [\gamma_s\cosh\beta_s\zeta + \delta_s\sinh\beta_s\zeta] = 0, \tag{16}$$

where γ_s and δ_s are known in terms of A_s, B_s, C_s and D_s. By expanding this expression in powers of ζ and using the linear independence of various powers of ζ, we obtain a series of homogeneous equations in γ_s or δ_s. We conclude that γ_s and δ_s must vanish if the determinant of coefficients is non-singular. This determinant is of van der Monde form[5] and we are thus able to say that Eq. (16) implies $\gamma_s = \delta_s \equiv 0$ if the β_s (which are still to be determined) are such that

$$\prod_{i>j}(\beta_i^2 - \beta_j^2) \neq 0$$

and no β_i is zero or, in other words, if no β is zero or the same as or the negative of another β. [All such cases are readily seen to be redundant: If any β_s is zero, the corresponding terms in Eq. (15) are constants, which can be ignored. Similarly, if any β_s is the same as or the negative of another, it can be eliminated from the sum simply by regrouping terms and redefining the coefficients A_s, B_s, C_s and D_s in Eq. (15)].

We now evaluate T_{rr} and T_{rz} on the circumference. Defining

$$\xi_s \equiv J_0(\beta_s)/J_1(\beta_s) \tag{17}$$

and using the result derived for γ_s and δ_s, we obtain the following equations:

$$\left(\frac{K^2}{\beta_s} + \xi_s\right)A_s - B_s = 0,$$

$$\left(\frac{K^2}{\beta_s} + \xi_s\right)C_s - D_s = 0,$$

$$\left(1 - \frac{\mu}{\lambda + \mu}\frac{\xi_s}{\beta_s}\right)A_s - \left(\frac{1}{\beta_s} - \xi_s\right)B_s = 0 \text{ and}$$

$$\left(1 - \frac{\mu}{\lambda + \mu}\frac{\xi_s}{\beta_s}\right)C_s - \left(\frac{1}{\beta_s} - \xi_s\right)D_s = 0.$$

These two sets of equations have nontrivial solutions only if

$$\beta_s^2(1 + \xi_s^2) = K^2, \tag{18}$$

in which case

$$\frac{B_s}{A_s} = \frac{D_s}{C_s} = \frac{K^2}{\beta_s} + \xi_s. \tag{19}$$

Equation (18) is the characteristic equation determining the permissible values of β_s over which we sum in Eq. (15) to construct η. Since $1 \leq K^2 \leq 2$ [see Eq. (9) and recall that the value of Poisson's ratio is between 0 and $\frac{1}{2}$], it is possible to show[3] that the β_s's are complex for all cases of interest. Furthermore, since Bessel functions are real functions and are either symmetric or antisymmetric, it follows that, if β_s satisfies Eq. (18), then β_s^*, $-\beta_s$ and $-\beta_s^*$ also satisfy Eq. (18). (The asterisk denotes complex conjugate.) According to the discussion following Eq. (16), we must omit $-\beta_s$ and $-\beta_s^*$ as well as $\beta_s = 0$, so our sums extend over all β_s (and their conjugates) with positive real part that satisfy Eq. (18).

The function η now takes the form

$$\eta = 2\mu R^3\alpha\Delta T \sum_s \left[x\, \mathrm{J}_1(\beta_s x) + \left(\frac{K^2}{\beta_s} + \xi_s\right) \mathrm{J}_0(\beta_s x)\right]$$
$$\times\ (A_s \sinh \beta_s \zeta + C_s \cosh \beta_s \zeta) \qquad (20)$$

and the remaining four boundary conditions become

$$\phi_a(x) \equiv \sum_s \beta_s^3 A_s$$
$$\times \left[\left(\frac{2}{\beta_s} - \xi_s\right)\mathrm{J}_0(\beta_s x) - x\mathrm{J}_1(\beta_s x)\right] = 0, \qquad (21a)$$

$$\phi_b(x) \equiv \sum_s \beta_s^3 C_s[x\mathrm{J}_0(\beta_s x) - \xi_s \mathrm{J}_1(\beta_s x)] = 0, \qquad (21b)$$

$$\phi_c(x) \equiv B + \sum_s \beta_s^2\left[\left(\frac{K^2}{\beta_s} - \xi_s\right)\mathrm{J}_0(\beta_s x) - x\,\mathrm{J}_1(\beta_s x)\right]$$
$$\times\ (A_s \sinh \beta_s l + C_s \cosh \beta_s l) = 0, \qquad (21c)$$

$$\phi_d(x) \equiv -x + \sum_s \beta_s^2\left[x\,\mathrm{J}_0(\beta_s x) - \left(\frac{K^2}{\beta_s} + \xi_s\right)\mathrm{J}_1(\beta_s x)\right]$$
$$\times\ (A_s \cosh \beta_s l + C_s \sinh \beta_s l) = 0, \qquad (21d)$$

all for $0 \le x \le 1$; these equations are, in order, the conditions that $T_{zz}(x, 0)$, $T_{rz}(x, 0)$, $w(x, l)$ and $u(x, l)$ be zero.

The constant B, which is to be found from Eq. (14), can now be given explicitly in terms of the A_s and C_s; we obtain

$$B = -2(K^2 - 2) \sum_s \mathrm{J}_1(\beta_s)$$
$$\times\ (A_s \sinh \beta_s l + C_s \cosh \beta_s l) \qquad (22)$$

and $\phi_c(x)$ becomes

$$\phi_c(x) = \sum_s \Bigg\{2(2 - K^2)\mathrm{J}_1(\beta_s)$$
$$+ \beta_s^2\left[\left(\frac{K^2}{\beta_s} - \xi_s\right)\mathrm{J}_0(\beta_s x) - x\,\mathrm{J}_1(\beta_s x)\right]\Bigg\}$$
$$\times\ (A_s \sinh \beta_s l + C_s \cosh \beta_s l). \qquad (21c')$$

It is easily verified that

$$\int_0^1 x\left[\left(\frac{2}{\beta_s} - \xi_s\right)\mathrm{J}_0(\beta_s x) - x\mathrm{J}_1(\beta_s x)\right] dx = 0,$$

which guarantees that Eq. (11) is satisfied. The constants A_s and C_s are both complex; Eqs. (21a) and (21b) furnish relations between their real and imaginary parts rather than a requirement that they be zero. From the structure of Eqs. (21) it is clear that

$$A_s^* \equiv A(\beta_s^*) = [A(\beta_s)]^*$$

and similarly for C_s. Thus all the ϕ's of Eqs. (21), as well as η itself, are real quantities.

- *Least-squares adjustment*

The coefficients A_s and C_s have to be determined numerically, since there is not much hope of solving Eqs. (21) analytically. Thus we have to approximate η by a finite number (say $2N$) of terms. To choose the coefficients in the best possible way, subject to this limitation, we minimize the error

$$\epsilon_N = \int_0^1 x(\phi_a^2 + \phi_b^2 + \phi_c^2 + \phi_d^2)\, dx \qquad (23)$$

incurred in trying to satisfy the boundary conditions with a finite number of terms. This procedure is necessary because the functions appearing in Eqs. (21) are not mutually orthogonal.

The minimization conditions

$$\frac{\partial \epsilon_N}{\partial A_s} = \frac{\partial \epsilon_N}{\partial C_s} = 0$$

lead to the linear equations

$$\sum_{s'=1}^{2N} (M_{ss'} A_{s'} + N_{ss'} C_{s'}) = \Gamma_s \cosh \beta_s l,$$
$$\sum_{s'=1}^{2N} (N_{s's} A_{s'} + Q_{ss'} C_{s'}) = \Gamma_s \sinh \beta_s l, \qquad (24)$$

in which

$$\Gamma_s = \mathrm{J}_1(\beta_s)\left(K^2\xi_s - \frac{4 + K^2}{\beta_s}\right),$$

$$M_{ss'} = \beta_s^2\beta_{s'}^2(U_{ss'} \cosh \beta_s l \cosh \beta_{s'} l$$
$$+ V_{ss'} \sinh \beta_s l \sinh \beta_{s'} l + \beta_s\beta_{s'} W_{ss'}),$$
$$N_{ss'} = \beta_s^2\beta_{s'}^2(U_{ss'} \cosh \beta_s l \sinh \beta_{s'} l$$
$$+ V_{ss'} \sinh \beta_s l \cosh \beta_{s'} l), \qquad (25)$$
$$Q_{ss'} = \beta_s^2\beta_{s'}^2(U_{ss'} \sinh \beta_s l \sinh \beta_{s'} l$$
$$+ V_{ss'} \cosh \beta_s l \cosh \beta_{s'} l + \beta_s\beta_{s'} X_{ss'}).$$

The matrices **U**, **V**, **W** and **X**, which are linear combinations of integrals of Bessel functions and are independent of l, are discussed further in the Appendix. It is interesting to note that the β_s's and **U**, **V**, **W** and **X** depend only on Poisson's ratio and can be tabulated as functions of this ratio without reference to cylinder dimensions.† Such a tabulation would be of use in solving any problem of cylindrical symmetry in which the circumference is a free surface. Different boundary conditions on the ends of the cylinder will still lead to a set of equations like (24), but with **M**, **N** and **Q** given by different linear combinations of **U**, **V**, **W** and **X**.

† Some values of β_s are presented later in Table 1.

Application

- *Computational procedure*

Our program consists of finding the roots of Eq. (18), solving Eqs. (24) for A_s and C_s and constructing η and any desired displacements and stresses using Eqs. (20) and (13). To compute the Bessel functions, the standard downward-recurrence algorithm,[6] modified to handle complex arguments, was used. One can show, using asymptotic (large argument) expansions of the Bessel functions, that the roots of Eq. (18) are given approximately[3] by

$$\beta_s \approx s\pi - \frac{\ln 4s\pi}{4s\pi} + \frac{(\frac{1}{4} - K^2)}{2s\pi} - \frac{i}{2}\left[1 - \frac{2(\frac{1}{4} - K^2)}{(2s\pi)^2}\right] \ln 4s\pi, \tag{26}$$

which is asymptotically exact for large s. [Equation (26) determines the value of β_1 to within 2%.] The algorithm for computing β_s uses the above expression as a first approximation and then iterates the calculation with Newton's method until β_s is determined to five significant figures. The values obtained in this way for the first ten roots of Eq. (18), as well as the values of $J_1(\beta_s)$ and ξ_s, are given in Table 1 for values of K^2 from 1 to 2 in steps of 0.1.

Standard algorithms for solving linear equations can be used if Eqs. (24) are first rewritten as real equations for the real and imaginary parts of A_s and C_s. Problems with large numbers arising from $\cosh \beta_s l$ or $\sinh \beta_s l$ for large s (Re $\beta_s \sim s\pi$) are best circumvented by solving for the quantities

$$\psi_s = A_s \cosh \beta_s l + C_s \sinh \beta_s l$$

and

$$\phi_s = (A_s - C_s) \exp(-\beta_s l) \sinh \beta_s l,$$

instead of for A_s and C_s; Eqs. (24) rewritten in terms of ψ_s and ϕ_s are much more tractable than the original equations and contain no terms that become exponentially large for large s.

- *Numerical results*

Calculations using up to 10 roots and their conjugates (i.e., up to 20 terms in the series for η) were made for $K^2 = 1.32$ and for various values of l. Results for the stresses on the top, bottom and side surfaces of the cylinder are shown in Figs. 1 through 5. All data in these figures are based on calculations using 10 roots and their conjugates. The actual computed stress curves show small oscillations around the smoothed curves given in the figures; these oscillations are presumed to be due to the use of a finite number of terms (much the same as in Fourier series) and were therefore omitted from the figures.

Table 1 Real and imaginary parts of β_s, $J_1(\beta_s)$ and $\xi_s \equiv J_0(\beta_s)/J_1(\beta_s)$, $s = 1, 2, 3, \cdots, 10$, where β_s† is a root of $\beta^2(1 + \xi^2) = K^2$.‡

BETA RE	BETA IM	J1(BETA) RE	J1(BETA) IM	XI RE	XI IM
		K**2=1.0			
0.2811E 01	-0.1340E 01	0.6420E 00	0.5762E 00	0.4140E-01	0.9679E 00
0.6095E 01	-0.1634E 01	-0.5958E 00	-0.5547E 00	0.6360E-02	0.9891E 00
0.9289E 01	-0.1827E 01	0.5835E 00	0.5550E 00	0.2125E-02	0.9948E 00
0.1246E 02	-0.1966E 01	-0.5780E 00	-0.5563E 00	0.9714E-03	0.9970E 00
0.1562E 02	-0.2076E 01	0.5749E 00	0.5574E 00	0.5265E-03	0.9981E 00
0.1877E 02	-0.2165E 01	-0.5730E 00	-0.5582E 00	0.3191E-03	0.9986E 00
0.2192E 02	-0.2242E 01	0.5716E 00	0.5589E 00	0.2095E-03	0.9990E 00
0.2507E 02	-0.2308E 01	-0.5706E 00	-0.5595E 00	0.1445E-03	0.9992E 00
0.2822E 02	-0.2366E 01	0.5698E 00	0.5599E 00	0.1043E-03	0.9994E 00
0.3137E 02	-0.2419E 01	-0.5692E 00	-0.5602E 00	0.7717E-04	0.9995E 00
		K**2=1.1			
0.2790E 01	-0.1346E 01	0.6574E 00	0.5716E 00	0.4650E-01	0.9648E 00
0.6086E 01	-0.1635E 01	-0.6015E 00	-0.5511E 00	0.7030E-02	0.9880E 00
0.9283E 01	-0.1827E 01	0.5871E 00	0.5525E 00	0.2343E-02	0.9943E 00
0.1245E 02	-0.1966E 01	-0.5806E 00	-0.5544E 00	0.1070E-02	0.9967E 00
0.1562E 02	-0.2076E 01	0.5769E 00	0.5558E 00	0.5813E-03	0.9979E 00
0.1877E 02	-0.2166E 01	-0.5746E 00	-0.5569E 00	0.3518E-03	0.9985E 00
0.2192E 02	-0.2242E 01	0.5730E 00	0.5577E 00	0.2294E-03	0.9989E 00
0.2507E 02	-0.2308E 01	-0.5718E 00	-0.5584E 00	0.1585E-03	0.9991E 00
0.2822E 02	-0.2366E 01	0.5709E 00	0.5590E 00	0.1147E-03	0.9993E 00
0.3136E 02	-0.2419E 01	-0.5701E 00	-0.5594E 00	0.8500E-04	0.9994E 00
		K**2=1.2			
0.2768E 01	-0.1351E 01	0.6733E 00	0.5663E 00	0.5184E-01	0.9617E 00
0.6078E 01	-0.1636E 01	-0.6072E 00	-0.5474E 00	0.7709E-02	0.9868E 00
0.9278E 01	-0.1828E 01	0.5906E 00	0.5500E 00	0.2563E-02	0.9938E 00
0.1245E 02	-0.1967E 01	-0.5831E 00	-0.5524E 00	0.1168E-02	0.9964E 00
0.1561E 02	-0.2076E 01	0.5789E 00	0.5542E 00	0.6330E-03	0.9977E 00
0.1877E 02	-0.2166E 01	-0.5763E 00	-0.5555E 00	0.3846E-03	0.9984E 00
0.2192E 02	-0.2242E 01	0.5744E 00	0.5566E 00	0.2509E-03	0.9988E 00
0.2507E 02	-0.2308E 01	-0.5730E 00	-0.5574E 00	0.1736E-03	0.9991E 00
0.2822E 02	-0.2366E 01	0.5719E 00	0.5580E 00	0.1253E-03	0.9993E 00
0.3136E 02	-0.2419E 01	-0.5711E 00	-0.5586E 00	0.9346E-04	0.9994E 00
		K**2=1.3			
0.2746E 01	-0.1357E 01	0.6897E 00	0.5603E 00	0.5742E-01	0.9587E 00
0.6069E 01	-0.1637E 01	-0.6128E 00	-0.5436E 00	0.8395E-02	0.9857E 00
0.9272E 01	-0.1828E 01	0.5941E 00	0.5473E 00	0.2782E-02	0.9932E 00
0.1245E 02	-0.1967E 01	-0.5857E 00	-0.5503E 00	0.1268E-02	0.9961E 00
0.1561E 02	-0.2076E 01	0.5809E 00	0.5525E 00	0.6856E-03	0.9975E 00
0.1876E 02	-0.2166E 01	-0.5779E 00	-0.5541E 00	0.4161E-03	0.9982E 00
0.2192E 02	-0.2242E 01	0.5757E 00	0.5554E 00	0.2717E-03	0.9987E 00
0.2507E 02	-0.2308E 01	-0.5742E 00	-0.5563E 00	0.1870E-03	0.9990E 00
0.2821E 02	-0.2366E 01	0.5730E 00	0.5571E 00	0.1354E-03	0.9992E 00
0.3136E 02	-0.2419E 01	-0.5720E 00	-0.5578E 00	0.1025E-03	0.9994E 00
		K**2=1.4			
0.2722E 01	-0.1362E 01	0.7064E 00	0.5533E 00	0.6327E-01	0.9557E 00
0.6060E 01	-0.1638E 01	-0.6184E 00	-0.5395E 00	0.9086E-02	0.9846E 00
0.9267E 01	-0.1828E 01	0.5975E 00	0.5446E 00	0.3002E-02	0.9927E 00
0.1244E 02	-0.1967E 01	-0.5882E 00	-0.5483E 00	0.1367E-02	0.9958E 00
0.1561E 02	-0.2076E 01	0.5829E 00	0.5509E 00	0.7409E-03	0.9973E 00
0.1876E 02	-0.2166E 01	-0.5795E 00	-0.5527E 00	0.4479E-03	0.9981E 00
0.2191E 02	-0.2242E 01	0.5771E 00	0.5542E 00	0.2928E-03	0.9986E 00
0.2506E 02	-0.2308E 01	-0.5754E 00	-0.5553E 00	0.2009E-03	0.9989E 00
0.2821E 02	-0.2367E 01	0.5740E 00	0.5562E 00	0.1461E-03	0.9991E 00
0.3136E 02	-0.2419E 01	-0.5730E 00	-0.5569E 00	0.1075E-03	0.9993E 00
		K**2=1.5			
0.2698E 01	-0.1367E 01	0.7236E 00	0.5454E 00	0.6940E-01	0.9528E 00
0.6051E 01	-0.1638E 01	-0.6239E 00	-0.5353E 00	0.9787E-02	0.9834E 00
0.9261E 01	-0.1829E 01	0.6009E 00	0.5418E 00	0.3225E-02	0.9922E 00
0.1244E 02	-0.1967E 01	-0.5906E 00	-0.5462E 00	0.1465E-02	0.9955E 00
0.1560E 02	-0.2076E 01	0.5848E 00	0.5492E 00	0.7937E-03	0.9971E 00
0.1876E 02	-0.2166E 01	-0.5811E 00	-0.5513E 00	0.4807E-03	0.9979E 00
0.2191E 02	-0.2242E 01	0.5785E 00	0.5529E 00	0.3142E-03	0.9985E 00
0.2506E 02	-0.2308E 01	-0.5766E 00	-0.5542E 00	0.2187E-03	0.9988E 00
0.2821E 02	-0.2367E 01	0.5751E 00	0.5552E 00	0.1572E-03	0.9991E 00
0.3136E 02	-0.2419E 01	-0.5739E 00	-0.5560E 00	0.1176E-03	0.9993E 00
		K**2=1.6			
0.2672E 01	-0.1372E 01	0.7412E 00	0.5364E 00	0.7586E-01	0.9500E 00
0.6042E 01	-0.1639E 01	-0.6294E 00	-0.5310E 00	0.1049E-01	0.9823E 00
0.9256E 01	-0.1829E 01	0.6043E 00	0.5390E 00	0.3446E-02	0.9917E 00
0.1243E 02	-0.1968E 01	-0.5931E 00	-0.5441E 00	0.1566E-02	0.9952E 00
0.1560E 02	-0.2077E 01	0.5868E 00	0.5475E 00	0.8479E-03	0.9969E 00
0.1876E 02	-0.2166E 01	-0.5827E 00	-0.5499E 00	0.5123E-03	0.9978E 00
0.2191E 02	-0.2242E 01	0.5799E 00	0.5517E 00	0.3349E-03	0.9984E 00
0.2506E 02	-0.2308E 01	-0.5777E 00	-0.5532E 00	0.2314E-03	0.9988E 00
0.2821E 02	-0.2367E 01	0.5761E 00	0.5543E 00	0.1673E-03	0.9990E 00
0.3136E 02	-0.2419E 01	-0.5749E 00	-0.5552E 00	0.1249E-03	0.9992E 00
		K**2=1.7			
0.2645E 01	-0.1377E 01	0.7591E 00	0.5262E 00	0.8266E-01	0.9472E 00
0.6033E 01	-0.1639E 01	-0.6348E 00	-0.5265E 00	0.1121E-01	0.9811E 00
0.9250E 01	-0.1829E 01	0.6077E 00	0.5361E 00	0.3671E-02	0.9911E 00
0.1243E 02	-0.1968E 01	-0.5955E 00	-0.5419E 00	0.1666E-02	0.9949E 00
0.1560E 02	-0.2077E 01	0.5887E 00	0.5458E 00	0.9023E-03	0.9967E 00
0.1875E 02	-0.2166E 01	-0.5843E 00	-0.5485E 00	0.5450E-03	0.9977E 00
0.2191E 02	-0.2242E 01	0.5812E 00	0.5505E 00	0.3563E-03	0.9983E 00
0.2506E 02	-0.2308E 01	-0.5789E 00	-0.5521E 00	0.2465E-03	0.9987E 00
0.2821E 02	-0.2367E 01	0.5772E 00	0.5533E 00	0.1780E-03	0.9990E 00
0.3135E 02	-0.2419E 01	-0.5758E 00	-0.5543E 00	0.1308E-03	0.9991E 00
		K**2=1.8			
0.2617E 01	-0.1381E 01	0.7774E 00	0.5148E 00	0.8983E-01	0.9445E 00
0.6024E 01	-0.1639E 01	-0.6402E 00	-0.5218E 00	0.1194E-01	0.9800E 00
0.9244E 01	-0.1829E 01	0.6110E 00	0.5331E 00	0.3895E-02	0.9906E 00
0.1243E 02	-0.1968E 01	-0.5980E 00	-0.5397E 00	0.1766E-02	0.9946E 00
0.1559E 02	-0.2077E 01	0.5906E 00	0.5441E 00	0.9550E-03	0.9965E 00
0.1875E 02	-0.2166E 01	-0.5858E 00	-0.5470E 00	0.5775E-03	0.9975E 00
0.2190E 02	-0.2242E 01	0.5825E 00	0.5493E 00	0.3767E-03	0.9982E 00
0.2506E 02	-0.2308E 01	-0.5801E 00	-0.5510E 00	0.2607E-03	0.9986E 00
0.2821E 02	-0.2367E 01	0.5782E 00	0.5523E 00	0.1884E-03	0.9989E 00
0.3135E 02	-0.2419E 01	-0.5766E 00	-0.5533E 00	0.1315E-03	0.9991E 00
		K**2=1.9			
0.2588E 01	-0.1385E 01	0.7960E 00	0.5020E 00	0.9742E-01	0.9419E 00
0.6015E 01	-0.1639E 01	-0.6455E 00	-0.5169E 00	0.1267E-01	0.9788E 00
0.9239E 01	-0.1829E 01	0.6142E 00	0.5301E 00	0.4121E-02	0.9901E 00
0.1242E 02	-0.1968E 01	-0.6004E 00	-0.5375E 00	0.1867E-02	0.9943E 00
0.1559E 02	-0.2077E 01	0.5925E 00	0.5423E 00	0.1010E-02	0.9963E 00
0.1875E 02	-0.2166E 01	-0.5874E 00	-0.5457E 00	0.6127E-03	0.9974E 00
0.2190E 02	-0.2242E 01	0.5838E 00	0.5480E 00	0.3951E-03	0.9981E 00
0.2505E 02	-0.2308E 01	-0.5812E 00	-0.5498E 00	0.2692E-03	0.9985E 00
0.2820E 02	-0.2367E 01	0.5792E 00	0.5513E 00	0.1920E-03	0.9988E 00
0.3135E 02	-0.2419E 01	-0.5776E 00	-0.5525E 00	0.1419E-03	0.9990E 00
		K**2=2.0			
0.2557E 01	-0.1389E 01	0.8148E 00	0.4876E 00	0.1055E 00	0.9394E 00
0.6006E 01	-0.1639E 01	-0.6507E 00	-0.5119E 00	0.1340E-01	0.9776E 00
0.9233E 01	-0.1829E 01	0.6175E 00	0.5270E 00	0.4348E-02	0.9895E 00
0.1242E 02	-0.1968E 01	-0.6027E 00	-0.5353E 00	0.1967E-02	0.9940E 00
0.1559E 02	-0.2077E 01	0.5944E 00	0.5405E 00	0.1065E-02	0.9961E 00
0.1875E 02	-0.2166E 01	-0.5890E 00	-0.5441E 00	0.6422E-03	0.9973E 00
0.2190E 02	-0.2243E 01	0.5852E 00	0.5469E 00	0.4244E-03	0.9980E 00
0.2505E 02	-0.2308E 01	-0.5824E 00	-0.5488E 00	0.2890E-03	0.9984E 00
0.2820E 02	-0.2367E 01	0.5803E 00	0.5504E 00	0.2068E-03	0.9988E 00
0.3135E 02	-0.2419E 01	-0.5785E 00	-0.5516E 00	0.1525E-03	0.9990E 00

† Note that β^*, $-\beta$ and $-\beta^*$ are also solutions of the equation.

‡ The parameter K depends only on Poisson's ratio for the material: $K^2 = 2(1 - \nu)$.

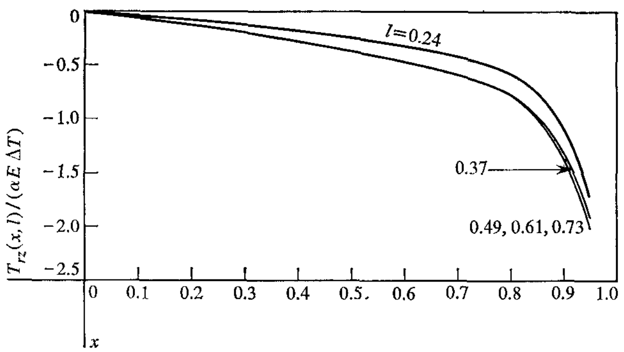


Figure 1 Shear stress on the constrained surface.

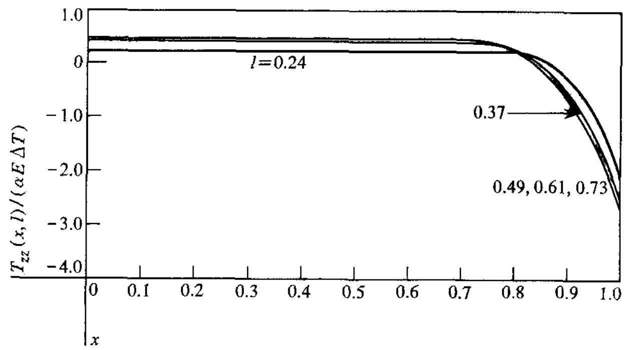


Figure 2 Axial stress on the constrained surface.

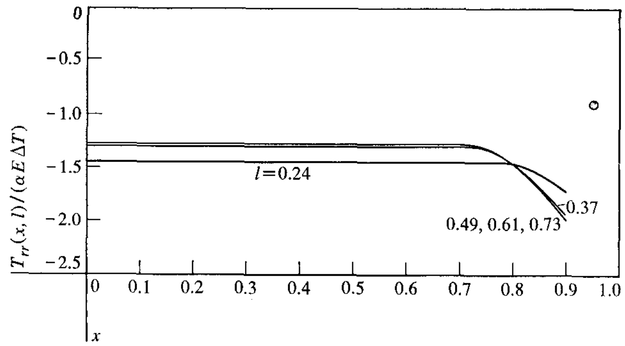


Figure 3 Radial stress on the constrained surface. The encircled point at $x = 0.95$ is the computed value of T_{rr} for $N = 10$; the exact solution should tend to $-\infty$ as $x \rightarrow 1$.

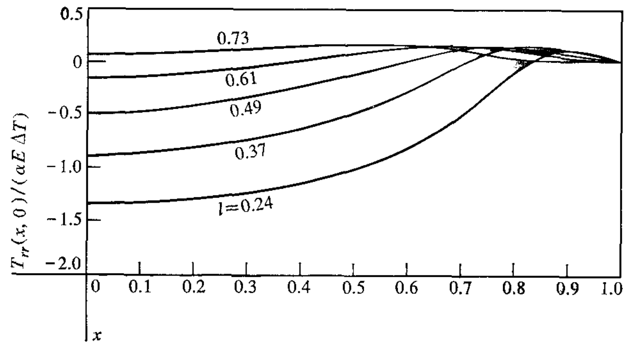


Figure 4 Radial stress on the free end surface.

The $\eta(x, \zeta)$ series is very slowly convergent near $x = 1$, $\zeta = l$ (i.e., near the circumference on the constrained surface) and many more terms would have to be included to improve substantially the results in this neighborhood. It appears, however, that the stress T_{zz} at $x = 1$ and $\zeta = l$ will tend to $-\infty$ as the number of terms becomes sufficiently large (see Fig. 6). Since only $\partial u/\partial z$ and $\partial w/\partial z$ are non-zero on the constrained surface, it follows that $T_{rr}(1, l)$ must also tend to $-\infty$, although the computed values of T_{rr} show no clear indication of this behavior for N less than 10 (see Fig. 3). A similar study of the maximum shear stress on the constrained surface, though more ambiguous, also suggests that $T_{rz}(x, l) \rightarrow -\infty$ as $x \rightarrow 1$ if a sufficiently large number of terms is included. (The presence of stress concentration points at the constrained edges seems to be a general feature of problems of this type.[1,2]) Because of the lack of orthogonality between the various functions in Eqs. (21), it is not obvious how the convergence can be improved.

In the limit $l \equiv h/R \gg 1$, all stresses vanish on the free surface ($z = 0$) and the stresses on the constrained surface ($z = l$) are much the same as shown in Figs. 1 to 3 for $l = 0.73$. The axial stress $T_{zz}(1, \zeta)$ is essentially zero unless z is closer than about R to the constrained end (i.e., $\zeta < 1$), where it will behave approximately as shown in Fig. 5. The effects of the constraints, in other words, are negligibly small beyond an axial distance R from the constrained end.

The error ϵ_N, defined in Eq. (23), is shown as a function of N in Fig. 7. This error decreases approximately as $N^{-3/2}$, suggesting that the solution is asymptotically exact as $N \rightarrow \infty$. Values of $T_{zz}(x, 0)$, $T_{rz}(x, 0)$, $w(x, l)$ and $u(x, l)$ for $N = 10$ and $l = 0.73$ are shown in Fig. 8. These quantities should be zero when $N = \infty$, so the figure gives some indication of the convergence that can be obtained with 20 terms in the series (corresponding to 10 roots and their conjugates).

Appendix

The matrices **U**, **V**, **W** and **X** were introduced in Eqs. (25). Each of these matrices is defined as an integral over x of the functions obtained by squaring each of Eqs. (21); **U** arises from the square of Eq. (21d), **V** from (21c), **W** from (21a) and **X** from (21b).

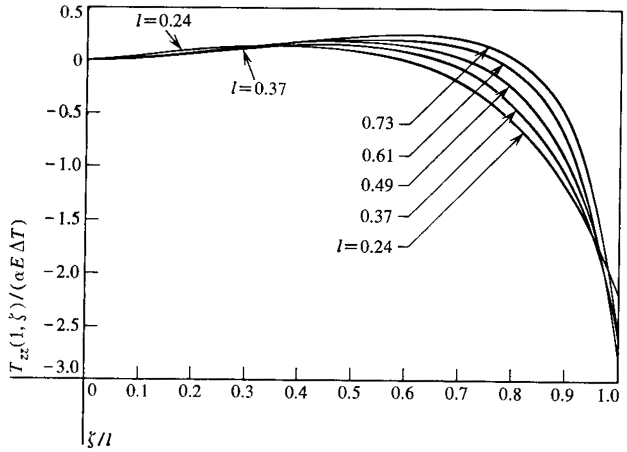


Figure 5 Axial stress on the circumference.

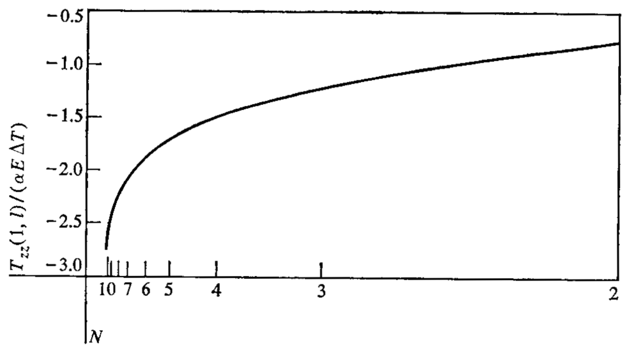


Figure 6 Axial stress at the junction line between cylinder and base as a function of the number of terms in the η series; $l = 0.65$.

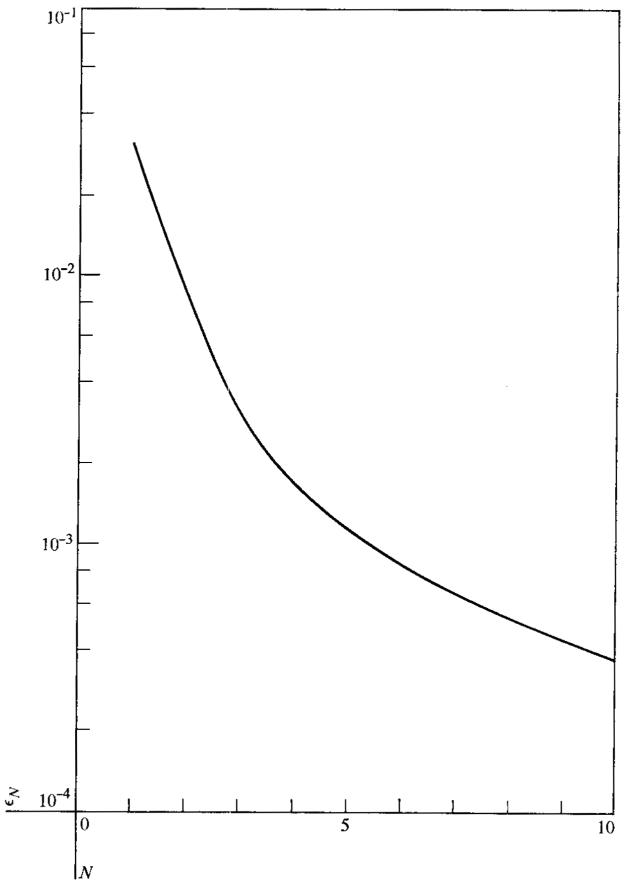
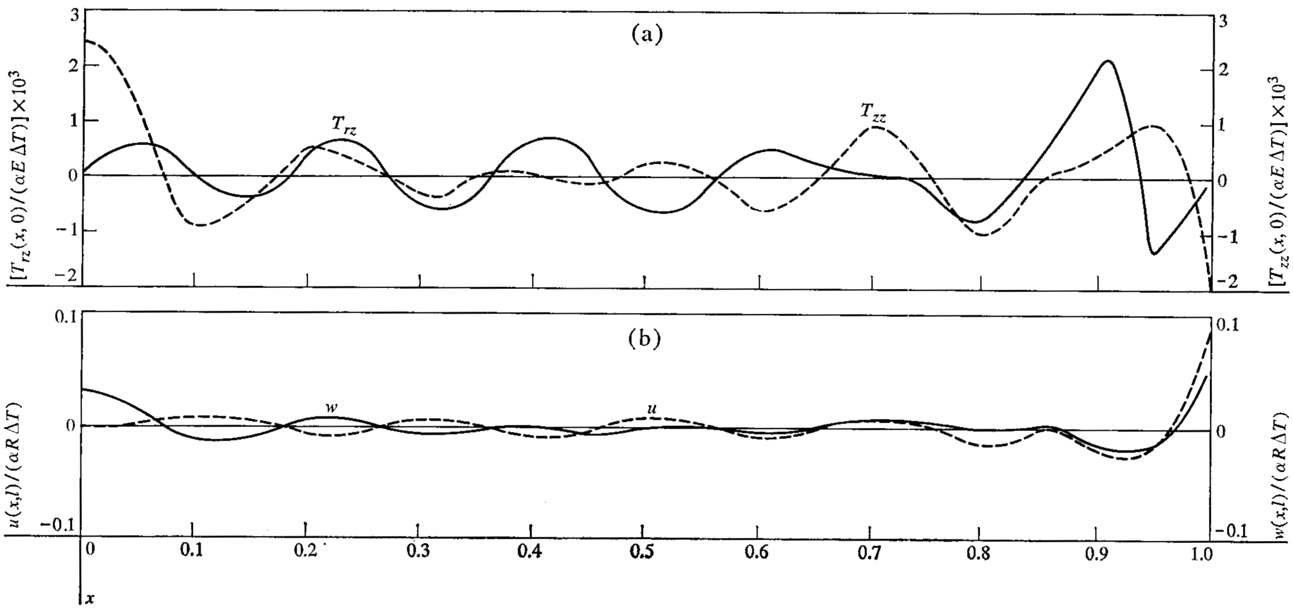


Figure 7 Least-squares error as a function of the number of terms in the η series; $l = 0.65$.

Figure 8 Radial dependence of (a) the shear and axial stresses on the free end surface and (b) the radial and axial displacements on the constrained surface; $l = 0.73$ and $N = 10$.



If we define

$$I_{kl}^{(p)}(ss') = \int_0^1 x^p \mathrm{J}_k(\beta_s x)\,\mathrm{J}_l(\beta_{s'} x)\,dx, \tag{A1}$$

we find

$$\begin{aligned} U_{ss'} = U_{s's} &= I_{00}^{(3)}(ss') \\ &- \left(\frac{K^2}{\beta_s} + \xi_s\right) I_{10}^{(2)}(ss') - \left(\frac{K^2}{\beta_{s'}} + \xi_{s'}\right) I_{10}^{(2)}(s's) \\ &+ \left(\frac{K^2}{\beta_s} + \xi_s\right)\left(\frac{K^2}{\beta_{s'}} + \xi_{s'}\right) I_{11}^{(1)}(ss'), \end{aligned} \tag{A2}$$

$$\begin{aligned} V_{ss'} = V_{s's} &= I_{11}^{(3)}(ss') \\ &- \left(\frac{K^2}{\beta_s} - \xi_s\right) I_{01}^{(2)}(ss') - \left(\frac{K^2}{\beta_{s'}} - \xi_{s'}\right) I_{01}^{(2)}(s's) \\ &+ \left(\frac{K^2}{\beta_s} - \xi_s\right)\left(\frac{K^2}{\beta_{s'}} - \xi_{s'}\right) I_{00}^{(1)}(ss') \\ &- \frac{2(2 - K^2)^2}{\beta_s^2 \beta_{s'}^2} \mathrm{J}_1(\beta_s)\,\mathrm{J}_1(\beta_{s'}), \end{aligned} \tag{A3}$$

$$\begin{aligned} W_{ss'} = W_{s's} &= I_{11}^{(3)}(ss') \\ &- \left(\frac{2}{\beta s} - \xi_s\right) I_{01}^{(2)}(ss') - \left(\frac{2}{\beta_{s'}} - \xi_{s'}\right) I_{01}^{(2)}(s's) \\ &+ \left(\frac{2}{\beta_s} - \xi_s\right)\left(\frac{2}{\beta_{s'}} - \xi_{s'}\right) I_{00}^{(1)}(ss') \text{ and} \end{aligned} \tag{A4}$$

$$\begin{aligned} X_{ss'} = X_{s's} &= I_{00}^{(3)}(ss') - \xi_s I_{10}^{(1)}(ss') \\ &- \xi_{s'} I_{10}^{(2)}(s') + \xi_s \xi_{s'} I_{11}^{(1)}(ss'). \end{aligned} \tag{A5}$$

The quantities $I_{kl}^{(p)}$ can all be obtained by direct integration and one finds

$$I_{00}^{(1)}(ss') = \mathrm{J}_1(\beta_s)\,\mathrm{J}_1(\beta_{s'})(\beta_s\xi_{s'} - \beta_{s'}\xi_s)/(\beta_s^2 - \beta_{s'}^2),$$

$$I_{11}^{(1)}(ss') = \mathrm{J}_1(\beta_s)\,\mathrm{J}_1(\beta_{s'})(\beta_{s'}\xi_{s'} - \beta_s\xi_s)/(\beta_s^2 - \beta_{s'}^2),$$

$$\begin{aligned} I_{10}^{(2)}(ss') = I_{01}^{(2)}(s's) &= [2\beta_s I_{00}^{(1)}(ss') \\ &- \mathrm{J}_1(\beta_s)\,\mathrm{J}_1(\beta_{s'})(\beta_{s'} + \beta_s\xi_s\xi_{s'})]/(\beta_s^2 - \beta_{s'}^2), \end{aligned}$$

$$\begin{aligned} I_{00}^{(3)}(ss') &= I_{00}^{(1)}(ss') \\ &+ 2[\beta_{s'} I_{01}^{(2)}(ss') - \beta_s I_{10}^{(2)}(ss')]/(\beta_s^2 - \beta_{s'}^2) \end{aligned}$$

and

$$\begin{aligned} I_{11}^{(3)}(ss') &= I_{11}^{(1)}(ss') \\ &+ 2[\beta_s I_{01}^{(2)}(ss') - \beta_{s'} I_{10}^{(2)}(ss')]/(\beta_s^2 - \beta_{s'}^2). \end{aligned}$$

Thus the matrices **U**, **V**, **W** and **X** can be constructed once a tabulation of β_s, ζ_s and $\mathrm{J}_1(\beta_s)$ is available. These matrices depend only on the parameter K^2 and are independent of the dimensions of the cylinder.

Acknowledgments

I am indebted to D. Seraphim for suggesting the problem and to V. Jagirdar and N. Bohidar for several discussions.

References

1. B. J. Aleck, *J. Appl. Mech.* **16,** 118 (1949).
2. P. S. Theocaris and K. Dafermos, *J. Appl. Mech.* **31,** 714 (1964).
3. A. I. Lur'e, *Three-Dimensional Problems of the Theory of Elasticity,* Interscience Publishers, New York 1964, Chapter 7.
4. A. E. H. Love, *The Mathematical Theory of Elasticity,* Dover Publications, New York 1944, esp. Sections 74 and 188.
5. P. J. Davis, *Interpolation and Approximation,* Blaisdell Publishing Co., New York 1963, p. 24.
6. F. W. J. Olver, "Bessel Functions of Integer Order," *Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun, Number 55, Applied Mathematics Series, National Bureau of Standards, Washington, D. C. 1964, Chapter 9.

Received September 3, 1968