Thermal Expansion in a Constrained Elastic Cylinder

Abstract: The stress developed in an elastic cylinder of finite length undergoing thermal expansion with one end clamped is expressed in terms of a series expansion of a biharmonic function, appropriate derivatives of which give the displacements and stresses within the cylinder. The coefficients in this series are determined by a least-squares fit to the boundary conditions at the ends of the cylinder and values of the stress on various surfaces are found as functions of the height-to-radius ratio. All components of the stress tensor become infinite at the circumference on the clamped end. A tabulation is included of quantities of interest in any cylindrical problem in which the curved surface is a free surface.

Introduction

There has been considerable interest in the problem of stress induced by thermal shrinkage or expansion in an elastic body with one or more surfaces constrained. Such problems arise, for example, in the encapsulation of electronic circuit components. The calculation of these stresses in a circular cylinder is the subject of this paper. The cylindrical symmetry reduces the problem to a twodimensional one, offering the possibility of a tractable solution in terms of a series of biharmonic functions. In previous work studies of rectangular plates have been carried out variationally and studies of long bars (constrained on a long edge) have been made using series expansions.² The cylindrical geometry, although permitting solution of a three-dimensional problem in two-dimensional terms, leads to considerable mathematical complexity in the sense that the terms in the series for various stresses are not orthogonal and the coefficients must be found by solving a set of linear equations. Certain mathematical constants involved in the solution are independent of the dimensions of the cylinder and can be given as functions of the elastic moduli. A considerably more extensive tabulation of these quantities than has heretofore appeared³ is given here.

Theory

The problem is to determine the displacements and stresses in a cylinder of radius R and height h that undergoes thermal expansion with one end (z = h) clamped to a rigid, nonexpanding plate. The other end (z = 0) and the curved

Let u be the displacement of a point in the cylinder in the radial direction and w the displacement in the z direction. If we assume that all quantities are independent of the angular coordinate θ , the strains are

$$\epsilon_{rr} = \frac{\partial u}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{u}{r}, \quad \epsilon_{zz} = \frac{\partial w}{\partial z},$$

$$\epsilon_{rz} = \epsilon_{zr} = \frac{1}{2} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right), \quad \epsilon_{r\theta} = \epsilon_{z\theta} = 0$$
(1)

and the dilation is

$$\Delta = \epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz} = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}.$$
 (2)

The cylinder temperature is assumed to be raised by a constant amount ΔT above the temperature at which no strain exists. If λ and μ are the Lamé constants at the elastically isotropic cylinder, in terms of which Young's modulus E and Poisson's ratio ν are expressed as

$$E = \mu(3\lambda + 2\mu)/(\lambda + \mu) \text{ and } \nu = \frac{1}{2}\lambda/(\lambda + \mu), \tag{3}$$

the stress-strain relations are4

 $T_{r\theta} = T_{z\theta} = 0,$

$$T_{rr} = 2\mu\epsilon_{rr} + \lambda\Delta - (3\lambda + 2\mu)\alpha\Delta T,$$

$$T_{\theta\theta} = 2\mu\epsilon_{\theta\theta} + \lambda\Delta - (3\lambda + 2\mu)\alpha\Delta T,$$

$$T_{zz} = 2\mu\epsilon_{zz} + \lambda\Delta - (3\lambda + 2\mu)\alpha\Delta T,$$

$$T_{rz} = 2\mu\epsilon_{rz} \text{ and}$$
(4)

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surface or circumference (r = R) of the cylinder are free surfaces, i.e., no forces are applied.

where α is the linear coefficient of thermal expansion; there is a constant strain $\alpha \Delta T$ in each principal direction.

The equations of equilibrium are

$$\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} + \frac{\partial T_{rz}}{\partial z} = 0$$
 (5a)

and

$$\frac{\partial T_{rz}}{\partial r} + \frac{T_{rz}}{r} + \frac{\partial T_{zz}}{\partial z} = 0. \tag{5b}$$

These are subject to the following boundary conditions: At z = h (on the plate) the cylinder is constrained to have zero displacement, so that

$$u(r, h) = w(r, h) = 0;$$
 (6a)

at r = R and at z = 0 no forces are applied, so that

$$T_{rr}(R,z) = T_{rz}(R,z) = 0$$
 and (6b)

$$T_{zz}(r, 0) = T_{rz}(r, 0) = 0.$$
 (6c)

Love⁴ has shown that problems of cylindrical symmetry can be solved in terms of a single biharmonic function. For example, let

$$\nabla^4 \Phi \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right)^2 \Phi = 0 \tag{7}$$

and define the following expressions (which are equivalent, but not identical, to those given by Love):

$$u = -\frac{1}{2\mu} \frac{\partial^2 \Phi}{\partial r \partial z} ,$$

$$w = \frac{1}{2\mu} \left(\frac{\lambda + 2\mu}{\lambda + \mu} \nabla^2 \Phi - \frac{\partial^2 \Phi}{\partial z^2} \right) .$$
(8)

The stresses obtained in terms of Φ from Eqs. (4) using Eqs. (1) and (2) automatically satisfy the equilibrium conditions, Eqs. (5), and the present problem is reduced to constructing the function Φ that satisfies the boundary conditions given in Eqs. (6).

We begin this construction by expanding Φ in a series of solutions of the biharmonic equation (7), each term of which automatically satisfies the boundary conditions on the circumference of the cylinder. The coefficients in this series are adjusted so that the remaining boundary conditions are satisfied when the number of terms in the series becomes infinite; inasmuch as the terms in the series are nonorthogonal, this adjustment is made using a least-squares procedure. The coefficients of certain trivial solutions of the biharmonic equation are chosen so that the boundary conditions are satisfied in an average sense and it turns out that the trivial solutions included in Φ serve to remove the constant thermal stresses and to add a constant to the axial displacement w.

We define the reduced quantities

$$x = r/R,$$

$$\zeta = z/R,$$

$$l = h/R,$$

$$K^{2} = \frac{\lambda + 2\mu}{\lambda + \mu} = 2(1 - \nu)$$
(9)

and write

$$\Phi = a_1 \zeta^2 + a_2 \zeta^3 + b_1 x^2 + b_2 x^2 \zeta + \eta(x, \zeta). \tag{10}$$

The first four terms are the trivial solutions of the biharmonic equation that are regular at x = 0 (the constant term and the term proportional to ζ have been omitted since they do not contribute to any displacements or stresses) and η represents the nontrivial part of Φ , which is still to be constructed. In terms of the coefficients a_1 , a_2 , b_1 and b_2 the displacements are

$$u = -\frac{b_2}{\mu R^2} x + \text{terms in } \eta,$$

$$w = \frac{1}{R^2} \left\{ \frac{a_1}{\lambda + \mu} + \frac{2(\lambda + 2\mu)b_1}{\mu(\lambda + \mu)} + \left[\frac{3a_2}{\lambda + \mu} + \frac{2(\lambda + 2\mu)b_2}{\mu(\lambda + \mu)} \right] \zeta \right\} + \text{terms in } \eta,$$

where the "terms in η " are obtained from Eqs. (8) with Φ replaced by η .

• Evaluation of coefficients

The stresses all involve derivatives of u and w, so it is clear that the a_1 and b_1 terms in Φ contribute constants to w and appear nowhere else. Without loss of generality we can choose $b_1=0$ and use a_1 to fix the constant term in w. The terms in a_2 and b_2 produce a displacement field like that due to unconstrained thermal expansion, i.e., constant strains with no shear; they add constants to the principal stresses and do not affect the shear stress. The boundary conditions require that T_{rr} vanish on the circumference and that T_{zz} vanish on the free end (z=0). It follows, then that

$$\int_0^1 x T_{zz}(x, 0) \ dx = \int_0^1 T_{rr}(1, \zeta) \ d\zeta = 0. \tag{11}$$

We shall choose η in such a way that its contribution to $T_{rr}(x, \zeta)$ automatically vanishes when x = 1. It will then turn out that η makes no contribution to the average of T_{zz} over the free end; thus Eqs. (11) will contain no terms in η . To satisfy Eqs. (11), a_2 and b_2 must be chosen to remove the thermal stress term $-(3\lambda + 2\mu) \times \alpha \Delta T$ from T_{rr} and T_{zz} . In this way $T_{rr}(1,\zeta)$ is made identically zero for all ζ , while $T_{zz}(x,0)$ presumably becomes

zero pointwise as the number of terms in the series for η becomes infinite. The required values of a_2 and b_2 are

$$a_2 = \frac{1}{3}R^3(3\lambda + 5\mu)\alpha\Delta T; \qquad b_2 = -R^3\mu\alpha\Delta T. \tag{12}$$

The displacements and stresses now become⁴

$$u = R\alpha \Delta T x - \frac{1}{2\mu} \frac{\partial^{2} \eta}{\partial r \partial z} ,$$

$$w = R\alpha \Delta T (\zeta - l + B)$$

$$+ \frac{1}{2\mu} \left(\frac{\lambda + 2\mu}{\lambda + \mu} \nabla^{2} \eta - \frac{\partial^{2} \eta}{\partial z^{2}} \right) ,$$

$$T_{rr} = \frac{\partial}{\partial z} \left[\frac{\lambda}{2(\lambda + \mu)} \nabla^{2} \eta - \frac{\partial^{2} \eta}{\partial r^{2}} \right] ,$$

$$T_{\theta\theta} = \frac{\partial}{\partial z} \left[\frac{\lambda}{2(\lambda + \mu)} \nabla^{2} \eta - \frac{1}{r} \frac{\partial \eta}{\partial r} \right] ,$$

$$T_{zz} = \frac{\partial}{\partial z} \left[\frac{3\lambda + 4\mu}{2(\lambda + \mu)} \nabla^{2} \eta - \frac{\partial^{2} \eta}{\partial z^{2}} \right] ,$$

$$T_{rz} = \frac{\partial}{\partial r} \left[\frac{\lambda + 2\mu}{2(\lambda + \mu)} \nabla^{2} \eta - \frac{\partial^{2} \eta}{\partial z^{2}} \right] ,$$

$$(13)$$

where we have rewritten a_1 so that the constant contribution to w is $R\alpha\Delta T(B-l)$. The constant B is chosen so that the boundary condition on w is satisfied on the average:

$$\int_0^1 x w(x, l) \ dx = 0. \tag{14}$$

• Construction of $\eta(x, \zeta)$ We write η in the form

$$\eta(x, \zeta) = 2\mu R^3 \alpha \Delta T
\times \left[\sum_{s} x J_1(\beta_s x) (A_s \sinh \beta_s \zeta + C_s \cosh \beta_s \zeta) \right]
+ \sum_{s} J_0(\beta_s x) (B_s \sinh \beta_s \zeta + D_s \cosh \beta_s \zeta),$$
(15)

where J_0 and J_1 are Bessel functions and A_s , B_s , C_s , D_s and β_s are constants to be determined. The quantities $J_0(\beta x)$ $\times \exp(\pm \beta \zeta)$ and $xJ_1(\beta x)\exp(\pm \beta \zeta)$ are well known solutions⁴ of the biharmonic equation (7).

The boundary conditions (6b) on the circumference of of the cylinder take the form

$$\sum_{\bullet} \left[\gamma_{\bullet} \cosh \beta_{\bullet} \zeta + \delta_{\bullet} \sinh \beta_{\bullet} \zeta \right] = 0, \tag{16}$$

where γ_* and δ_* are known in terms of A_* , B_* , C_* and D_* . By expanding this expression in powers of ζ and using the linear independence of various powers of ζ , we obtain a series of homogeneous equations in γ_* or δ_* . We conclude that γ_* and δ_* must vanish if the determinant of coefficients is non-singular. This determinant is of van der Monde

form⁵ and we are thus able to say that Eq. (16) implies $\gamma_s = \delta_s \equiv 0$ if the β_s (which are still to be determined) are such that

$$\prod_{i>i} (\beta_i^2 - \beta_i^2) \neq 0$$

and no β_i is zero or, in other words, if no β is zero or the same as or the negative of another β . [All such cases are readily seen to be redundant: If any β_i is zero, the corresponding terms in Eq. (15) are constants, which can be ignored. Similarly, if any β_i is the same as or the negative of another, it can be eliminated from the sum simply by regrouping terms and redefining the coefficients A_i , B_i , C_i and D_i in Eq. (15)].

We now evaluate T_{rr} and T_{rz} on the circumference. Defining

$$\xi_s \equiv J_0 (\beta_s) / J_1(\beta_s) \tag{17}$$

and using the result derived for γ_s and δ_s , we obtain the following equations:

$$\left(\frac{K^2}{\beta_s} + \xi_s\right) A_s - B_s = 0,$$

$$\left(\frac{K^2}{\beta_s} + \xi_s\right) C_s - D_s = 0,$$

$$\left(1 - \frac{\mu}{\lambda + \mu} \frac{\xi_s}{\beta_s}\right) A_s - \left(\frac{1}{\beta_s} - \xi_s\right) B_s = 0 \text{ and}$$

$$\left(1 - \frac{\mu}{\lambda + \mu} \frac{\xi_s}{\beta_s}\right) C_s - \left(\frac{1}{\beta_s} - \xi_s\right) D_s = 0.$$

These two sets of equations have nontrivial solutions only if

$$\beta_s^2(1+\xi_s^2)=K^2, \tag{18}$$

in which case

$$\frac{B_s}{A_s} = \frac{D_s}{C_s} = \frac{K^2}{\beta_s} + \xi_s. \tag{19}$$

Equation (18) is the characteristic equation determining the permissible values of β_{\bullet} over which we sum in Eq. (15) to construct η . Since $1 \leq K^2 \leq 2$ [see Eq. (9) and recall that the value of Poisson's ratio is between 0 and $\frac{1}{2}$], it is possible to show that the β_{\bullet} 's are complex for all cases of interest. Furthermore, since Bessel functions are real functions and are either symmetric or antisymmetric, it follows that, if β_{\bullet} satisfies Eq. (18), then β_{\bullet}^* , $-\beta_{\bullet}$ and $-\beta_{\bullet}^*$ also satisfy Eq. (18). (The asterisk denotes complex conjugate.) According to the discussion following Eq. (16), we must omit $-\beta_{\bullet}$ and $-\beta_{\bullet}^*$ as well as $\beta_{\bullet} = 0$, so our sums extend over all β_{\bullet} (and their conjugates) with positive real part that satisfy Eq. (18).

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The function η now takes the form

$$\eta = 2\mu R^3 \alpha \Delta T \sum_{s} \left[x J_1(\beta_s x) + \left(\frac{K^2}{\beta_s} + \xi_s \right) J_0(\beta_s x) \right]$$

$$\times (A_s \sinh \beta_s \zeta + C_s \cosh \overline{\beta_s} \zeta)$$
(20)

and the remaining four boundary conditions become

$$\phi_a(x) \equiv \sum_s \beta_s^3 A_s$$

$$\times \left[\left(\frac{2}{\beta_s} - \xi_s \right) J_0(\beta_s x) - x J_1(\beta_s x) \right] = 0, \quad (21a)$$

$$\phi_b(x) \equiv \sum_s \beta_s^3 C_s[x J_0(\beta_s x) - \xi_s J_1(\beta_s x)] = 0,$$
 (21b)

$$\phi_c(x) \equiv B + \sum_s \beta_s^2 \left[\left(\frac{K^2}{\beta_s} - \xi_s \right) J_0(\beta_s x) - x J_1(\beta_s x) \right]$$

$$\times (A_s \sinh \beta_s l + C_s \cosh \beta_s l) = 0, \qquad (21c)$$

$$\phi_d(x) \equiv -x + \sum_s \beta_s^2 \left[x J_0(\beta_s x) - \left(\frac{K^2}{\beta_s} + \xi_s \right) J_1(\beta_s x) \right]$$

$$\times (A_s \cosh \beta_s l + C_s \sinh \beta_s l) = 0, \qquad (21d)$$

all for $0 \le x \le 1$; these equations are, in order, the conditions that $T_{zz}(x, 0)$, $T_{rz}(x, 0)$, w(x, l) and u(x, l) be zero.

The constant B, which is to be found from Eq. (14), can now be given explicitly in terms of the A_s and C_s ; we obtain

$$B = -2(K^2 - 2) \sum_{s} \mathbf{J}_1(\beta_s)$$

$$\times (A_s \sinh \beta_s l + C_s \cosh \beta_s l) \tag{22}$$

and $\phi_c(x)$ becomes

$$\phi_{c}(x) = \sum_{s} \left\{ 2(2 - K^{2}) J_{1}(\beta_{s}) + \beta_{s}^{2} \left[\left(\frac{K^{2}}{\beta_{s}} - \xi_{s} \right) J_{0}(\beta_{s}x) - x J_{1}(\beta_{s}x) \right] \right\}$$

$$\times (A_{s} \sinh \beta_{s}l + C_{s} \cosh \beta_{s}l). \tag{21c'}$$

It is easily verified that

$$\int_0^1 x \left[\left(\frac{2}{\beta_s} - \xi_s \right) \mathbf{J}_0(\beta_s x) - x \mathbf{J}_1(\beta_s x) \right] dx = 0,$$

which guarantees that Eq. (11) is satisfied. The constants A_s and C_s are both complex; Eqs. (21a) and (21b) furnish relations between their real and imaginary parts rather than a requirement that they be zero. From the structure of Eqs. (21) it is clear that

$$A_s^* \equiv A(\beta_s^*) = [A(\beta_s)]^*$$

and similarly for C_s . Thus all the ϕ 's of Eqs. (21), as well as η itself, are real quantities.

• Least-squares adjustment

The coefficients A_s and C_s have to be determined numerically, since there is not much hope of solving Eqs. (21) analytically. Thus we have to approximate η by a finite number (say 2N) of terms. To choose the coefficients in the best possible way, subject to this limitation, we minimize the error

$$\epsilon_N = \int_0^1 x (\phi_a^2 + \phi_b^2 + \phi_c^2 + \phi_d^2) \ dx \tag{23}$$

incurred in trying to satisfy the boundary conditions with a finite number of terms. This procedure is necessary because the functions appearing in Eqs. (21) are not mutually orthogonal.

The minimization conditions

$$\frac{\partial \epsilon_N}{\partial A_s} = \frac{\partial \epsilon_N}{\partial C_s} = 0$$

lead to the linear equations

$$\sum_{s'=1}^{2N} \left(M_{ss'} A_{s'} + N_{ss'} C_{s'} \right) = \Gamma_s \cosh \beta_s l,$$

$$\sum_{s'=1}^{2N} \left(N_{s's} A_{s'} + Q_{ss'} C_{s'} \right) = \Gamma_s \sinh \beta_s l,$$
(24)

in which

$$\Gamma_{s} = J_{1}(\beta_{s}) \left(K^{2} \xi_{s} - \frac{4 + K^{2}}{\beta_{s}} \right),$$

$$M_{ss'} = \beta_{s}^{2} \beta_{s'}^{2} (U_{ss'} \cosh \beta_{s} l \cosh \beta_{s'} l + V_{ss'} \sinh \beta_{s} l \sinh \beta_{s'} l + \beta_{s} \beta_{s'} W_{ss'}),$$

$$+ V_{ss'} \sinh \beta_{s} l \sinh \beta_{s'} l + \beta_{s} \beta_{s'} W_{ss'}),$$

$$(25)$$

$$N_{ss'} = \beta_{s}^{2} \beta_{s'}^{2} (U_{ss'} \cosh \beta_{s} l \sinh \beta_{s'} l + V_{ss'} \sinh \beta_{s} l \cosh \beta_{s'} l),$$

$$Q_{ss'} = \beta_{s}^{2} \beta_{s'}^{2} (U_{ss'} \sinh \beta_{s} l \sinh \beta_{s'} l + V_{ss'} \cosh \beta_{s} l \cosh \beta_{s'} l + \beta_{s} \beta_{s'} X_{ss'}).$$

The matrices U, V, W and X, which are linear combinations of integrals of Bessel functions and are independent of l, are discussed further in the Appendix. It is interesting to note that the β_s 's and U, V, W and X depend only on Poisson's ratio and can be tabulated as functions of this ratio without reference to cylinder dimensions.† Such a tabulation would be of use in solving any problem of cylindrical symmetry in which the circumference is a free surface. Different boundary conditions on the ends of the cylinder will still lead to a set of equations like (24), but with M, N and Q given by different linear combinations of U, V, W and X.

[†] Some values of β_{\bullet} are presented later in Table 1.

Application

· Computational procedure

Our program consists of finding the roots of Eq. (18), solving Eqs. (24) for A_{\bullet} and C_{\bullet} and constructing η and any desired displacements and stresses using Eqs. (20) and (13). To compute the Bessel functions, the standard downward-recurrence algorithm, modified to handle complex arguments, was used. One can show, using asymptotic (large argument) expansions of the Bessel functions, that the roots of Eq. (18) are given approximately by

$$\beta_* \approx s\pi - \frac{\ln 4s\pi}{4s\pi} + \frac{(\frac{1}{4} - K^2)}{2s\pi} - \frac{i}{2} \left[1 - \frac{2(\frac{1}{4} - K^2)}{(2s\pi)^2} \right] \ln 4s\pi, \tag{26}$$

which is asymptotically exact for large s. [Equation (26) determines the value of β_1 to within 2%.] The algorithm for computing β_* uses the above expression as a first approximation and then iterates the calculation with Newton's method until β_* is determined to five significant figures. The values obtained in this way for the first ten roots of Eq. (18), as well as the values of $J_1(\beta_*)$ and ξ_* , are given in Table 1 for values of K^2 from 1 to 2 in steps of 0.1.

Standard algorithms for solving linear equations can be used if Eqs. (24) are first rewritten as real equations for the real and imaginary parts of A_s and C_s . Problems with large numbers arising from $\cosh \beta_s l$ or $\sinh \beta_s l$ for large s (Re $\beta_s \sim s\pi$) are best circumvented by solving for the quantities

$$\psi_s = A_s \cosh \beta_s l + C_s \sinh \beta_s l$$

and

$$\phi_s = (A_s - C_s) \exp(-\beta_s l) \sinh \beta_s l$$

instead of for A_* and C_* ; Eqs. (24) rewritten in terms of ψ_* and ϕ_* are much more tractable than the original equations and contain no terms that become exponentially large for large s.

• Numerical results

Calculations using up to 10 roots and their conjugates (i.e., up to 20 terms in the series for η) were made for $K^2=1.32$ and for various values of l. Results for the stresses on the top, bottom and side surfaces of the cylinder are shown in Figs. 1 through 5. All data in these figures are based on calculations using 10 roots and their conjugates. The actual computed stress curves show small oscillations around the smoothed curves given in the figures; these oscillations are presumed to be due to the use of a finite number of terms (much the same as in Fourier series) and were therefore omitted from the figures.

Table 1 Real and imaginary parts of β_s , $J_1(\beta_s)$ and $\xi_s \equiv J_0(\beta_s)/J_1(\beta_s)$, $s = 1, 2, 3, \cdots, 10$, where $\beta_s \dagger$ is a root of $\beta^2(1 + \xi^2) = K^2.\ddagger$

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0.2746E 01 -0.1377E 01	0.2507E	02	-0.2308E	01	-0.5730E	00	-0.5574E	00	0.1736	E-03	0.9991E 0.9993E	00
0.274-6E 01 -0.1375 01 -0.2876 00 -0.5502E 00 0.577-2E-01 0.587E 00 0.1501E 01 -0.1501E 01 -0.1501E 01 -0.1501E 01 -0.1501E 01 -0.1501E 01 0.5007E 00 0.5502E 00 0.5502E 00 0.5507E 00 0.2876 00 0.2	0.3136E	02	-0.2419E	01	-0.5711E			ÕÕ	0.9346	E-04	0.9994E	00
Care	0.2746E	01	-0.1357E	01	0.68976	00	0.5603E	00	0.5742	E-01	0.9587E	.00.
0.156.1E 02 -0.276.E 01	0.92726	01	-0.1828E	01	0.5941E	00	0.5473E	00	0.2782	E-02	0.9932E	00
0.2507E 02 -0.2308E 01 -0.5732E 00 -0.5531E 00 0.1876E-03 0.999E 00 0.7313E 02 -0.2419E 01 0.5732E 00 0.5531E 00 0.1055E-30 0.999E 00 0.999E 0	0.1561E	02	-0.2076E	01	0.5809E	00	0-5525F	0.0	0-6856	E=03	0.9975E 0.9982E	00
0.2321E 02 -0.2368E 01	0.2192E	02	-0.2242E	01	0.5757E -0.5742E	00	-0.5563E	00	0.2717	E-03	0.9990E	00
0.9327E 01 -0.1828E 01	0.28216	02	-0.2366E	01	0.5730E -0.5720E	00	0.5571E -0.5578E	00	0.1354	E-03	0.9992E	
0.9327E 01 -0.1828E 01	0.2722E	01	-0.1362E	01	0.7064E	00	1.4 0.5533E	00	0.6327	E-01	0.9557E	00
0.124-E 02 -0.1978 01 -0.5882 00 -0.5893 00 0.1397E-02 0.9978E 00 0.1297E 02 -0.2708 01 0.2978E 00 0.2197E 02 -0.2242E 01 0.5777E 00 -0.5592E 00 0.2708E-03 0.9988E 00 0.2208E-03 0.9988E 00 0.2508E 02 -0.2308E 01 0.5776E 00 -0.5592E 00 0.2109E-03 0.9988E 00 0.2308E-03 0.9988E 00 0.5756E 00 0.2508E-03 0.9988E 00 0.2208E-03 0.9988E 00 0.2508E-03 0.9988E-03 0.9988E	0.6060E 0.9267E	01	-0.1638E	01	-0.6184E 0.5975E	00	0.5446E	00	0.9086	E-02	0.9846E 0.9927E	00
0.7299E 01 -0.1307E 01 -0.239E 00 0.5454E 00 0.6990E-01 0.9228E 00 0.920E-01 -0.1638E 01 -0.6239E 00 -0.5353E 00 0.9702TE-02 0.9824E 00 0.920E-01 -0.1638E 01 -0.6239E 00 -0.5353E 00 0.9702TE-02 0.9824E 00 0.1244E 01 -0.1407E-02 0.992E 00 0.1246E 01 -0.5708E 01 -0.5708E 01 -0.5708E 01 -0.5708E 01 0.0246E 01 -0.5708E 01 0.0246E 01 -0.5708E 01 0.5708E 01 0.2246E 01 -0.5708E 01 0.5708E 01 0.2246E 01 -0.5708E 01 0.5708E 01 0.2246E 01 -0.5708E 01 0.5708E 01 0.5708E 01 0.5708E 01 0.2246E 01 -0.5708E 01 0.5708E 01 0.570	0.156 LE	02	-0.1967E	01 01			-0.5483E	00	0.1367	E-02		00
0.7299E 01 -0.1307E 01 -0.239E 00 0.5454E 00 0.6990E-01 0.9228E 00 0.920E-01 -0.1638E 01 -0.6239E 00 -0.5353E 00 0.9702TE-02 0.9824E 00 0.920E-01 -0.1638E 01 -0.6239E 00 -0.5353E 00 0.9702TE-02 0.9824E 00 0.1244E 01 -0.1407E-02 0.992E 00 0.1246E 01 -0.5708E 01 -0.5708E 01 -0.5708E 01 -0.5708E 01 0.0246E 01 -0.5708E 01 0.0246E 01 -0.5708E 01 0.5708E 01 0.2246E 01 -0.5708E 01 0.5708E 01 0.2246E 01 -0.5708E 01 0.5708E 01 0.2246E 01 -0.5708E 01 0.5708E 01 0.5708E 01 0.5708E 01 0.2246E 01 -0.5708E 01 0.5708E 01 0.570	0.1876E 0.2191E	02	-0.2242E	01	-0.5795E 0.5771E	00	-0.5527E 0.5542E	00	0.4479	E-03	0.9986E	00
0.7299E 01 -0.1307E 01 -0.239E 00 0.5454E 00 0.6990E-01 0.9228E 00 0.920E-01 -0.1638E 01 -0.6239E 00 -0.5353E 00 0.9702TE-02 0.9824E 00 0.920E-01 -0.1638E 01 -0.6239E 00 -0.5353E 00 0.9702TE-02 0.9824E 00 0.1244E 01 -0.1407E-02 0.992E 00 0.1246E 01 -0.5708E 01 -0.5708E 01 -0.5708E 01 -0.5708E 01 0.0246E 01 -0.5708E 01 0.0246E 01 -0.5708E 01 0.5708E 01 0.2246E 01 -0.5708E 01 0.5708E 01 0.2246E 01 -0.5708E 01 0.5708E 01 0.2246E 01 -0.5708E 01 0.5708E 01 0.5708E 01 0.5708E 01 0.2246E 01 -0.5708E 01 0.5708E 01 0.570	0.2821E	02	-0.2367E	01	0.5740E	00	0.5562E	00	0.1461	E-03	0.9991E	00
0.4051E 01 -0,1838E 01 -0.6239E 00 -0.5353E 00 0.7978E 02 0.793E 00 0.793E 0							11.5					
0.18060 02 -0.20660 1 -0.51860 0 0 .5408E 00 0 .7931E-03 0.9971E 00 0.18760 02 -0.21866 01 -0.51816 00 -0.5512E 00 0 .20187E-03 0.9981E 00 0.2281E 02 -0.2387E 01 -0.5751E 00 -0.552E 00 0.2187E-03 0.9981E 00 0.2187E-03 0.9991E 00 0.2187E-03 0.	0.60516	01	-0.1638E	01	0.7236E -0.6239E	00	0.5454E -0.5353E	00	0.6940	E-01	0.9528E 0.9834E	00
0.18766 02 -0.21846 01 -0.58186 00 -0.58138 00 0.4807E-03 0.4937E 00 0.21016 02 -0.22426 01 0.57826 00 0.52526 00 0.1147E-03 0.4938E 00 0.21016 02 -0.24476 01 0.57826 00 0.55526 00 0.1157E-03 0.4938E 00 0.2821E 02 -0.24196 01 0.57858 00 0.55526 00 0.1157E-03 0.4938E 00 0.2821E 02 -0.24196 01 0.57858 00 0.55526 00 0.1176E-03 0.4938E 00 0.2821E 01 -0.1372E 01 0.57396 00 -0.55526 00 0.1176E-03 0.4991E 00 0.4952E 01 -0.1898 01 -0.2416 01 0.57858 00 0.23364 00 0.7586E-01 0.9931E 00 0.2452E 01 -0.1898 01 -0.2596 00 0.23364 00 0.7586E-01 0.9931E 00 0.2452E 01 -0.1898 01 -0.5931E 00 0.2596 00 0.3346E 00 0.4947E 00 0.	0.12448	05	-0.1829E	01	-0.5906E	00	-0.5462E	00	0.1465	E-02	0.9955E	00
0.2821E 02 -0.2387E 01	0.1876E	0 Z	-0.2166E	01	-0.5B11E	00	-0.5513E	aa	0-4807	E-03	A. 9979E	00
0.2874E 01 -0.1372E 01 -0.732E 01 0.752E 00 0.5334E 00 0.758E-01 0.950E 00 0.632E 01 -0.1872E 01 -0.1872E 01 0.752E 00 0.330E 00 0.310E 00 0.1087E-01 0.9872E 00 0.1283E 02 -0.198E 01 -0.1872E 01 -0.593E 00 -0.544E 00 0.5390E 00 0.156E-02 0.993FE 00 0.1283E 02 -0.2977E 01 0.568E 00 0.4975E 00 0.156E-02 0.995FE 00 0.1878E 02 -0.2186E 01 -0.582FE 00 -0.544E 00 0.556E-02 0.995FE 00 0.1878E 02 -0.2186E 01 -0.582FE 00 -0.594E 00 0.5128E-03 0.995E 00 0.250E 02 -0.2787E 01 0.556E 00 0.556E 00 0.5128E-03 0.995E 00 0.280E 02 -0.2787E 01 0.5786E 00 0.556E 00 0.2384E 02 -0.2384E 01 0.5786E 00 0.556E 00 0.2186E 02 -0.2986E 00 0.2384E 01 0.998E 00 0.5786E 00 0.282E 02 -0.2387E 01 0.5766E 00 0.5565E 00 0.1286E-03 0.999E 00 0.3386E 02 -0.2387E 01 0.5766E 00 -0.5565E 00 0.1286E-03 0.999E 00 0.4925E 01 -0.1878E 01 0.5766E 00 -0.5565E 00 0.1286E-01 0.999E 00 0.4925E 01 -0.1878E 01 0.5766E 00 -0.5565E 00 0.1286E-01 0.999E 00 0.4925E 01 -0.1878E 01 0.5766E 00 -0.5565E 00 0.1286E-01 0.999E 00 0.4925E 01 -0.1878E 01 0.5766E 00 -0.5565E 00 0.1286E-01 0.999E 00 0.4925E 01 -0.1878E 01 0.5766E 00 -0.5565E 00 0.1126E-01 0.999E 00 0.4925E 01 -0.1878E 01 0.5778E 00 -0.5565E 00 0.1126E-01 0.999E 00 0.1286E 01 0.999E 01 0.0778E 00 -0.5565E 00 0.3676E-02 0.999E 00 0.1286E 01 0.999E 01 0.0778E 00 -0.5565E 00 0.3676E-02 0.999E 00 0.1286E 01 0.999E 01 0.0778E 00 0.5555E 00 0.1286E-01 0.999E 00 0.2566E 01 -0.189E 01 0.0778E 00 0.5555E 00 0.1286E-01 0.999E 00 0.2566E 01 -0.189E 01 0.5778E 00 0.5555E 00 0.189E-01 0.999E 00 0.2566E 01	0.2506E 0.2821E	02	-0.2308E	01 01	-0.5766E 0.5751E	00	-0.5542E	00	0.1572	E-03	0.99916	00
0.1500E 02 -0.207TE 01	0.31366	92	-0.24196	01					0.11 /6	E-03	0.44436	00
0.1500E 02 -0.207TE 01	0.6042F	01	-0.1372E	01	0.7412E -0.6294E	00	0.5364E -0.5310E	00	0.1049	E-01	0.9823E	00
0.250.66 02 -0.230.76 01 -0.37176 00 -0.59328 00 0.220.76 03 0.490.76 00 0.330.77 00 0.330	0.9256E 0.1243E	20	-0.1968E	91	0.6043E -0.5931E	00	0.5390E -0.5441E	00	0.3446	E-02	0.9917E	00
0.250.66 02 -0.230.76 01 -0.37176 00 -0.59328 00 0.220.76 03 0.490.76 00 0.330.77 00 0.330	0.1876E	02	-0.2077E	01	0.5868F -0.5827E	00	0.5475E -0.5499E	00	0.5123	E-03	0.9978E	00
C.2645E 01 -0.1377E 01	0.25068	02	-0.2308E	01	~0.5777E	00	-0.5532E	00	0.2314	E-03	0.9988E	00
0.4033E 01 -0.1639E 01 -0.639E 00 -0.5269E 00 0.1161E-00 0.949E 00 0.1161E-00 0.949E 00 0.1166E-00 0.949E 00 0.166E-00 0.949E 00 0.949E 00 0.166E-	0.3136E	02	-0.2419E	01	-0.5749E				0.1249	E-03	0.9992E	00
0.124.8	0.2645E	01	-0.1377E	01	0.7591E	00	0.5262E	00	0.8266	E-01	0.94726	
0.1500E 02 -0.2077E 01	0-1243F	02	-0.1829E	01	-0.5955E	00	0.5361E -0.5419F	00	0.3671	E-02 E-02	0.9911E	00
0.250ec 02 -0.230ec 01 -0.5776e 00 -0.5521E 00 0.2405E-03 0.998TE 00 0.281E 02 -0.2307c 01 0.57776 00 0.5538 00 0.130ec 03 0.999TE 00 0.33356 02 -0.2419C 01 -0.5756c 00 -0.5549E 00 0.130ec 03 0.999TE 00 0.33356 02 -0.2419C 01 -0.5756c 00 -0.5549E 00 0.130ec 03 0.999TE 00 0.130ec 03 0.999TE 00 0.130ec 03 0.999TE 00 0.150ec 03 0.999TE 00 0.9	0-1560F	02	-0.2077E	01	0.5887E	00	0.5458E	00	0.9023	E-03	0.9967E 0.9977E	00
0.2821E 02 -0,2337E 01	0.2191E 0.2506E	02	-0.2242E	01 01	0.5812E -0.5789E	00	0.5505E -0.5521E	00	0.2465	E-03	0.9987E	00
0.2617E 01 -0.1381E 01 0.77746 00 0.5146E 00 0.8993E-01 0.9445E 00 0.900E 00 0.9244E 01 -0.1639E 01 0.6602E 00 -0.5218E 00 0.3995E-02 0.9800E 00 0.9244E 01 -0.1829E 01 0.6110E 00 0.5331E 00 0.3995E-02 0.9800E 00 0.900E 00 0.1747E 02 -0.1948E 01 0.9800E 00 0.9997E 00 0.1800E-02 0.9996E 00 0.1800E 00 0.9996E 00 0.1800E 00 0.9996E 00 0.1800E 00 0.9996E 00 0.1800E 00 0.9996E 00 0.2100E 02 -0.2242E 01 0.5252E 00 0.9996E 00 0.5775E-03 0.9996E 00 0.2206E 02 -0.2308E 01 -0.5801E 00 -0.5710E 00 0.3776E-03 0.9996E 00 0.2800E 02 -0.2308E 01 -0.5801E 00 -0.5510E 00 0.2507E-03 0.9996E 00 0.2800E 02 -0.2419E 01 -0.5782E 00 0.5252E 00 0.3767E-03 0.9998E 00 0.3135E 02 -0.2419E 01 -0.5782E 00 -0.5533E 00 0.1886E-03 0.9998E 00 0.3135E 02 -0.2419E 01 -0.5782E 00 -0.5533E 00 0.1305E-03 0.9998E 00 0.3135E 02 -0.2419E 01 -0.5555E 00 -0.5533E 00 0.1305E-03 0.9998E 00 0.9998E 00 0.9259E 01 -0.1839E 01 -0.5555E 00 -0.5533E 00 0.1207E-01 0.9788E 00 0.9998E 00 0.9259E 01 -0.1839E 01 -0.5555E 00 -0.5530E 00 0.4121E-02 0.9998E 00 0.9259E 01 -0.1839E 01 -0.5555E 00 -0.5530E 00 0.4121E-02 0.9998E 00 0.9259E 01 -0.1839E 01 -0.5799E 00 0.5907E 00 0.9998E 00 0.9998E 00 0.9259E 00 0.9998E 00 0	0.2821E 0.3135E	02 02	-0.2367F	01	0.5777E	00 00	0.5533E -0.5543E	00	0.1780	E-03	0.9990E	
0.1399E 02 -0.201E 01 -0.599E 00 -0.599E 00 -0.379E 03 -0.399EE 00 -0.250E 02 -0.2242E 01 -0.5252E 00 -0.5510E 00 -0.379E-03 0.399EE 00 -0.250E 02 -0.230E 01 -0.550E 00 -0.5510E 00 -0.5510E 00 -0.379E-03 0.399EE 00 -0.3135E 02 -0.2419E 01 -0.578E 00 -0.553E 00 -0.553E 00 -0.1315E-03 0.999EE 00 -0.3135E 02 -0.2419E 01 -0.578E 00 -0.553E 00 -0.1315E-03 0.999EE 00 -0.533E 02 -0.2419E 01 -0.578E 00 -0.553E 00 -0.1315E-03 0.999EE 00 -0.533E 00 -0.1315E-03 0.999EE 00 -0.533E 00 -0.533E 00 -0.130E-03 0.999EE 00 -0.535E 00 -0.535E 00 -0.535E 00 -0.535E 00 -0.545E-00 0.999EE 00 -0.535E 00 -0.535E 00 -0.535E 00 -0.545E-00 0.999EE 00 -0.535E 00 -0.535E 00 -0.545E-03 0.999EE 00 -0.555E 00 -0.535E 00 -0.555E 00 -0.535E 00 -0.535E 00 -0.545E-03 0.999EE 00 -0.555E 00 -0.555E 00 -0.545E-03 0.999EE 00 -0.555E 00 -0.	0.2617E	01	-0.1381E	01	0.77746	**2= 00	1.8 0.5148E	00	0.8983	E-01	0.9445E	00
0.1399E 02 -0.201E 01 -0.599E 00 -0.599E 00 -0.379E 03 -0.399EE 00 -0.250E 02 -0.2242E 01 -0.5252E 00 -0.5510E 00 -0.379E-03 0.399EE 00 -0.250E 02 -0.230E 01 -0.550E 00 -0.5510E 00 -0.5510E 00 -0.379E-03 0.399EE 00 -0.3135E 02 -0.2419E 01 -0.578E 00 -0.553E 00 -0.553E 00 -0.1315E-03 0.999EE 00 -0.3135E 02 -0.2419E 01 -0.578E 00 -0.553E 00 -0.1315E-03 0.999EE 00 -0.533E 02 -0.2419E 01 -0.578E 00 -0.553E 00 -0.1315E-03 0.999EE 00 -0.533E 00 -0.1315E-03 0.999EE 00 -0.533E 00 -0.533E 00 -0.130E-03 0.999EE 00 -0.535E 00 -0.535E 00 -0.535E 00 -0.535E 00 -0.545E-00 0.999EE 00 -0.535E 00 -0.535E 00 -0.535E 00 -0.545E-00 0.999EE 00 -0.535E 00 -0.535E 00 -0.545E-03 0.999EE 00 -0.555E 00 -0.535E 00 -0.555E 00 -0.535E 00 -0.535E 00 -0.545E-03 0.999EE 00 -0.555E 00 -0.555E 00 -0.545E-03 0.999EE 00 -0.555E 00 -0.	0.6024E	01	-0.1639E	01	-0.6402E 0.6110E	00	-0.5218F 0.5331E	00	0.1194	E-01	0.9800E 0.9906E	00
0.250cE 02 -0.2308E 01 -0.5501E 00 -0.5510E 00 0.2607E-03 0.9986E 00 0.2801E 02 -0.2307E 01 0.5782E 00 0.5253E 00 0.1886E-03 0.9986E 00 0.3135E 02 -0.2419E 01 -0.5786E 00 -0.5533E 00 0.1886E-03 0.9986E 00 0.3135E 02 -0.2419E 01 0.7960E 00 0.5902E 00 0.1315E-03 0.9981E 00 0.06015E 01 -0.1639E 01 -0.5555E 00 -0.5159E 00 0.1207E-01 0.9788E 00 0.9239E 01 -0.1829E 01 0.6142E 00 0.5301E 00 0.4121E-02 0.9901E 00 0.1207E 02 -0.1207E 01 0.5555E 00 0.5301E 00 0.4121E-02 0.9901E 00 0.1207E-01 0.9788E 00 0.1207E-01 0.9788E 00 0.1207E-01 0.9788E 00 0.1207E-01 0.9788E 00 0.1207E-01 0.9991E 00 0.9991E	0.1243E	02 02	-0.1968E	01 01	-0.5980E				0.9550	F-03	0.9946E	00
0.2821E 02 -0,2438T 01	0.2190E	02	-0.2242E	Oι	0.5825E	00	0.54936	00	0.2607	F-03	0.9982E	00
0.2558E 01 -0.1355E 01 0.7900E 00 0.5020E 00 0.9742E-01 0.9419E 00 0.0015E 01 -0.1459E 01 -0.4499E 00 0.5020E 00 0.1257E-01 0.9419E 00 0.0253E 01 0.01252E 01 0.4499E 00 0.1559E 00 0.1257E-01 0.9708E 00 0.1257E-01 0.9708E 00 0.1257E 01 0.01262E 01 0.01262E 01 0.01359E 00 0.1257E-01 0.9901E 00 0.1559E 02 -0.207TE 01 0.5252E 00 0.2523E 00 0.100E-02 0.9943E 00 0.1559E 02 -0.277E 01 0.5252E 00 0.5457E 00 0.100E-02 0.9945E 00 0.21900E 02 -0.2242E 01 0.5898E 00 0.4557E 00 0.4127E-03 0.9951E 00 0.2505E 02 -0.2308E 01 -0.5812E 00 -0.5496E 00 0.2697E-03 0.9951E 00 0.2505E 02 -0.2308E 01 -0.5812E 00 -0.5496E 00 0.2697E-03 0.9951E 00 0.2700E 02 -0.2337E 01 -0.5776E 00 -0.5525E 00 0.1470E-03 0.9951E 00 0.33135E 02 -0.2419E 01 -0.5776E 00 -0.5525E 00 0.1470E-03 0.9968E 00 0.2557E 01 -0.1397E 01 0.8149E 00 0.5525E 00 0.1470E-03 0.9968E 00 0.2557E 01 -0.1397E 01 0.8149E 00 0.5525E 00 0.1470E-03 0.9998E 00 0.2557E 01 -0.1397E 01 0.8149E 00 0.4876E 00 0.1557E 01 -0.4386E 01 0.05776E 00 0.4876E 00 0.1457E-03 0.9998E 00 0.9776E 00 0.97334E 01 0.4596E 01 0.05776E 00 0.55276E 00 0.1457E-03 0.9998E 00 0.9776E 00 0.97334E 01 0.9776E 00 0.9735E 01 0.9776E 00 0.9736E 00 0.9736E 00 0.9740E 00 0.9736E 00 0.9740E 00 0.9736E 00 0.9740E 00 0.9	0.2821E	02	-0.2367E	01	0.5782E	00	0.5523E	00	0.1884	E-03	0.9989E	00
0.9239E 01 -0.1827E 01 0.6142E 00 0.5301E 00 0.4121E-02 0.9901E 00 0.1222E 02 -0.1988E 01 -0.6004E 00 -0.5375E 00 0.1807E-02 0.9949E 00 0.1859E 02 -0.1247E 01 0.5952E 00 0.3425E 00 0.187E-03 0.9949E 00 0.1859E 00 0.2400E 02 -0.2247E 01 0.5838E 00 0.4800E 00 0.397E-03 0.998E 00 0.2800E 02 -0.2337E 01 0.5952E 00 -0.5952E 00 0.127E-03 0.995E 00 0.2800E 02 -0.2337E 01 0.5797E 00 0.5535E 00 0.1207E-03 0.998E 00 0.2800E 02 -0.2337E 01 0.5797E 00 0.5535E 00 0.1207E-03 0.998E 00 0.2800E 02 -0.2337E 01 -0.579E 00 0.5535E 00 0.1207E-03 0.998E 00 0.2800E 01 -0.1809E 01 0.8189E 00 0.5555E 00 0.1409E-03 0.999BE 00 0.2800E 01 -0.1809E 01 0.8189E 00 0.5555E 00 0.1409E-03 0.999BE 00 0.2800E 01 -0.1809E 01 -0.507E 00 -0.5535E 00 0.1409E-03 0.999BE 00 0.2555E 01 -0.1809E 01 -0.507E 00 0.9939E 00 0.1409E-03 0.999BE 00 0.9233E 01 -0.1809E 01 -0.507E 00 0.9353E 00 0.1809E-02 -0.9490E 00 0.1809E 02 -0.207EE 01 -0.509E 00 0.9395E 00 0.1809E-02 -0.9490E 00 0.509E 00 0.9395E 00 0.9	0.2500=	01		01	0.79605	00	0.5020E	00	0.9749	E-01	0.94195	00
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	 0.0135E	02	-0.2419E	01	-0.9103E		-0.55105		V. 1523		30,7700	

[†] Note that β^* , $-\beta$ and $-\beta^*$ are also solutions of the equation. ‡ The parameter K depends only on Poisson's ratio for the material: $K^2 = 2(1 - \nu)$.

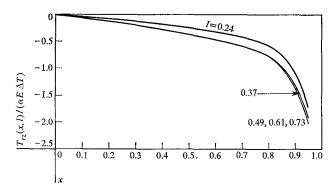


Figure 1 Shear stress on the constrained surface.

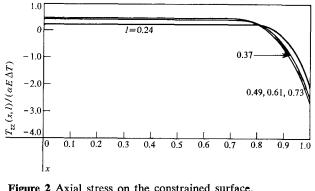


Figure 2 Axial stress on the constrained surface.

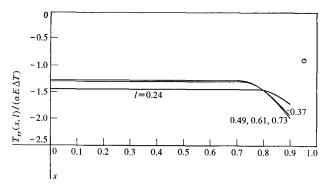


Figure 3 Radial stress on the constrained surface. The encircled point at x = 0.95 is the computed value of T_{rr} for N=10; the exact solution should tend to $-\infty$ as $x\to 1$.

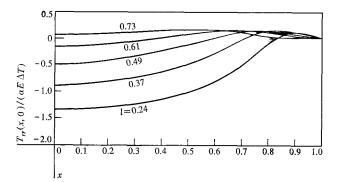


Figure 4 Radial stress on the free end surface.

The $\eta(x,\zeta)$ series is very slowly convergent near x=1, $\zeta = l$ (i.e., near the circumference on the constrained surface) and many more terms would have to be included to improve substantially the results in this neighborhood. It appears, however, that the stress T_{zz} at x = 1 and $\zeta = l$ will tend to $-\infty$ as the number of terms becomes sufficiently large (see Fig. 6). Since only $\partial u/\partial z$ and $\partial w/\partial z$ are non-zero on the constrained surface, it follows that $T_{rr}(1, l)$ must also tend to $-\infty$, although the computed values of T_{rr} show no clear indication of this behavior for N less than 10 (see Fig. 3). A similar study of the maximum shear stress on the constrained surface, though more ambiguous, also suggests that $T_{rz}(x, l) \rightarrow -\infty$ as $x \rightarrow 1$ if a sufficiently large number of terms is included. (The presence of stress concentration points at the constrained edges seems to be a general feature of problems of this type. 1,2) Because of the lack of orthogonality between the various functions in Eqs. (21), it is not obvious how the convergence can be improved.

In the limit $l \equiv h/R \gg 1$, all stresses vanish on the free surface (z = 0) and the stresses on the constrained surface (z = l) are much the same as shown in Figs. 1 to 3 for

l = 0.73. The axial stress $T_{zz}(1, \zeta)$ is essentially zero unless z is closer than about R to the constrained end (i.e., $\zeta < 1$), where it will behave approximately as shown in Fig. 5. The effects of the constraints, in other words, are negligibly small beyond an axial distance R from the constrained end.

The error ϵ_N , defined in Eq. (23), is shown as a function of N in Fig. 7. This error decreases approximately as $N^{-3/2}$, suggesting that the solution is asymptotically exact as $N \to \infty$. Values of $T_{zz}(x, 0)$, $T_{rz}(x, 0)$, w(x, l) and u(x, l)for N = 10 and l = 0.73 are shown in Fig. 8. These quantities should be zero when $N = \infty$, so the figure gives some indication of the convergence that can be obtained with 20 terms in the series (corresponding to 10 roots and their conjugates).

Appendix

The matrices U, V, W and X were introduced in Eqs. (25). Each of these matrices is defined as an integral over x of the functions obtained by squaring each of Eqs. (21); U arises from the square of Eq. (21d), V from (21c), W from (21a) and X from (21b).

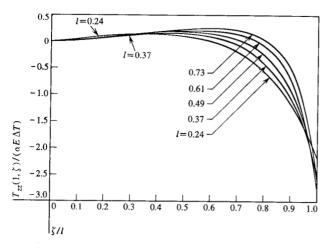


Figure 5 Axial stress on the circumference.

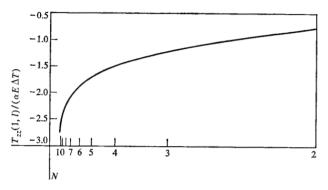


Figure 6 Axial stress at the junction line between cylinder and base as a function of the number of terms in the η series; l=0.65.

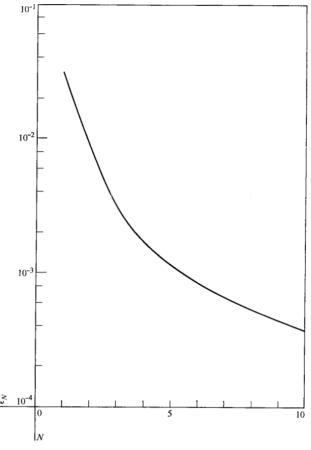
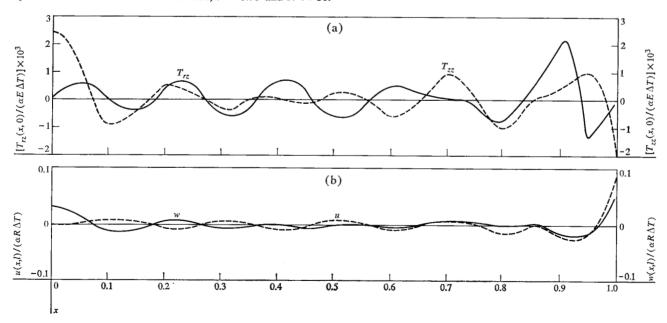


Figure 7 Least-squares error as a function of the number of terms in the η series; l=0.65.

Figure 8 Radial dependence of (a) the shear and axial stresses on the free end surface and (b) the radial and axial displacements on the constrained surface; l = 0.73 and N = 10.



If we define

$$I_{kl}^{(p)}(ss') = \int_0^1 x^p J_k(\beta_s x) J_l(\beta_{s'} x) dx, \qquad (A1)$$

we find

$$U_{ss'} = U_{s's} = I_{00}^{(3)}(ss')$$

$$-\left(\frac{K^2}{\beta_s} + \xi_s\right) I_{10}^{(2)}(ss') - \left(\frac{K^2}{\beta_{s'}} + \xi_{s'}\right) I_{10}^{(2)}(s's)$$

$$+ \left(\frac{K^2}{\beta_s} + \xi_s\right) \left(\frac{K^2}{\beta_{s'}} + \xi_{s'}\right) I_{11}^{(1)}(ss'), \tag{A2}$$

$$V_{ss'} = V_{s's} = I_{11}^{(3)}(ss')$$

$$-\left(\frac{K^2}{\beta_s} - \xi_s\right)I_{01}^{(2)}(ss') - \left(\frac{K^2}{\beta_{s'}} - \xi_{s'}\right)I_{01}^{(2)}(s's)$$

$$+\left(\frac{K^2}{\beta_s} - \xi_s\right)\left(\frac{K^2}{\beta_{s'}} - \xi_{s'}\right)I_{00}^{(1)}(ss')$$

$$-\frac{2(2 - K^2)^2}{\beta_s^2\beta_{s'}^2} J_1(\beta_s)J_1(\beta_{s'}), \tag{A3}$$

$$W_{ss'} = W_{s's} = I_{11}^{(3)}(ss')$$

$$-\left(\frac{2}{\beta s} - \xi_{s}\right) I_{01}^{(2)}(ss') - \left(\frac{2}{\beta_{s'}} - \xi_{s'}\right) I_{01}^{(2)}(s's)$$

$$+ \left(\frac{2}{\beta_{s}} - \xi_{s}\right) \left(\frac{2}{\beta_{s'}} - \xi_{s'}\right) I_{00}^{(1)}(ss') \text{ and } (A4)$$

$$X_{\bullet\bullet'} = X_{\bullet'\bullet} = I_{00}^{(3)}(ss') - \xi_{\bullet} I_{10}^{(1)}(ss') - \xi_{\bullet'} I_{10}^{(2)}(s') + \xi_{\bullet} \xi_{\bullet'} I_{11}^{(1)}(ss').$$
 (A5)

The quantities $I_{kl}^{(p)}$ can all be obtained by direct integration and one finds

$$I_{00}^{(1)}(ss') = J_{1}(\beta_{s})J_{1}(\beta_{s'})(\beta_{s}\xi_{s'} - \beta_{s'}\xi_{s})/(\beta_{s}^{2} - \beta_{s'}^{2}),$$

$$I_{11}^{(1)}(ss') = J_{1}(\beta_{s})J_{1}(\beta_{s'})(\beta_{s'}\xi_{s'} - \beta_{s}\xi_{s})/(\beta_{s}^{2} - \beta_{s'}^{2}),$$

$$I_{10}^{(2)}(ss') = I_{01}^{(2)}(s's) = [2\beta_{\bullet}I_{00}^{(1)}(ss') - J_{1}(\beta_{\bullet})J_{1}(\beta_{\bullet'})(\beta_{\bullet'} + \beta_{\bullet}\xi_{\bullet}\xi_{\bullet'})]/(\beta_{\bullet}^{2} - \beta_{\bullet'}^{2}),$$

$$I_{00}^{(3)}(ss') = I_{00}^{(1)}(ss') + 2[\beta_{\bullet'}I_{01}^{(2)}(ss') - \beta_{\bullet}I_{10}^{(2)}(ss')]/(\beta_{\bullet}^{2} - \beta_{\bullet'}^{2})$$

and

$$I_{11}^{(3)}(ss') = I_{11}^{(1)}(ss') + 2[\beta_s I_{01}^{(2)}(ss') - \beta_{s'} I_{10}^{(2)}(ss')]/(\beta_s^2 - \beta_{s'}^2).$$

Thus the matrices U, V, W and X can be constructed once a tabulation of β_* , ζ_* and $J_1(\beta_*)$ is available. These matrices depend only on the parameter K^2 and are independent of the dimensions of the cylinder.

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References

- 1. B. J. Aleck, J. Appl. Mech. 16, 118 (1949).
- P. S. Theocaris and K. Dafermos, J. Appl. Mech. 31, 714 (1964).
- A. I. Lur'e, Three-Dimensional Problems of the Theory of Elasticity, Interscience Publishers, New York 1964, Chapter 7.
- A. E. H. Love, The Mathematical Theory of Elasticity, Dover Publications, New York 1944, esp. Sections 74 and 188.
- P. J. Davis, Interpolation and Approximation, Blaisdell Publishing Co., New York 1963, p. 24.
- F. W. J. Olver, "Bessel Functions of Integer Order,"
 Handbook of Mathematical Functions, edited by M.
 Abramowitz and I. A. Stegun, Number 55, Applied
 Mathematics Series, National Bureau of Standards,
 Washington, D. C. 1964, Chapter 9.

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