

Algebraically Generalized Recursive Function Theory*

Abstract: The Uniformly Reflexive Structure (URS) introduced by E. G. Wagner is, for this paper, a nonassociative algebra consisting of a domain and a binary operation satisfying the following axioms:

0. $(\exists *) (\forall a) [a \cdot * = * \cdot a = *];$
1. $(\exists \psi) (\forall a, b, c, d) [\psi \neq * \ \& \ ((a \neq * \ \& \ b \neq * \ \& \ c \neq * \ \& \ d \neq *) \rightarrow ((a = d \ \& \ (((\psi \cdot a) \cdot b) \cdot c) \cdot d = b) \text{ or } (a \neq d \ \& \ (((\psi \cdot a) \cdot b) \cdot c) \cdot d = c)))];$ and
2. $(\exists \alpha) (\forall b, c, d) [\alpha \neq \psi \ \& \ ((b \neq * \ \& \ c \neq * \ \& \ d \neq *) \rightarrow ((\alpha \cdot b) \cdot c \neq * \ \& \ ((\alpha \cdot b) \cdot c) \cdot d = (b \cdot d) \cdot (c \cdot d))].$

Wagner showed that these structures generalize much of Recursive Function Theory (RFT).

In this paper the functions "computed" by a URS are the functions given by left multiplications by elements of the URS. A family of functions is said to form a URS if it is the family of left multiplications of some URS. Axioms for Basic Recursive Function Theory are given characterizing those families of functions which form URS's. The Partial Metarecursive Functions and the Computable Functionals of McCarthy are shown to form URS's.

An investigation of notions analogous to the "recursively enumerable" notion in RFT shows that if any splinter ("successor set") of a URS is semicomputable, then all are. A partial analogue to the Rice-Myhill-Shapiro Theorem is proved for URS's satisfying an axiom corresponding to Kleene's "indefinite description." Finally, a study of pairing functions leads to work analogous to Rogers' on Gödel numberings and generalizes similar work of Wagner.

Introduction

This paper is an investigation of algebraic structures satisfying a set of axioms for a generalized theory of recursive functions. The axiom system was introduced by Wagner¹ who conceived of these structures as pairs, $\langle U, G \rangle$, such that U is an "arbitrary infinite domain" and G is a map from U to a family of functions from U into U . He called the pairs, $\langle U, G \rangle$, satisfying the axioms Uniformly Reflexive Structures (URS's) and showed that the partial A -recursive functions (for any subset A of the natural numbers) could be made to form a URS.

The axioms presented in this paper have been rewritten so that they are satisfied by nonassociative algebras consisting of a domain, U , and a binary operation, $\cdot$. The relationship between the binary operation, $\cdot$, and the indexing, G , is $u \cdot v = (G(u))(v)$; so the set of functions involved becomes the set of left multiplications of the algebra. Thus there is a natural one-to-one correspondence between Wagner's URS's and the algebras satisfying the axioms of this paper. Hence these algebras will be called URS's.

The difference between the axioms presented at the beginning of the next section and those of Ref. 1 is one of notation (except for the one point discussed after their presentation). Once the problem is overcome of conceiving of the notation, x , as representing both an element of a URS and the function, ϕ_x , denoting multiplication on the left by this element, the notation shortens several of the proofs of theorems from Recursive Function Theory (RFT).

In Section 1 some of the basic results of the theory of recursive functions are obtained for URS's, paralleling those of Ref. 1. In Section 2 necessary and sufficient conditions are given for a family of functions to be associated with a URS (as the set of left multiplications). These conditions are stated in terms of properties of partial recursive functions which are more familiar than the "blend" operation of Wagner's axioms. They are given in the form of axioms for Basic Recursive Function Theory (BRFT).

Often in the literature RFT has been informally axiomatized by means of the Enumeration and S-m-n properties (see Section 2). Usually the axioms have described the Gödel numbering while properties of the family of partial recursive functions, such as closure under composition, have been taken for granted. Basic RFT combines the usual

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axioms of Gödel numberings with a short list of additional properties of the family of partial recursive functions in an attempt to axiomatize RFT without explicitly identifying the domain of the family of functions or postulating a successor function which exhausts that domain.

It is convenient to incorporate the property of closure under "tests for equality with conditional branching" into the axioms of BRFT in a form paralleling Axiom 1 of the theory of URS's. However, BRFT is not a rewriting of the axioms for a URS. The blend operation and the Enumeration and S-m-n properties are dissimilar in both form and origin. In various approaches to generalizing RFT, most of the work has been done to show that the approach satisfies BRFT. Thus the theorem of Section 2, that BRFT characterizes the families of functions which form URS's, easily links such generalizations as the Partial Metarecursive Functions and the Computable Functionals² with the theory of URS's.

Results similar to those of RFT are pursued in the third section, which ends with a partial analogue to the Rice-Myhill-Shapiro Theorem.³ Section 4 continues exploring the nature of families of functions forming URS's. Some results with particular emphasis on pairing functions are obtained, similar to those of Wagner⁴ on Gödelizations.

Throughout this paper, when not otherwise determined by context, the following notational conventions have been observed:

- (1) The symbol $*$ denotes a particular distinguished element of a URS (see next section);
- (2) lower case Greek letters (without subscripts) denote important elements of URS's which are used in the same way in every section, e.g., $\psi, \alpha, \beta, \iota, \mu, \phi, \delta$;
- (3) m and n denote natural numbers;
- (4) f, g, h, j , and k denote functions;
- (5) other lower case italic letters denote arbitrary non- $*$ elements of URS's and are used as variables;
- (6) I and J denote URS's;
- (7) N denotes the natural numbers;
- (8) F, G, H , and K denote families of functions;
- (9) other capital italic letters denote sets and families of sets; and
- (10) $\square$ indicates the end of a proof.

1. Axioms, definitions, and preliminary results

The algebra, $I = \langle A, \cdot \rangle$, with domain, A , and binary operation, $\cdot$, is a URS if it satisfies the following axioms:

0. $(\exists *) (\forall a) [a \cdot * = * \cdot a = *]$;
1. $(\exists \psi) (\forall a, b, c, d) [\psi \neq * \& ((a \neq * \& b \neq * \& c \neq * \& d \neq *) \rightarrow ((a = d \& (((\psi \cdot a) \cdot b) \cdot c) \cdot d = b) \text{ or } (a \neq d \& (((\psi \cdot a) \cdot b) \cdot c) \cdot d = c)))]$; and
2. $(\exists \alpha) (\forall b, c, d) [\alpha \neq \psi \& ((b \neq * \& c \neq * \& d \neq *) \rightarrow ((\alpha \cdot b) \cdot c \neq * \& ((\alpha \cdot b) \cdot c) \cdot d = (b \cdot d) \cdot (c \cdot d))]$.

(In order to axiomatize the first-order theory of URS's without constants, the three axioms are conjoined into one with the three initial existential quantifiers moved to the front.) Note that Axiom 2 is stronger than Wagner's axiom for α since it stipulates $\alpha \neq \psi$, providing immediately two distinct elements of $A - \{*\}$. Wagner¹ has shown that the existence of at least two distinct elements of $A - \{*\}$ eliminates all finite models. Conversely, it can be shown that if $\alpha = \psi$ replaces $\alpha \neq \psi$ in Axiom 2, there must be exactly one element in $A - \{*\}$.

In an algebra satisfying Axioms 0, 1, and 2, the element $*$ is unique. In the rest of this paper the non- $*$ elements of a URS, rather than the whole domain, will be named explicitly: the notation $I = \langle A \cup \{*\}, \cdot \rangle$ will mean that the domain of the algebra I is $A \cup \{*\}$ and $*$ is not an element of A . The binary operation $\cdot$ will, of course, be a function from $(A \cup \{*\}) \times (A \cup \{*\})$ to $A \cup \{*\}$.

If e is an element of the URS $I = \langle A \cup \{*\}, \cdot \rangle$, then e is an index of the partial function ϕ_e from A into A whose value at x is $e \cdot x$ ($*$ is equated with "undefined"). The set of those partial functions (of one variable) indexed by elements of A is denoted $F_1(I)$. The element e also indexes partial functions of several variables: $\phi_e^n: A^n \rightarrow A$ is given by $\phi_e^n(x_1, \dots, x_n) = ((\dots ((e \cdot x_1) \cdot x_2) \dots) \cdot x_{n-1}) \cdot x_n$.

The parentheses quickly get out of hand; so, since the operation is most often performed from the left, i.e., in the order $(a \cdot b) \cdot c$, the expression $(a \cdot b) \cdot c$ is abbreviated abc ; while $a \cdot (b \cdot c)$ is abbreviated $a(bc)$. With the convention that lower case letters represent elements of the URS not equal to $*$, the axioms become:

0. $*a = a* = ** = *$;
1. $\psi \neq * \& \psi abcd = \begin{cases} b & \text{if } a = d; \\ c & \text{if } a \neq d; \end{cases}$
2. $\alpha \neq \psi \& \alpha ab \neq * \& \alpha abc = ac(bc)$.

The set of partial functions of n arguments indexed by elements of A (in the URS $I = \langle A \cup \{*\}, \cdot \rangle$) is denoted $F_n(I)$, and

$$F(I) = \bigcup_{n=1}^{\infty} F_n(I)$$

is called the family of partial functions computed by I . Let N be the natural numbers and let the map $n \rightarrow \phi_n$ be a Gödel numbering of the partial recursive functions of one variable. Define $a \cdot b = \phi_a(b)$ if $\phi_a(b)$ is defined and let $*$ represent "undefined" ($* \cdot a = a \cdot * = *$). Wagner¹ has shown that $I = \langle N \cup \{*\}, \cdot \rangle$ is a URS; and, of course, $F(I)$ is the family of partial recursive functions. In Section 2 a proof similar to Wagner's will be used to characterize the families $F(I)$ of partial functions computed by URS's.

Following Wagner¹ two useful constants are singled out besides α and ψ in order to prove a general existence meta-theorem for functions in any URS (i.e., computed by any URS). Let $\iota = \alpha(\psi\alpha\psi)(\psi\psi\psi\psi)$. Then for each a , $\iota a = a$,

so ι indexes the identity function. Let $\beta = \alpha(\psi\psi)\iota$. Then for each a and b , $\beta ab = a$; so for each a , βa indexes the constant function with value a . The next result makes possible the construction of many useful functions without explicitly writing indices as products of α and ψ .

Let $\{x_i\}$ be a collection of variables which range over all non- $*$ elements. Define a *term* as follows: α, ψ , and any x_i are terms, and if t_1 and t_2 are terms, so is $(t_1 \cdot t_2)$.

Theorem 1.1

(General Existence Metatheorem) For each term $t(x_1, \dots, x_n)$, with at most $x_1, \dots, x_n$ as variables, there is a term $t'(x_1, \dots, x_{n-1})$, with at most $x_1, \dots, x_{n-1}$ as variables, such that $(\forall x_1, \dots, x_n)[t'(x_1, \dots, x_{n-1}) \neq * \& t'(x_1, \dots, x_{n-1}) \cdot x_n = t(x_1, \dots, x_n)]$ is a consequence of the axioms for a URS.

Proof of Theorem 1.1

The proof is by induction on the number of parentheses in $t(x_1, \dots, x_n)$. Its outline follows: $x_n = \iota \cdot x_n$; if t_0 is a term with no occurrence of x_n , $t_0 = (\beta \cdot t_0) \cdot x_n$, also

$$((t_1 \cdot x_n) \cdot (t_2 \cdot x_n)) = ((\alpha \cdot t_1) \cdot t_2) \cdot x_n.$$

Note that if t_0, t_1 , and t_2 are not equal to $*$, then βt_0 and $\alpha t_1 t_2$ are not equal to $*$. Using these facts, one can easily construct the desired $t'(x_1, \dots, x_{n-1})$ and show that, by induction, it is not equal to $*$. $\square$

It is a good exercise to check that if $t(x_1, x_2) = (((\alpha \cdot (\psi \cdot x_1)) \cdot x_2) \cdot x_1)$ —abbreviated $\alpha(\psi x_1)x_2x_1$ —then the $t'(x_1)$ constructed is abbreviated $\alpha(\alpha(\alpha(\beta\alpha))(\alpha(\beta\psi)(\beta x_1)))\iota(\beta x_1)$. In this construction ψ was replaced by $((\beta \cdot \psi) \cdot x_2$; α , by $((\beta \cdot \alpha) \cdot x_2)$; etc.

The following example shows the strength of 1.1 in exhibiting functions computable in a URS: let ϕ_a^2, ϕ_b^n , and ϕ_c^n be partial functions in $F(I)$ for some URS I . An index for the function formed from these functions by composition ($f(x_1, \dots, x_n) = \phi_a^2(\phi_b^n(x_1, \dots, x_n), \phi_c^n(x_1, \dots, x_n))$) can be obtained by n applications of 1.1: there is a term d , depending only on α, ψ, a, b , and c such that $d \neq *$ and

$$dx_1 \dots x_n = a(bx_1 \dots x_n)(cx_1 \dots x_n).$$

Theorem 1.1 is stronger than Theorem 2.11 of Ref. 1 since it contains the result $t'(x_1, \dots, x_{n-1}) \neq *$. Otherwise these theorems are used in the same way. The notation of this paper allows a slightly simpler proof. Wagner postpones the rest of the strength of 1.1 to the level of the Iteration Theorem, his Theorem 2.15.¹

Corollary 1.2

(Iteration or S-m-n Theorem) For all natural numbers $m \geq 0$ and $n \geq 1$, there is a term S_n^m involving only α and ψ such that, for all $x_0, x_1, \dots, x_m, y_1, \dots, y_n$, $S_n^m x_0 \dots x_m \neq *$ and $S_n^m x_0 \dots x_m y_1 \dots y_n = x_0 \dots y_n$. Moreover, S_1^m can be used for any S_n^m .

Proof of Corollary 1.2

By 1.1 there is a term t involving only $\alpha, \psi, x_0, \dots$, and x_m such that $t \neq *$ and $ty_1 = x_0 \dots x_m y_1$. By m applications of 1.1, there is a term S_1^m involving only α and ψ such that $S_1^m x_0 \dots x_m = t$; so $S_1^m x_0 \dots x_m \neq *$ and $S_1^m x_0 \dots x_m y_n = x_0 \dots y_n$. $\square$

From now on Theorem 1.1 will be used implicitly in proofs without being mentioned.

Theorem 1.3

(Definition by cases) (cf. 2.12 to 2.14 of Ref. 1) There is an η such that $\eta abcdx = cx$ if $bx = a$, and $\eta abcdx = dx$ if $bx \neq a$ and $bx \neq *$.

Proof of Theorem 1.3

Let η be such that $\eta abcdx = \psi acd(bx)x$. Then

$$\eta abcdx = \begin{cases} cx & \text{if } bx = a; \\ dx & \text{if } bx \neq a, *. \end{cases} \quad \square$$

The next theorem is an inductively proved generalization of 1.3. It is more general than, but in the same spirit as, Wagner's 2.14.¹

Theorem 1.4

For each pair (m, n) of natural numbers ≥ 1 , there is a term η_n^m involving only α and ψ such that, for all $a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n, x_1, \dots, x_m$, $\eta_n^m a_1 \dots x_{m-1} \neq *$ and

$$\eta_n^m a_1 \dots x_m = \begin{cases} c_1 x_1 \dots x_m & \text{if } b_1 x_1 \dots x_m = a_1; \\ c_2 x_1 \dots x_m & \text{if } b_2 x_1 \dots x_m = a_2 \text{ and } \\ & b_1 x_1 \dots x_m \neq a_1, *; \\ \vdots & \\ c_n x_1 \dots x_m & \text{if } b_n x_1 \dots x_m = a_n \text{ and } \\ & b_1 x_1 \dots x_m \neq a_1, *, \dots, \\ & b_{n-1} x_1 \dots x_m \neq a_{n-1}, *; \\ *, & \text{otherwise.} \end{cases}$$

Proof of Theorem 1.4

Let η_1^m be such that $\eta_1^m a_1 b_1 c_1 x_1 \dots x_{m-1} \neq *$ and $\eta_1^m a_1 \dots x_m = \psi a_1 c_1 \theta(b_1 x_1 \dots x_m) x_1 \dots x_m$. If η_n^m has been constructed to satisfy 1.4, let η_{n+1}^m be such that $\eta_{n+1}^m a_1 \dots a_{n+1} \dots x_{m-1} \neq *$ and $\eta_{n+1}^m a_1 \dots x_m = \psi a_1 c_1 (\eta_n^m a_2 \dots a_{n+1} \dots c_{n+1})(b_1 x_1 \dots x_m) x_1 \dots x_m$. By induction on n , the proof of 1.3 suffices for 1.4. $\square$

Define $[a] = \{\langle x, y \rangle \mid ax = y, x \neq *, y \neq *\}$, the *graph* of the function indexed by a . It is easy to construct (see 2.1 of Ref. 1) a $\theta \neq *$ such that $[\theta]$ is the empty set. The next theorem is a uniform version of Wagner's 2.16.¹

Theorem 1.5

(Uniform Recursion or Fixed Point Theorem) There is a ϕ such that, for each a , $\phi a \neq *$ and $[\phi a] = [a(\phi a)]$.

Proof of Theorem 1.5

Let w be such that, for all a, b , and x , $wab \neq *$ and $wabx = a(bb)x$. Let ϕ be such that, for each a , $\phi a = wa(wa)$. Then, for each a and x , $\phi a \neq *$ and $\phi ax = wa(wa)x = a(wa(wa))x = a(\phi a)x$. $\mathcal{A}$

Corollary 1.6

(Minimalization) (cf. 4.3 of Ref. 1) There is a μ such that, for all o, s , and p , if there is an element a of the sequence $\{o, so, s(so), \dots\}$ such that $pa = o$ and, for each b preceding a in the sequence, $pb \neq *$ and $pb \neq o$, then $\mu osp = a$.

Proof of Corollary 1.6

Let e be such that, for each t, w, o, s , and p ,

$$etwosp = \begin{cases} w & \text{if } pw = o; \\ t(sw)osp & \text{if } pw \neq o, *; \end{cases}$$

[A formal construction of e would proceed as follows: let b_1, b_2, c_1, c_2 , and e be such that $b_1twosp = pw$, $b_2twosp = o$, $c_1twosp = w$, $c_2twosp = t(sw)osp$, and $etwosp = \eta_2^5 oob_1b_2c_1c_2 twosp$.] Let μ be such that, for each o, s , and p , $\mu osp = \phi eosp$. Then

$$\mu osp = e(\phi e)osp = \begin{cases} o & \text{if } po = o; \\ \phi e(so)osp & \text{if } po \neq o, *; \text{ etc.} \end{cases} \quad \mathcal{A}$$

A predicate $P(x_1, \dots, x_n)$ with exactly $x_1, \dots, x_n$ as free variables is said to be *semicomputable* in a URS if there are elements a and b such that $P(x_1, \dots, x_n)$ if and only if $bx_1 \dots x_n = a$. Let $P_1, \dots, P_n$ be n predicates with $x_1, \dots, x_m$ as free variables which are semicomputable in I , and let $f_1, \dots, f_n$ be partial functions in $F_m(I)$. The notation $(P_1 \rightarrow f_1(x_1, \dots, x_m), \dots, P_n \rightarrow f_n(x_1, \dots, x_m))$ will be used for the function f which results from these predicates and functions by application of 1.4 (cf. Ref. 2). The value of f on $x_1, \dots, x_m$ is not $*$ if and only if, proceeding from left to right, a predicate P_k is found to hold before $*$ is introduced in checking the predicates and the corresponding function value $f_k(x_1, \dots, x_m) \neq *$. In this case $f(x_1, \dots, x_m) = f_k(x_1, \dots, x_m)$. Another notation for f will be

$$f(x_1, \dots, x_m) = \begin{cases} f_1(x_1, \dots, x_m) & \text{if } P_1; \\ \cdot & \\ \cdot & \\ \cdot & \\ f_n(x_1, \dots, x_m) & \text{if } P_n. \end{cases}$$

Note that " $f_k(x_1, \dots, x_m)$ if P_k " in this notation abbreviates " $f_k(x_1, \dots, x_m)$ if P_k and not $P_1, \dots$, and not P_{k-1} ."

2. Basic recursive function theory

Let $I = \langle A \cup \{*\}, \cdot \rangle$ be a URS. Recall that, for each integer $n > 0$, each element a of A indexes the partial function of n variables $\phi_a^n = \lambda x_1 \dots x_n [ax_1 \dots x_n]$. $F_n(I) = \{\phi_a^n | a \in A\}$ and $F(I) = \bigcup_n F_n(I)$.

Let A be a set with at least two elements. Suppose, for each $n > 0$, F_n is a family of partial functions from A^n into A and $F = \bigcup_n F_n$. The following axioms will be shown to characterize the families of partial functions on A computed by URS's.

• Axioms of basic recursive function theory (BRFT)

- (1) F contains (for each $x \in A$) the constant functions $(C_x^n(y_1, \dots, y_n) = x)$ and the projection functions $(U_n^m(x_1, \dots, x_m) = x_n)$;
- (2) $(\exists \Psi \in F_4)(\forall a, b, c, x \in A) [\Psi(a, b, c, x) = b \text{ if } x = a; c, \text{ otherwise}]$; and
- (3) F is closed under composition.
- (4) (Enumeration) For each integer $m > 0$, $(\exists \Phi_m \in F_{m+1}) [F_m = \{\lambda x_1 \dots x_m [\Phi_m(x, x_1, \dots, x_m)] | x \in A\}]$; and
- (5) (S-m-n) for each $m, n > 0$, $(\exists S_n^m \in F_{m+1}) (\forall x, x_1, \dots, x_m, y_1, \dots, y_n \in A) [S_n^m(x, x_1, \dots, x_m)$ is defined and $\Phi_n(S_n^m(x, x_1, \dots, x_m), y_1, \dots, y_n) = \Phi_{m+n}(x, x_1, \dots, x_m, y_1, \dots, y_n)]$.

Theorem 2.1

If A and F are as above, there is some $URS I = \langle A \cup \{*\}, \cdot \rangle$ such that $F = F(I)$ if and only if F satisfies the axioms of BRFT.

Proof of Theorem 2.1

$\Leftarrow$

Suppose $I = \langle A \cup \{*\}, \cdot \rangle$ is a URS with ψ, α , and ι as in Section 1, and $F = F(I)$. Then ϕ_ψ^4 satisfies (2), ϕ_ι^{m+1} satisfies (4), (5) is Corollary 1.2, and (1) and (3) are consequences of the General Existence Metatheorem 1.1.

$\Leftarrow$

Suppose F satisfies conditions (1) through (5). Define the binary operation $\cdot$ on $(A \cup \{*\})^2$ as follows: if $a \neq *$, $b \neq *$, and $\Phi_1(a, b)$ is defined, $a \cdot b = \Phi_1(a, b)$; otherwise, $a \cdot b = *$. Axiom 0 is obviously satisfied. Let $\Phi_4(e_1, a, b, c, x) = \Psi(a, b, c, x)$. [Ψ is an element of F_4 so there is an e_1 in A such that, for all a, b, c , and x in A , $\Phi_4(e_1, a, b, c, x) = \Psi(a, b, c, x)$.] Let

$$\begin{aligned} \Phi_3(e_2, a, b, c) &= S_1^3(C_{e_1}^3(a, b, c), U_1^3(a, b, c), \\ &\quad U_2^3(a, b, c), U_3^3(a, b, c)), \\ \Phi_2(e_3, a, b) &= S_1^2(C_{e_2}^2(a, b), U_1^2(a, b), U_2^2(a, b)), \\ \Phi_1(e_4, a) &= S_1^1(C_{e_3}^1(a), U_1^1(a)). \end{aligned}$$

Then e_4abcx [the abbreviation for $((e_1 \cdot a) \cdot b) \cdot c \cdot x$] = $\Psi(a, b, c, x)$ and e_4 satisfies Axiom 1. Let

$$\begin{aligned} \Phi_3(a_1, b, c, x) &= \Phi_1(\Phi_1(U_1^3(b, c, x), U_3^3(b, c, x)), \\ &\quad \Phi_1(U_2^3(b, c, x), U_3^3(b, c, x))), \\ \Phi_2(a_2, b, c) &= S_1^2(C_{a_1}^2(b, c), U_1^2(b, c), U_2^2(b, c)), \\ \Phi_1(a_3, b) &= S_1^1(C_{a_2}^1(b), U_1^1(b)). \end{aligned}$$

Then $a_3bc = S_1^2(a_1, b, c) \neq *$ for all b and c in A ; and $a_3bcx = bx(cx)$ for all b, c , and x in A .

Since A has more than one element, it is easy to show that $a_3 \neq e_4$. Thus the algebra $I = \langle A \cup \{*\}, \cdot \rangle$ is a URS.

Now $F_1 = \{\lambda x_1[\Phi_1(x, x_1)] | x \in A\} = \{\lambda x_1[x \cdot x_1] | x \in A\} = F_1(I)$. Suppose $F_{n-1}(I) = F_{n-1}$ for some $n \geq 2$. If f is in $F_n(I)$, $f = \phi_a^n$ for some a in A and $f(x_1, \dots, x_n) = \Phi_1(\phi_a^{n-1}(U_1^n(x_1, \dots, x_n), \dots, U_{n-1}^n(x_1, \dots, x_n)), U_n^n(x_1, \dots, x_n))$; so, since ϕ_a^{n-1} is in F_{n-1} and F is closed under composition, f is in F_n . If $g \in F_n$, $g = \lambda x_1 \dots x_n[\Phi_n(b, x_1, \dots, x_n)] = \phi_{s_{i,n-1}, b}^n \in F_n(I)$, for some $b \in A$. So $F_n(I) = F_n$; and, by induction on n , $F(I) = F$. $\square$

The following corollaries relating URS's to other approaches to generalizing the theory of recursive functions are stated with, at most, a bare hint toward proof.

Corollary 2.2

(Wagner, Ref. 1) Let B be a subset of the natural numbers N . There is a URS $I = \langle N \cup \{*\}, \cdot \rangle$ such that $F(I)$ is the family of partial B -recursive functions. [Note that here $x \cdot y$ is $\phi_x^B(y)$ or $\{x\}^B(y)$ (cf. Ref. 5).]

Corollary 2.3

Let L be the recursive ordinals. There is a URS $I = \langle L \cup \{*\}, \cdot \rangle$ such that $F(I)$ is the family of partial metarecursive functions. [Axioms (1), (2), and (3) are easy to verify using Kripke's equation calculus (Ref. 6) to characterize partial metarecursive functions. A slight modification of the proof of (4), the Enumeration Theorem as given by Owings in Ref. 6, allows an easy proof of (5), the S-m-n Theorem.]

Let $C(G)$ represent the functions computable from the family G of base functions and predicates in McCarthy's formulation of computability in Ref. 2. Let D be the union of the domains of all the functions of $C(G)$. Let $Cp(G)$ be the predicates computable from G .

Corollary 2.4

There is a URS, $I = \langle D \cup \{*\}, \cdot \rangle$ such that $F(I) = C(G)$ if and only if $Cp(G)$ contains eq , the predicate for equality on D , and $C(G)$ contains a partial function Φ of two variables such that, for all partial functions $f: D^m \rightarrow D$ in $C(G)$, there is an $x \in D$ such that, for all $x_1, \dots, x_m \in D$, $\Phi(\Phi(\dots \Phi(\Phi(x, x_1), x_2), \dots, x_{m-1}), x_m) = f(x_1, \dots, x_m)$.

Given, for example, the λ -scheme for naming the functions of $C(G)$, if all the names for functions of $C(G)$ are in D , $Cp(G)$ contains eq , and $C(G)$ contains Φ such that $\Phi(f, x) = \lambda x_1 \dots x_{m-1}[f(x, x_1, \dots, x_{m-1})]$ when f names a partial function from D^m into D , then $C(G)$ satisfies the axioms of BRFT.

Corollary 2.5

If F satisfies the axioms of BRFT with enumerating functions Φ_1 , then $F = C(\{eq, \Phi_1\})$.

3. Results of basic recursive function theory in URS's

In this section notions analogous to those of "recursive set" and "recursively enumerable set" will be explored. Let $I = \langle A \cup \{*\}, \cdot \rangle$ be a URS. A subset B of A is *computable* in I if there are elements b, c , and d such that $c \neq d$ and, for each $x, x \in B$ implies $bx = c$ and $x \in A - B$ implies $bx = d$, i.e., if a characteristic function of B is in $F(I)$. This notion (due to Wagner) seems to be the only reasonable analogue for "recursive"; however, there are several choices open for an analogue to "recursively enumerable," including Wagner's "semicomputable."

A subset B of A is *semicomputable* (sc) in I if there are elements b and c such that, for each $x, x \in B$ implies $bx = c$ and $x \in A - B$ implies $bx = *$, i.e., if a "semicharacteristic" function of B is in $F(I)$.

If f is in $F_n(I)$, the *domain* of f is the subset of A^n on which it is defined (takes values in A). The *range* of f is defined similarly. Each function f in $F(I)$ is a *partial* function (with respect to the domain A). If f is defined on each element of A^n , then f is said to be a *total* function. The word *partial* will be used to emphasize the fact that a function f may not be total. It is easy to show that B is sc if and only if it is the domain of a function in $F_1(I)$.

Wagner⁷ considered several other possible analogues including "range of a partial function in $F(I)$ " and "empty or range of a total function in $F(I)$." He has shown that a sc set need not be the range of a total function in $F(I)$ although each sc set is the range of a partial function in $F(I)$. His Theorem 6.7 of Ref. 7 shows that a sc set with sc complements need not be computable. In Ref. 8 an example is given of a URS I such that the range of every total function in $F(I)$ is either finite or co-finite (the complement of a finite set). It is easy to show that the intersection of sc sets is sc and that restrictions of computable functions to sc sets are computable functions.

Another notion similar to "recursively enumerable," but not discussed by Wagner in this context, is the notion of a *splinter* (Wagner's "successor set"). Let $I = \langle A \cup \{*\}, \cdot \rangle$ be a URS. The subset B of A is a *splinter* (of I) if there are elements s and x in A such that $B = \{x, sx, s(sx), \dots\}$. Note that B may be finite but does not contain $*$ since it is a subset of A .

The notation $s^{(0)}x$ for x , $s^{(1)}x$ for sx , and $s^{(n+1)}x$ for $s(s^{(n)}x)$ for each $n \geq 1$ will be useful; $\{s^{(n)}x\}$ will represent the splinter $\{x, sx, s(sx), \dots\}$. Every splinter is countable; so, since there are URS's of every cardinality (Ref. 7), sc sets need not be splinters and ranges of partial or total functions in $F(I)$ need not be splinters.

The following questions are open:

- (1) Must every range of a partial function in $F(I)$ be sc?
- (2) Must every range of a total function in $F(I)$ be sc?
- (3) Must every splinter be sc?

Wagner¹ has shown that the existence of two distinct, non- $*$ elements in the domain of a URS (guaranteed by

$\alpha \neq \psi$ in the axioms of this paper) implies the existence of infinite splinters, e.g., $\{\beta^{(n)}\beta\}$ and $\{(\psi\psi\alpha)^{(n)}\iota\}$.

Theorem 3.1

(Wagner's Theorem 4.5 of Ref. 1) Let $S = \{s^{(n)}o\}$ be an infinite splinter in the URS $I = \langle A \cup \{\ast\}, \cdot \rangle$. Let g be any recursive function of m variables (on natural numbers). There is a function f in $F_m(I)$ such that, for all natural numbers $n_1, \dots, n_m$, $f(s^{(n_1)}o, \dots, s^{(n_m)}o) = s^{(g(n_1, \dots, n_m))}o$. If S is computable, g may be any partial recursive function.

Proof of Theorem 3.1

(For a complete proof see Section 4 of Ref. 1. What follows is only an outline of that work and an indication of how it can be translated into the notation of this paper.)

It suffices to show that the restrictions of functions in $F(I)$ to S form a family of functions closed under composition, primitive recursion, and minimalization of "regular" functions. (The last sentence of Theorem 3.1 will be a Corollary of Theorem 3.2.)

Closure under composition follows immediately from 1.1 (see also the example on composition after the proof of 1.1). The other two properties are results of Theorem 1.5 and Corollary 1.6 (Recursion and Minimalization) and will be proved here.

Suppose g is an element of $F_n(I)$ with index b [i.e., $bx_1 \dots x_n = g(x_1, \dots, x_n)$] and f is an element of $F_{n+2}(I)$ with index a . The restriction of h in $F_{n+1}(I)$ to S is obtained from the restrictions of f and g to S by *primitive recursion* if, for all $x_1, \dots, x_n$, and y in S , $h(x_1, \dots, x_n, o) = g(x_1, \dots, x_n)$ and $h(x_1, \dots, x_n, sy) = f(x_1, \dots, x_n, y, h(x_1, \dots, x_n, y))$. Let p be an element of A such that, for all x and y in A , $pxy = o$ if $sy = x$ and $pxy = so$ if $sy \neq x$ and $sy \neq \ast$. Let c be an element of A such that, for all $d, x_1, \dots, x_n$, and x in A , $cdx_1 \dots x_n x = bx_1 \dots x_n$, if $x = o$; and $cdx_1 \dots x_n x = ax_1 \dots x_n(\mu os(px)) (dx_1 \dots x_n(\mu os(px)))$ if $x \neq o, \ast$. (Recall that μ is the index defined in 1.6.) Then ϕc is an index for a function satisfying the requirements for h where ϕ is the index defined in 1.5 so that $[c(\phi c)] = [\phi c]$.

If $f \in F_{n+1}(I)$ so that $(\forall x_1, \dots, x_n \in S) (\exists x \in S) [f(x_1, \dots, x_n, x) = o]$ (i.e., if f is regular) and a is an index for f , then an index for the minimalization of f is b such that $bx_1 \dots x_n = \mu os(ax_1 \dots x_n)$. $\mathcal{A}$

Wagner noted that the method of proof of 1.6 did not allow the value of μosp to be determined if p were defined but did not take the value o on $S = \{s^{(n)}o\}$. If 3.1 were to hold for partial functions, this value should be $\ast$. Then the above proof would provide minimalization of "total" functions (total on S). Wagner showed that this could be done if S were computable. An example of a URS with no computable infinite splinter is given in Ref. 8.

The next theorem is a strengthening of Wagner's result (3.1 here).

Theorem 3.2

Let $S = \{s^{(n)}o\}$ be any sc infinite splinter and let g be any partial recursive function of m variables. Then there is a function f in $F_m(I)$ such that, for all natural numbers $n_1, \dots, n_m$, if $g(n_1, \dots, n_m)$ is defined, $f(s^{(n_1)}o, \dots, s^{(n_m)}o) = s^{(g(n_1, \dots, n_m))}o$ and if $g(n_1, \dots, n_m)$ is undefined, $f(s^{(n_1)}o, \dots, s^{(n_m)}o) = \ast$.

Proof of Theorem 3.2

Let d be such that, for all a, p , and x , $dapx = x$ if $px = a$; $\ast$, otherwise. Let c be such that $cx \neq \ast$ if and only if x is in S . Let μ' be such that $\mu'asp = dap(\alpha\beta c(\mu asp))$. Then $\mu'osp \neq \ast$ only if $\mu'osp$ is in S and $p(\mu'osp) = o$. The method of proof of 3.1 now suffices for 3.2. $\mathcal{A}$

If every splinter were sc (or if each URS contained a sc splinter, see 3.11), then each URS would contain the partial recursive functions in the sense of 3.2. If non-sc splinters exist, then another interesting candidate for the role of analogue to the recursively enumerable sets would be the collection of unions of sc (or computable) sets and splinters.

Because there are URS's with sc, co-sc sets which are not computable, it is necessary to consider a special class of URS's in order to obtain some of the more interesting results of RFT. If the only axioms added to 0, 1, and 2 are axioms satisfied by the partial recursive functions, then the theory is still in some sense a generalized theory of recursive functions.

A URS is said to be *well-ordered* if it satisfies Axiom 3:

3. $(\exists \delta)(\forall x, y)[x(\delta x) = \ast \rightarrow xy = \ast]$.

Wagner⁹ gives a "choice function" axiom:

$(\exists c)(\forall a, b, x)[(cab = \ast \rightarrow bx \neq a) \& (cab \neq \ast \rightarrow b(cab) = a)]$.

It is easy to see that a URS is well-ordered if and only if it "has a choice function." In the above axioms δ can be chosen so that $\delta x = cx(\alpha(\beta(\beta(x)))x)$; and, conversely, c can be chosen so that $cab = \delta(qab)$, where $qabx = a$ if $bx = a$; $\ast$, otherwise.

Lemma 3.3

In any well-ordered URS there is a κ such that κpqx is in $\{px, qx\}$ and, if $\kappa pqx = \ast$, then $px = qx = \ast$.

Wagner calls any URS with such a κ "cohesive." He has shown that, in the presence of such a κ , sc, co-sc sets are computable.

Proof of Lemma 3.3

Let t be such that $tpqxy = yx$ if x is in $\{p, q\}$; $\ast$, otherwise. Then if $\delta(tpqx) = \ast$, $px = qx = \ast$; and, in any case, $\delta(tpqx)$ is in $\{p, q, \ast\}$. Let κ be such that $\kappa pqx = \delta(tpqx)x$. Then κ satisfies the requirements of 3.3. $\mathcal{A}$

Theorem 3.4

Let $I = \langle A \cup \{*\}, \cdot \rangle$ be a well-ordered URS and let B be a sc, co-sc subset of A . Then B is computable.

Theorem 3.4 (together with Theorem 6.7 et seq. of Ref. 7) answers negatively Wagner's question about whether every URS "has a choice function." However, the partial recursive functions [any URS $I = \langle N \cup \{*\}, \cdot \rangle$ such that $F(I)$ is the partial recursive functions] are well-ordered (see Kleene's Theorem of the Indefinite Description⁵).

The following results about splinters hold in any URS, but will be applied in particular to well-ordered URS's.

Lemma 3.5

Let I be any URS. Let $P(x_1, \dots, x_n)$ be any computable predicate. There is a function $f \in F_{n+1}(I)$ such that $f(s, a, x_1, \dots, x_{n-1}) =$ the "least" y in $\{s^{(n)}a\}$ such that $P(x_1, \dots, x_{n-1}, y)$ if there is such a y .

The predicate $P(x_1, \dots, x_n)$ is *computable* (cf. the discussion following 1.6) if and only if there is a total function $g \in F_n(I)$ such that $g(x_1, \dots, x_n) = \psi$ if and only if $P(x_1, \dots, x_n)$. The notation for f will be $f(s, a, x_1, \dots, x_{n-1}) = \mu y \in \{s^{(n)}a\} [P(x_1, \dots, x_{n-1}, y)]$. There should be no confusion between this use of μ and the constant μ of 1.6.

Proof of Lemma 3.5

Let $uax_1 \dots x_n = a$ if $P(x_1, \dots, x_n)$; $\psi\alpha\psi\alpha$, otherwise. Let $bsax_1 \dots x_{n-1} = \mu as(uax_1 \dots x_{n-1})$. Then b is an index of the desired function. $\mathcal{A}$

Note that $\mu y \in \{s^{(n)}a\} [P(x_1, \dots, x_{n-1}, y)]$ need not be $*$ even though there is no such "least" y . This notation will be used even when $P(x_1, \dots, x_n)$ is not computable if instead there is a function $g \in F_n(I)$ such that $(\forall x_1, \dots, x_{n-1}) (\exists y \in \{s^{(n)}a\}) [g(x_1, \dots, x_{n-1}, y) \text{ is defined}]$ and g is a "semicharacteristic" function for P .

Lemma 3.6

Let I be any URS. If $f \in F_1(I)$ and S is a splinter in the domain of f , then $f(S)$ is also a splinter.

Proof of Lemma 3.6

Let $S = \{s^{(n)}a\}$ and let u be an index of f . Let $vrtbxz = \alpha$ if $b = z$; ψ , if $b = x \neq z$; $rt(tb)xz$, otherwise. Let $wtx = u(\mu y \in S[\phi vrt(ua)x(uy) = \psi])$. If $f(S)$ is infinite, then $f(S) = \{(\phi w)^{(n)}(ua)\}$ and all finite sets are splinters. $\mathcal{A}$

Lemma 3.7

Let I be any URS and let S and T be infinite splinters. There is an $f \in F_1(I)$ such that S is in the domain of f and f restricted to S is one-to-one and onto T .

Proof of Lemma 3.7

Let w be such that $wcasbtix = b$ if $x = a$; $c(sa)s(tb)tx$, otherwise. Let $u = \phi wasbt$ (ϕ as in 1.5). Then $u(s^{(n)}a) =$

$t^{(n)}b$ for all natural numbers n such that $s^{(n)}a \neq *$ and $t^{(n)}b \neq *$. $\mathcal{A}$

Note that ϕw above is an index of a function $g \in F_1(I)$ such that, if $\{s^{(n)}a\}$ and $\{t^{(n)}b\}$ are infinite, $g(a, s, b, t)$ is the index of an "isomorphism" between them.

Theorem 3.8

If S and T are splinters, then their union is a splinter.

Proof of Theorem 3.8

The union of any finite set and a splinter is obviously a splinter (see 1.7). Assume $S = \{s^{(n)}a\}$ and $T = \{t^{(n)}b\}$ are infinite. The proof will use a combination of 3.6, (methods similar to those of) 3.7, and the pairing function with index $\psi\psi$ introduced by Wagner.¹

Let w be such that $wx_1x_2x_3x_4x_5x = \psi\psi x_4x_5$ if $x = \psi\psi x_2x_3$; $x_1x_4x_5(x_3x_2)x_3x$, otherwise. Let $W = \{(\phi wasbt)^{(n)}(\psi\psi as)\}$. Then $W = \{\psi\psi xs | x \in S\} \cup \{\psi\psi xt | x \in T\}$. Let e be such that $ex = x\psi$. Then e is an index of a function in $F_1(I)$ such that $f(W) = S \cup T$ and W is contained in the domain of f . Since W is a splinter, $S \cup T$ must be a splinter by 3.6. $\mathcal{A}$

Lemma 3.9

If S is a splinter, then $\{\psi\psi xy | x, y \in S\}$ is a splinter.

Proof of Lemma 3.9

Let $S = \{s^{(n)}a\}$ be a splinter. If S is finite the Lemma is obvious. Assume S is infinite. There are recursive functions f and g on the natural numbers N such that $(\forall n_1, n_2 \in N) (\exists m \in N) [f(m) = n_1 \text{ and } g(m) = n_2]$ (for instance, take f and g to be the inverses of one of the standard pairing functions). Hence by 3.1 there are elements q and r such that $(\forall n_1, n_2 \in N) (\exists m \in N) [q(s^{(m)}a) = s^{(n_1)}a \text{ and } r(s^{(m)}a) = s^{(n_2)}a]$. Let u be such that $ux = \psi\psi(qx)(rx)$. Then u is the index of a function h such that S is in the domain of h and $h(S) = \{\psi\psi xy | x, y \in S\}$. $\mathcal{A}$

Theorem 3.10

If S and T are splinters, then their intersection is a splinter.

Proof of Theorem 3.10

If $S \cap T$ is finite, then it is a splinter. Assume $S \cap T$ is infinite. Let W be the splinter $\{\psi\psi xs | x \in S\} \cup \{\psi\psi xt | x \in T\}$ as in 3.8 (where $S = \{s^{(n)}a\}$ and $T = \{t^{(n)}b\}$). Note that $s \neq t$ is not assumed. Let e be such that $ex = x\psi$ and a be such that $ax = x\alpha$.

By 3.9 the set $V = \{\psi\psi xy | x, y \in W\}$ is an infinite splinter. Let $V = \{v^{(n)}d\}$. Let u be such that $ux = e(e\mu y \in \{v^{(n)}x\} [e(ey) = e(ay) \& a(ey) = s \& a(ay) = t])$. Then u is the index of a function $f \in F_1(I)$ such that V is in the domain of f and $f(V) = S \cap T$. $\mathcal{A}$

Theorem 3.11

If S and T are infinite splinters and S is sc, then T is sc.

Proof of Theorem 3.11

If S is sc, Lemma 3.7 provides f and g in $F_1(I)$ with $S = \text{domain } f$, $T \subseteq \text{domain } g$, and $x \in T$ if and only if $f(g(x)) = x$. $\mathcal{A}$

The next theorem states that a URS is well-ordered if and only if it "contains" a uniform procedure for finding an infinite splinter in each infinite set. Hence to show that a URS is not well-ordered, it is sufficient to exhibit an infinite sc set which contains no infinite splinter.

Theorem 3.12

Let I be a URS. Then I is well-ordered if and only if there are functions f and g in $F_1(I)$ such that $\{(f(u))^{(n)}(g(u))\}$ is contained in $\{x | ux \neq *\}$ and infinite if $\{x | ux \neq *\}$ is infinite.

Proof of Theorem 3.12

$\Rightarrow$

Suppose I is well-ordered and $u(\delta u) = *$ implies $ux = *$ for all x . Let t be such that $tu \neq *$ for all u and $tux = ux$ if $x \neq \delta u$; $*$, otherwise. Then if $\{x | ux \neq *\}$ is infinite, $\{t^{(n)}u\}$ is an infinite splinter and $\{\delta y | y \in \{t^{(n)}u\}\}$ is an infinite splinter and a subset of $\{x | ux \neq *\}$. To make this construction uniform, let v be (a term involving only δ , α , and ψ) such that $vux = \mu y \in \{t^{(n)}u\} [\delta y = x]$; and let s be such that $sux = \delta(t(vx))$. Note that $su(\delta u) = \delta(tu)$, $su(\delta(tu)) = \delta(t(tu))$, ..., $su(\delta(t^{(n)}u)) = \delta(t^{(n+1)}u)$, etc. Thus s and δ are indices for the desired functions f and g respectively.

$\Leftarrow$

Suppose $F_1(I)$ contains functions f and g as described above. Let δ be an index of g . Then δ satisfies Axiom 3. $\mathcal{A}$

The existence of a non-well-ordered URS in which every infinite sc set contains an infinite splinter is still open.

Let $I = \langle A \cup \{*\}, \cdot \rangle$ be a URS. A subset C of A is said to be *closed* if $(\forall x, y)[x \in C \ \& \ [x] = [y] \Rightarrow y \in C]$. (Recall that $[u] = \{\langle x, y \rangle | ux = y\}$ is the graph of the function indexed by u .) Let $\leq$, $<$, etc., be the relations induced on A by the relations $\subseteq$, $\subset$, etc., on $\{[x] | x \in A\}$. The *closure* of a set C will be $\{x | (\exists y)[y \in C \ \& \ [x] = [y]]\}$. Wagner¹ has shown that the closure of $\{x\}$ is infinite for each $x \in A$. For well-ordered URS's this will be a corollary of the following partial analogue of the Rice-Myhill-Shapiro Theorem.³

Theorem 3.13

Let $I = \langle A \cup \{*\}, \cdot \rangle$ be a well-ordered URS and let C be a closed sc subset of A . Then $x \in C$ and $y > x \Rightarrow y \in C$.

Proof of Theorem 3.13

The proof will depend on the fact (due to Wagner) that

$\{x | xx = *\}$ is not sc (if $vx = * \Leftrightarrow xx \neq *$, then $vv = * \Leftrightarrow vv \neq *$, a contradiction). Suppose C is closed and sc, $x \in C$, $y > x$, and $y \notin C$. Let p be such that $pzw = \beta y(zz)w$. Let q be such that $qzw = xw$. Let r be such that $rzw = \kappa(pz)(qz)w$ (κ as in 3.3) and $rz \neq *$. Then $rzw = yw$ if $zz \neq *$ or $xw \neq *$; $*$, otherwise. So $[rz] = [x] \Leftrightarrow zz = * \Leftrightarrow rz \in C$; and $\{z | zz = *\}$ must be sc since C is sc, which is a contradiction. Hence $x \in C$ and $y > x \Rightarrow y \in C$. $\mathcal{A}$

Corollary 3.14

If $I = \langle A \cup \{*\}, \cdot \rangle$ is a well-ordered URS and C is a closed computable subset of A , then $C = \phi$ or $C = A$.

Proof of Corollary 3.14

If C is closed and sc, then $\theta \in C \Leftrightarrow C = A$ since each $y \in A$ is $\geq \theta$. But $\theta \in C$ or $\theta \in A - C$ and C is closed and computable if and only if $A - C$ is closed and computable. $\mathcal{A}$

Corollary 3.15

In a well-ordered URS, the closure of a finite set is not sc and hence it must be infinite and co-infinite.

4. Pairing functions

Pairing functions for URS's have already been heavily used in Section 3. By means of a pairing function, the necessary and sufficient conditions given in Section 2 for a family of partial functions F to be $F(I)$ for some URS I can be translated into a set of requirements for F_1 alone.

These requirements can refer to the pairing function by means of its inverses: partial functions K and L from A into A are inverses of a pairing function P from $A \times A$ into A if K and L are defined on the range of P and, for each x and y in A , $K(P(x, y)) = x$ and $L(P(x, y)) = y$. If $I = \langle A \cup \{*\}, \cdot \rangle$ is a URS, then the function P with index $\psi\psi$ is a pairing function in $F_2(I)$ and inverses for it are the functions K , such that $K(x) = x \cdot \psi$, and L , such that $L(x) = x \cdot \alpha$.

Wagner has shown that if two URS's have the same one-ary functions then they have the same functions for any number of variables. His proof depends on the use of the pairing function with index $\psi\psi$ present in each URS. Another proof (depending indirectly on the same pairing function) will be given after the analogue to Theorem 2.1 which follows. For the rest of this section A is assumed to be a set with at least two distinct elements.

Theorem 4.1

Let F_1 be a family of partial functions from A into A . There is some URS $I = \langle A \cup \{*\}, \cdot \rangle$ such that $F_1 = F_1(I)$ if and only if F_1 satisfies the following conditions:

- (1) F_1 contains the constant functions $[C_x(y) = x]$, the identity function $[U(x) = x]$, and the inverses K and L

of a pairing function P (a one-to-one function from $A \times A$ into A) such that for each f and $g \in F_1$, $\lambda x[P(f(x), g(x))] \in F_1$;

- (2) $(\exists \Psi_1 \in F_1) (\forall a, b, c, d \in A) [\Psi_1(P(P(a, b), c), d) = b, \text{ if } x = a; c, \text{ otherwise}]$;
- (3) F_1 is closed under composition;
- (4) (Enumeration-1) $(\exists \Phi \in F_1) [F_1 = \{\lambda y[\Phi(P(x, y))]|x \in A\}]$; and
- (5) (S-1-1) $(\exists S \in F_1) (\forall a, b, c \in A) [S(a) \text{ is defined and } \Phi(P(S(P(a, b)), c)) = \Phi(P(a, P(b, c)))]$.

Proof of Theorem 4.1

The proof can be given directly or by application of Theorem 2.1. Since the application of 2.1 exposes the important relationship between the pairing function P and the functions of many variables it is given here.

$\Rightarrow$

If $F_1 = F_1(I)$ for some URS I , then (1) through (5) are consequences of Theorem 1.1 where P is the function with index $\psi\psi$.

$\Leftarrow$

Suppose F_1 satisfies (1) through (5) above. Define $P_{n+1}(x_1, \dots, x_{n+1}) = P(P_n(x_1, \dots, x_n), x_{n+1})$ for each $n \geq 2$ ($P_2 = P$). Let $F_2 = F_1 \circ P = \{f \circ P | f \in F_1\}$, $F_n = F_1 \circ P_n$ for $n > 2$, and $F = \bigcup_n F_n$ (where $\circ$ indicates composition).

It is easy to show that (1) and (2) of 4.1 imply (1) and (2) of 2.1 (BRFT) for F ; so (3), (4), and (5) of 2.1 will be shown here.

Suppose $f, g_1, \dots, g_m$ are in F_1 so that $f \circ P_m$ is in F_m , $g_1 \circ P_n$ is in F_n , ..., and $g_m \circ P_n$ is in F_n . Then $\lambda x[P(g_1(x), g_2(x))] \in F_1$, $\lambda x[P(P(g_1(x), g_2(x)), g_3(x))] \in F_1$, etc.; so $\lambda x[P_m(g_1(x), \dots, g_m(x))] \in F_1$; and finally, $\lambda x[f(P_m(g_1(x), \dots, g_m(x)))] \in F_1$ so the composition of $f \circ P_m$ with $g_1 \circ P_m, \dots, g_m \circ P_m$ is in F_n . Thus F is closed under composition.

If $f \in F_1$ then $f(y) = \Phi(P(a, y))$ for some $a \in A$; and $f(P_m(x_1, \dots, x_m)) = \Phi(P(a, P_m(x_1, \dots, x_m)))$. For each $m \geq 2$, let $g_m(x)$ be constructed from K, L , and P so that $g_m \in F_1$ and $g_m(P_{m+1}(x, x_1, \dots, x_m)) = P(x, P_m(x_1, \dots, x_m))$. If $f = \lambda y[\Phi(P(a, y))]$, then $f(P_m(x_1, \dots, x_m)) = \Phi(g_m(P_{m+1}(a, x_1, \dots, x_m)))$ and $\Phi \circ g_m \in F_1$. Let g_1 be the identity. Then $F_m = \{\lambda x_1 \dots x_m[(\Phi \circ g_m \circ P_{m+1})(x, x_1, \dots, x_m)] | x \in A\}$ and $\Phi_m = \Phi \circ g_m \circ P_{m+1} \in F_{m+1}$ for each $m > 0$.

For each $n > 0$ let f_n be constructed from K, L , and P so that $f_n \in F_1$ and $f_n(P(z, P_n(y_1, \dots, y_n))) = P_{n+1}(z, y_1, \dots, y_n)$. Then $f_n(P_{m+1}(x, x_1, \dots, x_m), P_n(y_1, \dots, y_n)) = P_{m+n+1}(x, x_1, \dots, y_n)$; and for each m and $n > 0$, $\Phi \circ g_{m+n} \circ f_n \in F_1$. Let a_n^m be an element of A such that $\Phi(g_{m+n}(f_n(x))) = \Phi(P(a_n^m, x))$. The function $\lambda x[S(P(a_n^m, x))]$ is in F_1 ; so the function $S_n^m = \lambda x x_1 \dots x_m[S(P(a_n^m, P_{m+1}(x, \dots, x_m)))]$ is in F_{m+1} . Now $\Phi_n(S_n^m(x, x_1, \dots, x_m), y_1, \dots, y_n) = \Phi(P(S(P(a_n^m, P_{m+1}(x, \dots, x_m))), P_n(y_1, \dots,$

$y_n))) = \Phi(P(a_n^m, P_{m+1}(x, x_1, \dots, x_m), P_n(y_1, \dots, y_n))) = \Phi_{m+n}(x, x_1, \dots, x_m, y_1, \dots, y_n)$; and S_n^m satisfies the S-m-n requirement of 2.1. $\square$

Corollary 4.2

If I is a URS then there is a pairing function P in $F_2(I)$ such that, for each $m \geq 2$, $F_m(I) = F_1(I) \circ P_m$ and P has inverses in $F_1(I)$.

The proof of 4.1 depends on the existence of a "standard" pairing function in each URS. The pairing function required in the conditions for F_1 to be $F_1(I)$ of some URS I was any arbitrary one-to-one function from $A \times A$ into A with inverses in F_1 satisfying the rest of the somewhat arbitrary properties (1) through (5). The following lemmas are intended to provide a definition for P in terms of $F_1(I)$ alone so that the class of pairing functions satisfying the definition is uniquely determined by $F_1(I)$ and, if P and P' are two members of this class, $F_1(I) \circ P_m = F_1(I) \circ P_m'$ for all $m \geq 2$.

Let $I = \langle A \cup \{\ast\}, \cdot \rangle$ be a URS. Consider the collection of pairs of functions $\langle K, L \rangle$ (with the same domain) that are inverses of pairing functions on A . Since I has the standard pairing function, the intersection $\mathcal{O}$ of this collection with $F_1(I) \times F_1(I)$ is not empty.

Note that if P is the standard pairing function, the usual inverses $f(x) = x \cdot \psi$ and $g(x) = x \cdot \alpha$ do not have the same domains, so $\langle f, g \rangle$ is not in $\mathcal{O}$. However, the intersection of sc sets is sc and the restriction of any function in $F_1(I)$ to a sc set is a function in $F_1(I)$. So if K is the restriction of f and L is the restriction of g to the intersection of the domain of f with the domain of g , then $\langle K, L \rangle$ is a pair of inverses for P and is in $\mathcal{O}$.

Define $\langle K_1, L_1 \rangle \leq \langle K_2, L_2 \rangle$ if there is an f in $F_1(I)$ such that $K_1 = K_2 \circ f$ and $L_1 = L_2 \circ f$. Define $\langle K_1, L_1 \rangle \equiv \langle K_2, L_2 \rangle$ if $\langle K_1, L_1 \rangle \leq \langle K_2, L_2 \rangle$ and $\langle K_2, L_2 \rangle \leq \langle K_1, L_1 \rangle$. It is easy to check that $\equiv$ is an equivalence relation on $\mathcal{O}$. Let M be the set of pairs in $\mathcal{O}$ that are inverses of some pairing function in $F_2(I)$. Then M is not empty.

Lemma 4.3

The set M is the maximum class with respect to the ordering induced by $\leq$ on the $\equiv$ classes of $\mathcal{O}$.

Proof of Lemma 4.3

Suppose $\langle K, L \rangle \in M$ is a pair of inverses for $P \in F_2(I)$. Let $\langle K_1, L_1 \rangle$ be any pair in $\mathcal{O}$. Then for each $x \in A$, $K_1(x) = K(P(K_1(x), L_1(x)))$ and $L_1(x) = L(P(K_1(x), L_1(x)))$; and $\lambda x[P(K_1(x), L_1(x))]$ is in $F_1(I)$ so $\langle K_1, L_1 \rangle \leq \langle K, L \rangle$. Let $\langle K_2, L_2 \rangle$ be a pair in $\mathcal{O}$ such that $\langle K, L \rangle \leq \langle K_2, L_2 \rangle$ via f in $F_1(I)$: $K = K_2 \circ f$, $L = L_2 \circ f$. Then $f \circ P$ is in $F_2(I)$ and, for each x and y in A , $K_2(f(P(x, y))) = x$ and $L_2(f(P(x, y))) = y$; so $\langle K_2, L_2 \rangle$ is in M . [Note that if I and J are

URS's, $F_1(I) = F_1(J)$, and P is a pairing function in $F_2(I)$ with inverses in $F_1(I)$, then, by 4.3, P is in $F_2(J)$ since the set M is uniquely determined by $F_1(I) = F_1(J)$.] $\mathcal{A}$

Let $\mathcal{Q}$ be the set of pairing functions on A with inverses in $\mathcal{P}$. Define $P_a \leq P_b$ if there is an f in $F_1(I)$ such that $P_a = f \circ P_b$; and $P_a \equiv P_b$ if $P_a \leq P_b$ and $P_b \leq P_a$. Let $R = \mathcal{Q} \cap F_2(I)$.

Lemma 4.4

The set R is the minimum class with respect to the ordering induced by $\leq$ on the $\equiv$ classes of $\mathcal{Q}$.

Proof of Lemma 4.4

Suppose $P \in R$ has inverses $\langle K, L \rangle \in M$. Let P_a be any pairing function in $\mathcal{Q}$ with inverses $\langle K_a, L_a \rangle$ in $\mathcal{P}$. Then $\langle K_a, L_a \rangle \leq \langle K, L \rangle$ via $f = \lambda x[P(K_a(x), L_a(x))]$; and $P = f \circ P_a$ so $P \leq P_a$. If $P_b \in \mathcal{Q}$ and $P_b = g \circ P$ with $g \in F_1(I)$, then $P_b \in F_2(I)$; so $P_b \in R$. $\mathcal{A}$

Lemma 4.5

If $P \in R$, then for each $m \geq 2$, $F_m(I) = F_1(I) \circ P_m$.

Proof of Lemma 4.5

The standard pairing function is in R and has the desired property (see 4.2). If P and Q are in $\mathcal{Q}$ and $P = f \circ Q$, then, for each $m \geq 2$, there is a function f_m constructed from f and the inverses for Q such that $P_m = f_m \circ Q_m$. If $F_m(I) = F_1(I) \circ P_m$, then $F_m(I) \subseteq F_1(I) \circ Q_m$. If $F_m(I) = F_1(I) \circ Q_m$, then $F_1(I) \circ P_m \subseteq F_m(I)$. Since each pairing function in R is equivalent ($\equiv$) to the standard pairing function, $F_m(I) = F_1(I) \circ P_m$ for each P in R . $\mathcal{A}$

Implicit in 4.5 is the fact that each $F_m(I)$ is uniquely determined by $F_1(I)$.

Corollary 4.6

(Wagner¹) If I and J are URS's and $F_1(I) = F_1(J)$ then $F(I) = F(J)$.

The next results are analogous to the work of Rogers¹⁰ and the terminology follows that of this reference.

The map π from A onto $F_1(I)$ given by $\pi(x) = \phi_x = \lambda y[x \cdot y]$ is called the *standard numbering* for the URS $I = \langle A \cup \{*\}, \cdot \rangle$. A *numbering* (for I) will be any map from A into $F_1(I)$. The function ϕ_i^2 in $F_2(I)$ has the property that $\pi(x) = \lambda y[\phi_i^2(x, y)]$. A numbering ρ is said to be *semi-effective* if there is a function h in $F_2(I)$ such that $\rho(x) = \lambda y[h(x, y)]$. Thus a numbering ρ is semi-effective if and only if there is a function f in $F_1(I)$ and a pairing function P in R (see 4.4) such that $\rho(x) = \lambda y[f(P(x, y))]$. If one particular pairing function P in R is fixed, there is a map Γ from $F_1(I)$ onto the class of semi-effective numberings for I given by $(\Gamma f) = \lambda x[\lambda y[f(P(x, y))]]$.

Define (on the class of semi-effective numberings) $\rho \leq \sigma$ if there is an f in $F_1(I)$ such that $\rho = \sigma \circ f$. Define $\rho \equiv \sigma$ if $\rho \leq \sigma$ and $\sigma \leq \rho$. Let Π be the equivalence class of numberings containing the standard numbering π ; Π is called the class of *fully effective numberings* for I .

Lemma 4.7

The set Π is the maximum class with respect to the ordering induced by $\leq$ on the $\equiv$ classes of semi-effective numberings.

Proof of Lemma 4.7

Suppose $\rho = \Gamma(f)$ is a semi-effective numbering for I . Let $f = \phi_a$ and $P = \phi_p^2$. Then, if $b \in A$ such that $b \cdot x = \alpha(\beta a)(px)$, $\rho = \pi \circ \phi_b$ and $\rho \leq \pi$. $\mathcal{A}$

Let $\rho \in \Pi$ and define $x!y = (\rho(x))(y)$; $(*!x = x!* = *!* = *)$.

Lemma 4.8

$J = \langle A \cup \{*\}, ! \rangle$ is a URS and $F(I) = F(J)$.

Proof of Lemma 4.8

For a proof of Lemma 4.8 see the proof of Wagner's Theorem 3.7 in Ref. 4. (Lemma 4.8 is a generalization of this theorem in which the notion of fully effective numbering is isolated. Similarly Lemma 4.9 is a restatement of Wagner's Theorem 3.6.)

Lemma 4.9

(Wagner⁴) If J is a URS and $F(J) = F(I)$, then the class of fully effective numberings for J is the class Π of fully effective numberings for I ; and, hence, the standard numbering for J is in Π .

Proof of Lemma 4.9

By 4.7 the class of fully effective numberings for J is uniquely determined by $F_1(J) = F_1(I)$; so it must be Π . $\mathcal{A}$

The next theorem summarizes the results in Lemmas 4.7, 4.8, and 4.9.

Theorem 4.10

Let I be a URS. The set of standard numberings for URS's J such that $F(J) = F(I)$ is the set of fully effective numberings for I .

Corollary 4.11

(Wagner⁴) The standard numbering for any URS I with $F(I) =$ the partial recursive functions is a fully effective numbering (or Gödel numbering) in the sense of Rogers.¹⁰

Summary

Uniformly Reflexive Structures (URS's) provide a generalized theory of computability which is independent of any particular domain. The generalization is a concise, finitely-axiomatized, first-order theory (with equality) with models that can be studied equally well as algebras and as collections of functions.

The algebra is similar to combinatory logic derived from the work of Schonfinkel.^{11,12} The algebraically-oriented notation makes possible very brief statements of proofs. It also emphasizes the algebraic structure of the proofs of such theorems as the Recursion Theorem and the part of the Rice-Myhill-Shapiro Theorem numbered 3.13 here.

In Section 2 a kind of equivalence was demonstrated between a characterization of families of functions [Basic Recursive Function Theory (BRFT)] and the theory of URS's. In one direction the equivalence provided the tie between URS's and the other approaches to a generalized theory of computability mentioned in Section 2 (which are concerned with classes of functions). In the other direction, the equivalence made possible the study of what parts of RFT follow from the axioms of BRFT by means of the study of the algebras which are models of the theory of URS's.

The theory can be shown to be undecidable by means of a weak representation of the partial recursive functions. In spite of the fact that the question, "Does every URS have an infinite semicomputable splinter?," is still open (so that 3.2 cannot be applied), it can be shown that, for each partial recursive function h , there is a term $\bar{h}$ and, for each natural number n , there is a term $\bar{n}$ ($s^{(n)}o$ for suitable s and o), such that $\bar{h} \cdot \bar{n} = \bar{m}$ is a theorem whenever $h(n) = m$. If the theory were decidable, then the function,

$$f(n) = \begin{cases} 1, & \text{if } \bar{h} \cdot \bar{n} = \bar{0} \text{ is a theorem,} \\ 0, & \text{otherwise,} \end{cases}$$

would be recursive, where $\bar{h}$ represents the function $h(n) = \phi_n(n)$. The function f would then be ϕ_m for some m , so that $\phi_m(m) = 1$ if and only if $\phi_m(m) = 0$.

In Section 4 the work of Rogers on numberings of partial recursive functions was generalized to URS's. A one-to-one correspondence was demonstrated between the class of URS's on some fixed domain which compute the same functions and the class of fully effective numberings from that domain to the class of functions (of one variable) computed. This class of fully effective numberings was shown to be a maximal class of numberings under a natural

ordering uniquely determined by the class of functions of one variable. Thus, for example, there is one URS for each Gödel numbering of the partial recursive functions.

The study of splinters in URS's brings into focus those places where RFT depends on the existence and computability of the successor function. The author has explored the structures of particular URS's (especially the families of computable sets, semicomputable sets, and splinters): in Ref. 8 the Boolean algebras of computable sets of URS's on the natural numbers are characterized and a URS with no computable splinter is presented. Although the theory can be viewed as studying computation from the black-box point of view, a study of particular models should provide information about the structure of computations.

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