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Small-signal Stability Criterion for Electrical Networks Containing Lossless Transmission Lines*

Abstract: A stability criterion is derived for networks containing lossless transmission lines as well as the usual lumped electrical elements. The criterion is stated in terms of the transmission line parameters and scattering matrix measurements made at the terminals of the lumped part of the network. The mathematical proof of the stability theorem involves some new results concerning a special system of difference-differential equations. Another stability criterion is derived in terms of more general input-output measurements.

Introduction

In this paper we derive a small-signal stability criterion for electrical networks containing lossless transmission lines. The criterion given in Theorem 2 is similar to that obtained by Nyquist[1] for feedback systems. The results are also valid for networks without transmission lines and give some well-known results. This criterion can be used, for example, in the design of high-speed computer networks to determine the lengths of interconnecting transmission lines so that the equilibrium states of the networks are stable. In fact, the motivation for this study is the problem of designing computer switching circuits such that when they are interconnected according to the logic design by means of transmission lines, the resulting array is stable at its equilibrium points. The theory is linear, but applies to nonlinear systems linearized about an equilibrium point.

The paper is divided into four sections. In the first the general formulation of the equations is described starting with the hyperbolic partial differential equations of the transmission lines. By using a well-known result about the wave equation, the partial differential equations are replaced by difference equations and these are combined with the linearized ordinary differential equations describing the remainder of the network. This yields a system of difference-differential equations. The stability criterion for such a system is stated in the second section in terms of the roots of its characteristic equation. This criterion is restated in terms of small-signal measurements (scattering matrix measurements) which can be made on the networks.

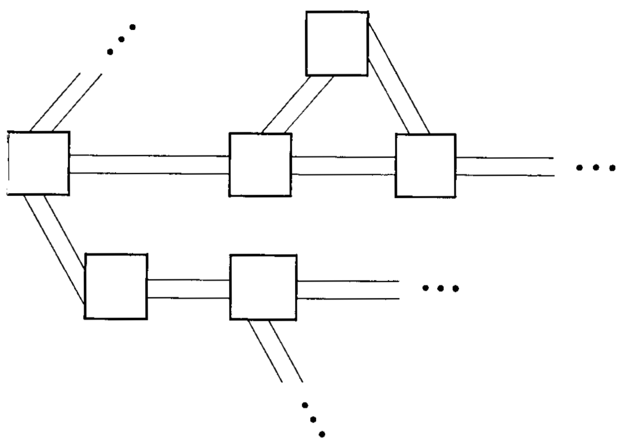

Figure 1 General electrical network of lumped-element circuits interconnected by transmission lines.

The test for stability uses the argument principle in the theory of complex variables. One dissimilarity with the standard Nyquist criterion is noted, i.e., that the complete change in argument does not occur only on the imaginary axis. In the third section two examples are considered. Some practical considerations in using this criterion as a design tool are discussed in the final section.

There are three appendices in this paper. The first furnishes the proof for Theorem 1, which states that the roots of the characteristic equation of the system determine its asymptotic behavior. In establishing this result, it was necessary to prove the existence of solutions, to determine the asymptotic behavior of characteristic roots, and to establish a representation of solutions using the Laplace

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transform for special systems of difference-differential equations of the type given in Eqs. (A1) and (A2). Such systems arise naturally in these applications and it seems that they have not been dealt with in the literature. The second appendix discusses the measurement of the scattering matrix and the third appendix derives the characteristic equation for more general input-output variables.

General formulation

We consider the most general network of lumped-element circuits interconnected by lossless transmission lines. In Fig. 1 the boxes represent the lumped-element circuits and the parallel lines represent the lossless transmission lines. Throughout the paper all the elements are linear and time invariant. The length of each line can be normalized to unity without loss of generality and the behavior in the kth line can be described by the following pair of partial differential equations:

$$L_k \frac{\partial i_k}{\partial t} = -\frac{\partial v_k}{\partial x}, \quad C_k \frac{\partial v_k}{\partial t} = -\frac{\partial i_k}{\partial x}, \tag{1}$$

where $i_k(x, t)$, $v_k(x, t)$ are the current in the line and the voltage to ground, respectively, at the point x and at time t, and L_k, C_k are, respectively, the inductance and capacitance of the line per unit length.

It is well known that solutions of (1) can be represented in terms of two waves traveling in opposite directions at the same speed $\gamma_k = (L_k C_k)^{-\frac{1}{2}}$. These waves are

$$\phi_k(x - \gamma_k t) = v_k(x, t) + Z_k i_k(x, t), \tag{2}$$

$$\psi_k(x + \gamma_k t) = v_k(x, t) - Z_k i_k(x, t), \tag{3}$$

where $Z_k = (L_k/C_k)^{\frac{1}{2}}$. By evaluating (2) and (3) at $x = 0$ and $x = 1$, we arrive at the difference relations

$$\begin{aligned} v_k(1, t) + Z_k i_k(1, t) &= v_k(0, t - \tau_k) \\ &\quad + Z_k i_k(0, t - \tau_k), \end{aligned} \tag{4}$$

$$\begin{aligned} v_k(0, t) - Z_k i_k(0, t) &= v_k(1, t - \tau_k) \\ &\quad - Z_k i_k(1, t - \tau_k), \end{aligned} \tag{5}$$

where $\tau_k = (L_k C_k)^{\frac{1}{2}}$. These relations are stated in terms of the unknowns v_k and i_k at both ends of the line.

We assume that the equations describing the lumped-element circuits can be written in normal form[2] as a system of first-order ordinary differential equations

$$\dot{x} = Ax + By, \tag{6}$$

where x is an n-vector* and describes the internal state of the lumped circuits (no further use is made of x as a spatial coordinate).† For example, some of the components of x may be currents in inductors while other components may be voltages across capacitors. The equation represents either a linear network or a nonlinear network which has been linearized about some equilibrium point. Also, the equilibrium point has been assumed to be $x = 0$ without loss of generality. The vector y is $2m$-dimensional where m is the total number of transmission lines in the network. The component y_k for $1 \leq k \leq m$ is some linear combination of $i_k(0, t)$ and $v_k(0, t)$, while y_{m+k} is a linear combination of $i_k(1, t)$ and $v_k(1, t)$. The choice for y_k is somewhat arbitrary and in this paper we will make the choice given in Eq. (9) below.

To complete the description we need to specify the relation between x, y, and the complementary variable of y which will be designated by z. The choice for z is given in Eq. (10) below and again this choice is somewhat arbitrary. The relation between x, y, and z is written in the general form

$$z = E'x + Fy, \tag{7}$$

where E and F are matrices of dimension $n \times 2m$ and $2m \times 2m$, respectively, and $'$ denotes the transposed matrix. The vector y can be viewed as the input to the lumped circuits while z can be viewed as the output. Thus (7) is the relation giving the output as a linear combination of the input and the internal state.

Finally we write the $2m$ relations (4) and (5), $k = 1, \cdots, m$, in vector form in terms of the vectors y and z as

$$Dy(t) + Cz(t) = J[\hat{D}(y)(t - \tau) - \hat{C}(z)(t - \tau)], \tag{8}$$

where D, C, $\hat{D}$, and $\hat{C}$ are diagonal matrices of dimension $2m$ and

$$J = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}.$$

The notation I_m represents the $m \times m$ identity matrix. Also in Eq. (8), τ is a $2m$-vector, $(\tau_1, \cdots, \tau_m, \tau_1, \cdots, \tau_m)'$; similarly $t - \tau$ is a $2m$-vector, the jth component being $t - \tau_j$. We have used the convention in writing Eq. (8) that

$$(y)(t - \tau) = \begin{bmatrix} y_1(t - \tau_1) \\ \cdot \\ \cdot \\ \cdot \\ y_m(t - \tau_m) \\ y_{m+1}(t - \tau_1) \\ \cdot \\ \cdot \\ \cdot \\ y_{2m}(t - \tau_m) \end{bmatrix}.$$

* For notational simplicity, lightface type is used to represent vectors and matrices, as well as scalar quantities.

† It might be more appropriate for some electrical networks to write the system as $P\dot{x} = Ax + By$. The theory derived in this paper is also valid for this case if P is non-singular and is substituted in the obvious places in the formulas.

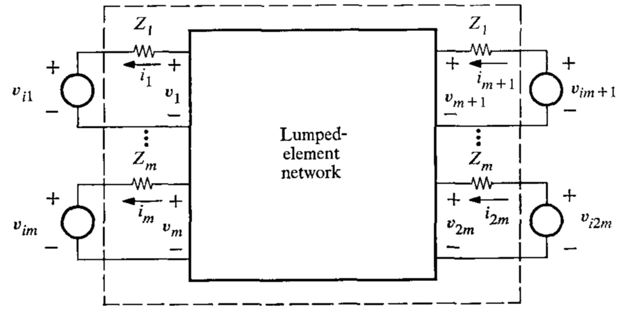

Figure 2 Network with transmission lines omitted and with incident voltages as inputs.

In general, a vector u evaluated at a vector ζ will be written $(u)(\zeta)$ and will mean: Evaluate the first component of the vector in the left parentheses at the first component of the vector in the right parentheses. Thus $(Au)(\zeta)$ will mean to multiply u by A and then to evaluate the resulting vector at ζ. The form of (8) can be readily seen by ordering the $2m$ equations (4), (5) for $k = 1, \cdots, m$ and grouping linear combinations of i and v to form y and z.

The matrix formulation of the problem is complete except for making a convenient choice for y and z. The most convenient choice leads to the so-called scattering matrix formulation (see, e.g., Carlin[3]). For each transmission line we form the new variables

$$y_k = v_{ik} \equiv \begin{cases} v_k - Z_k i_k\,, & k \leq m\,; \\ v_k + Z_k i_k\,, & k > m\,; \end{cases} \tag{9}$$

$$z_k = v_{rk} \equiv \begin{cases} v_k + Z_k i_k\,, & k \leq m\,; \\ v_k - Z_k i_k\,, & k > m\,. \end{cases} \tag{10}$$

Then the relations (4) and (5) are written simply as

$$v_i(t) = J(v_r)(t - \tau) \text{ or } y(t) = J(z)(t - \tau)\,. \tag{11}$$

The subscripts i and r stand for incident and reflected, respectively.

The differential equations for the internal states x with v_i as an input and v_r as output can be viewed as the network shown in Fig. 2. Since there is a positive impedance inserted at every port, there is no problem in writing the equations for the $2m$-port inside the dashed lines. The equations in normal form are:

$$\begin{aligned} \dot{x} &= Ax + Bv_i\,, \\ v_r &= E'x + Fv_i\,. \end{aligned} \tag{12}$$

The choice $y = v_i$ and $z = v_r$ also simplifies the form of the difference equations (11).

Stability criterion

We want to determine under what conditions the equilibrium state, $v_i = v_r = 0$ and $x = 0$, is stable. By eliminating v_i we obtain

$$\dot{x}(t) = Ax(t) + BJ(v_r)(t - \tau)\,, \tag{13}$$

$$v_r(t) = E'x(t) + FJ(v_r)(t - \tau)\,, \tag{14}$$

which is a system of difference-differential equations, (13), coupled with a system of difference equations, (14). In terms of a new vector $\zeta = (x, v_r)'$, we have

$$\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \dot{\zeta}(t) + \begin{pmatrix} -A & 0 \\ -E' & I_{2m} \end{pmatrix} \zeta(t) + \begin{pmatrix} 0 & -BJ \\ 0 & -FJ \end{pmatrix} (\zeta)(t - \tau) = 0\,. \tag{15}$$

The theory of systems of linear difference-differential equations with constant coefficients [i.e., having the form

$$\sum_{i=0}^{N} A_i \dot{y}(t - \omega_i) + B_i y(t - \omega_i) = f(t)\,,$$

where A_i, B_i are constant matrices and $0 = \omega_0 < \omega_1 < \cdots < \omega_N$] is well developed in Ref. 4 provided that the matrix of the leading term A_0 is nonsingular. Unfortunately, in the equations considered here the leading matrix (coefficient of $\dot{\zeta}$) is singular. Of course if one could eliminate the v_r variables, for example if $F = 0$, then the system would reduce to one with A_0 nonsingular. In general this is not possible without introducing an infinite number of delays, which complicates the equations unnecessarily. We would like to obtain a theorem stating that if all the roots of the characteristic equation associated with (15), i.e., the roots s_j of

$$\left| \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} s + \begin{pmatrix} -A & 0 \\ -E' & I_{2m} \end{pmatrix} + \begin{pmatrix} 0 & -BJe^{-sT} \\ 0 & -FJe^{-sT} \end{pmatrix} \right| = 0\,, \tag{16}$$

where $T = \text{diag}\,(\tau)$, satisfy $\text{Re}\,(s) \leqslant -c_1 < 0$, then any solution of (15) decays like $e^{-c_2 t}$ where $c_2 < c_1$.* Such a theorem is obtained (Theorem A4) in Appendix 1 for a system of the form of (15) and is restated here. We note that Eq. (16) is an exponential polynomial and as such generally has an infinite number of roots.

- *Theorem 1*

If the roots s_j of (16) lie in the half plane $\text{Re}\,(s) < \delta$, then given any $\delta' > \delta$ there exists a constant, $K(\delta)$, such that

$$\|\zeta(t)\| \leqslant Ke^{\delta' t} \quad \text{as} \quad t \rightarrow \infty\,. \tag{17}$$

For a vector the form $\|x\|$ is equivalent to $(x, x)^{\frac{1}{2}}$.

* The notation $|A|$, where A is a square matrix, will be used for the determinant of A.

Equation (16) can be rewritten as

$$h(s) \equiv \begin{vmatrix} s I_n - A & -B J e^{-sT} \\ -E' & I_{2m} - F J e^{-sT} \end{vmatrix} = 0 \tag{18}$$

and $h(s)$ can be simplified as follows:

$$\begin{aligned} h(s) &= |s I_n - A| \begin{vmatrix} I_n & -(s I_n - A)^{-1} B J e^{-sT} \\ -E' & I_{2m} - F J e^{-sT} \end{vmatrix} \\ &= |s I_n - A| \\ &\times |I_{2m} - F J e^{-sT} - E'(s I_n - A)^{-1} B J e^{-sT}| \\ &= |s I_n - A| \\ &\times |I_{2m} - [E'(s I_n - A)^{-1} B + F] J e^{-sT}| \,. \end{aligned} \tag{19}$$

With the definition that

$$W(s) \equiv E'(s I_n - A)^{-1} B + F \,, \tag{20}$$

we have

$$\begin{aligned} h(s) &= |s I_n - A| \, |I_{2m} - W(s) J e^{-sT}| \\ &= |s I_n - A| \, |I_{2m} - J e^{-sT} W(s)| \,. \end{aligned} \tag{21}$$

The matrix $W(s)$ is called the scattering matrix of the $2m$-port system consisting of the network without the transmission lines. Furthermore, as shown in Appendix 2, $W(s)$ can be measured for $s = i\omega$, $-\infty < \omega < \infty$. We want a stability criterion in terms of the measurable quantities, W, J, and T.

We need to determine whether $h(s)$ has any roots in the half-plane $\mathrm{Re}(s) \geqslant 0$. The argument principle from the theory of complex variables will be used:

$$\frac{1}{2\pi} \arg h(s) \Big|_{\mathcal{C}} = Z - P \,, \tag{22}$$

where $\mathcal{C}$ is a closed contour taken in the counterclockwise direction and $Z - P$ is the number of zeros minus the number of poles contained in the interior of $\mathcal{C}$.

The specific contour we take is $\mathcal{C}_\rho$ shown in Fig. 3 where ρ is large. Since $W(i\omega)$ can be measured, the change in argument of $g(s) \equiv |I_{2m} - Je^{-sT} W(s)|$ on the ω-axis from $i\rho$ to $-i\rho$ is a matter of measurement. On the part of $\mathcal{C}_\rho$ where $s = \rho e^{i\theta}$, $-\pi/2 \leqslant \theta \leqslant \pi/2$, since $W(s) = E'(sI_n - A)^{-1}B + F = F + \mathcal{O}(|s|^{-1})$ as $|s| \to \infty$, we need only test $|I_{2m} - Je^{-sT}F|$. Note: If $F = 0$, then we just have $|I_{2m}| = 1$. Hence no test, since there is no change in argument. It is clear that this function changes from $|I_{2m} - Je^{i\rho T}F|$ to $|I_{2m} - Je^{-\rho T}F|$ to $|I_{2m} - Je^{-i\rho T}F|$, so the total change in argument on this part of the contour is $2 \times \arg |I_{2m} - Je^{-i\rho T}F| \pm 2k\pi$ for some integer k. It is possible to determine k since the matrices J and T are known from the transmission lines and $F = W(i\infty)$. The fact that the

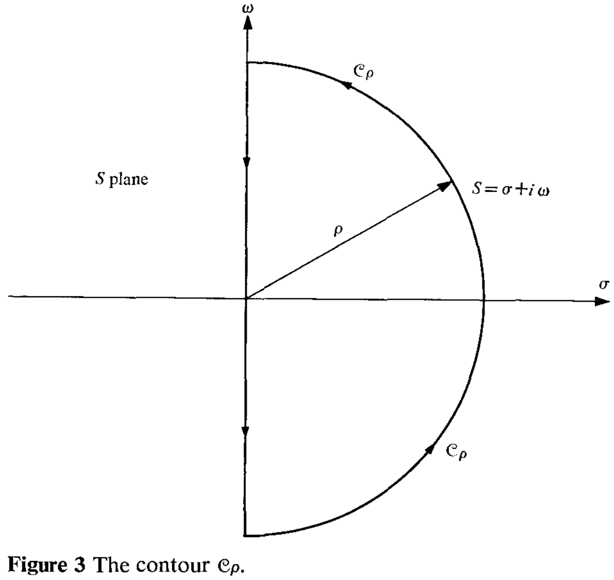

Figure 3 The contour $\mathcal{C}_\rho$.

function $g(s)$ has a change in its argument on the part of the contour in the right-half plane is due to the presence of the e^{-sT} factor (i.e., due to the transmission lines) and is a result not normally encountered in Nyquist criterion.

There is one difficulty with using the argument principle for meromorphic functions that was pointed out by Desoer.[5] Consider the function $e^t \sin(e^t)$ which has a Laplace transform analytic for all finite s. By taking a contour such as $\mathcal{C}_\rho$ for ρ sufficiently large, we would never encircle any poles and, therefore, might conclude that $e^t \sin(e^t) \to 0$ as $t \to \infty$. Of course the trouble is that $\mathcal{L}[e^t \sin(e^t)]$ ($\mathcal{L}$ = Laplace transform) has poles at $s = \infty$. We must therefore be careful about isolated zeros at infinity in our situation. The question is: Can $|I_{2m} - Je^{-sT}W(s)|$ have an isolated zero at $s = \infty$? The answer is no, because of the special form of the function $|I_{2m} - Je^{-sT}W(s)|$. Since $W(s) = F + \mathcal{O}(|s|^{-1})$ as $|s| \to \infty$, by Theorem 12.7, Ref. 4, we have that the zeros of $|I_{2m} - Je^{-sT}W(s)|$ are asymptotic to the zeros of $|I_{2m} - Je^{-sT}F|$ for $|s| \to \infty$. By Theorem 12.4, Ref. 4, all the zeros lie in a strip $|\mathrm{Re}(s)| < C_1$ where C_1 is some finite positive constant. If $|I_{2m} - Je^{-sT}F|$ has a zero at infinity, then there must exist some constant γ, $|\gamma| < C_1$, such that $g(\omega) \equiv |I_{2m} - Je^{-(\gamma + i\omega)T}F| \to 0$ as $\omega \to \infty$. However, $g(\omega)$ is an almost (quasi-) periodic function of ω and as such can't approach zero as $\omega \to \infty$ [unless, of course, $g(\omega) \equiv 0$, which is not the case here].

Finally, the total change in argument of $g(s)$ gives $N = Z - P$ for $g(s)$. However, $g(s)$ has a pole at some point s_j only if $|s_j I_n - A| = 0$. Thus $P \leqslant Z_1$, where Z_1 is the number of zeros of $|sI_n - A|$ in the right-half plane. Hence the number of zeros Z of $g(s)$ in the right-half plane is bounded by $Z = N + P \leqslant N + Z_1$. However, the fact that $W(i\omega)$ can be measured means essentially* that the lumped circuits are stable when there is no input, i.e., when $y = 0$. Therefore it is not unreasonable to assume that $Z_1 = 0$, which we now do. Thus the number Z_2 of zeros of $h(s)$ in the right-half plane is $Z_2 \leqslant Z_1 + Z = Z \leqslant N$. If $N = 0$, then $Z_2 = 0$. We therefore have:

• *Theorem 2*

If $|sI_n - A|$ has no zeros in the right-half plane, $\mathrm{Re}\,(s) \geqslant 0$, and if $|g(i\omega)| \geqslant \delta > 0$ for $\omega \longrightarrow \infty$, then the system (9) is exponentially stable if $\arg |I_{2m} - Je^{-sT}W(s)|_{\mathcal{C}_\rho} = 0$ for all ρ sufficiently large.

Note that in order to apply Theorem 1 in obtaining Theorem 2 we must require that the zeros of $h(s)$ do not accumulate on the imaginary axis. There probably is no general method for determining this a priori. However, if this did happen, it would be practically impossible to measure $\arg g(s)|_{\mathcal{C}_\rho}$ since it would mean that $g(i\omega_k) \rightarrow 0$ for some sequence $\{\omega_k\}$, $\omega_k \longrightarrow \infty$. Thus the measurements would have to be impossibly precise for large ω. The condition in Theorem 2 that $|g(i\omega)| \geqslant \delta > 0$ for $\omega \longrightarrow \infty$ is a practical requirement for ruling out this situation.

The requirement that the roots of $h(s)$ be in the left-half plane, $\mathrm{Re}\,(s) \leqslant -\delta < 0$, is necessary in general as pointed out by Snow.[7] He constructed an example of a linear, homogeneous, difference-differential equation where the equilibrium point is unstable (solutions grow like some power of t) even though the condition $\mathrm{Re}\,(s_j) < 0$ is met. For practical applications this is a fine point since one would very rarely encounter a system where stability is such a delicate matter.

For completeness we give the stability formulation in terms of a general input y and output z because in many cases it is more convenient, in terms of writing the equations for a particular system, to have some freedom in choosing these variables. If $v_k = a_k y_k + b_k z_k$ and $i_k = c_k y_k + d_k z_k$, and the determinant of coefficients is nonsingular for $k = 1, \cdots, 2m$, then Eqs. (4) and (5) can be written as

$$\begin{aligned}(\alpha - K\gamma)y(t) + (\beta - K\delta)z(t) \\ = J[(\alpha + K\gamma)(y)(t-\tau) \\ + (\beta + K\delta)(z)(t-\tau)]\,,\end{aligned} \tag{23}$$

where $\alpha = \mathrm{diag}\,(a_1, \cdots, a_{2m})$, $\beta = \mathrm{diag}\,(b_1, \cdots, b_{2m})$, $\gamma = \mathrm{diag}\,(c_1, \cdots, c_{2m})$, $\delta = \mathrm{diag}\,(d_1, \cdots, d_{2m})$, and $K = \mathrm{diag}\,(Z_1, \cdots, Z_m, -Z_1, \cdots, -Z_m)$. Note that in the

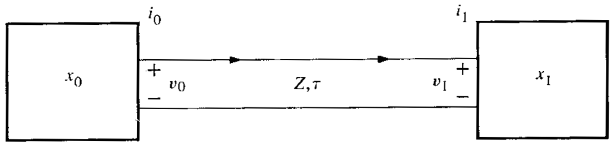

Figure 4 One-ports connected by transmission line.

scattering matrix description $\alpha = \beta = \frac{1}{2}I_{2m}$ and $-\gamma = \delta = \frac{1}{2}K^{-1}$. The remaining equations have the general form†

$$\begin{aligned}\dot{x} &= \hat{A}x + \hat{B}y\,, \\ z &= \hat{E}'x + \hat{F}y\,.\end{aligned} \tag{24}$$

The stability criterion is then stated in terms of the roots of the function

$$\begin{aligned}\Delta(s) = |sI_n - \hat{A}|\,|\alpha - K\gamma - J(\alpha + K\gamma)e^{-sT} \\ + [(\beta - K\delta) - J(\beta + K\delta)e^{-sT}]\hat{W}(s)|\,,\end{aligned} \tag{25}$$

where $\hat{W}(s) = \hat{E}'(sI_n - \hat{A})^{-1}\hat{B} + \hat{F}$ is the transfer matrix of the $2m$-port. This equation is derived in Appendix 3.

If $a_k = 1$, $b_k = c_k = 0$, and $d_k = 1$, then the input is voltage and the output is current. Hence $\hat{W}(s) = Y(s)$, the admittance matrix. Of course some combination of voltages and currents could be chosen and then one obtains $\hat{W}(s)$ as some hybrid matrix. For practical considerations we may not want to use the scattering matrix since it entails choosing the Z_k before measuring this matrix.

Examples

In the first example we consider two one-ports connected as shown in Fig. 4. We suppose that the equations of the terminating networks can be written as

$$\begin{aligned}\dot{x}_0 &= A_0x_0 + b_0i_0\,, \\ \dot{x}_1 &= A_1x_1 + b_1i_1\,,\end{aligned} \tag{26}$$

i.e., with i_0 and i_1 as input. The output v_0 and v_1 are obtained as

$$\begin{aligned}v_0 &= e_0'x_0 + f_0i_0\,, \\ v_1 &= e_1'x_1 + f_1i_1\,.\end{aligned} \tag{27}$$

(Here we are not using the scattering matrix because it requires an a priori choice of Z before measuring the scattering matrix.) These are the equations corresponding to (6) and (7); here

$$x = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, \quad A = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}, \quad B = \begin{pmatrix} b_0 & 0 \\ 0 & b_1 \end{pmatrix},$$

$$E = \begin{pmatrix} e_0 & 0 \\ 0 & e_1 \end{pmatrix}, \quad \text{and} \quad F = \begin{pmatrix} f_0 & 0 \\ 0 & f_1 \end{pmatrix}.$$

* It is possible that $|sI_n - A|$ could have a zero with $\mathrm{Re}\,(s) \geq 0$ and yet, from measurements at the $2m$ ports, this could not be observed. This question is related to the concept of complete controllability and observability, as defined by Kalman,[8] of the $2m$-port. A result of Kalman is that a system is irreducible if and only if it is completely controllable and observable. If this condition does not hold, then it would mean that there is a smaller system which would have the same behavior. Since such a zero is not observed in the output, we would not observe this oscillation in the output and hence it would not affect the stability of the quantities of interest.

† It is not always possible to write the equations in this form for an arbitrary choice of α, β, γ, and δ. However, the scattering matrix description always exists (see Ref. 3) as well as some hybrid description.

Since $y = (i_0, i_1)'$ and $z = (v_0, v_1)'$, we write Eqs. (4) and (5) in the form (8) as

$$Dy(t) + Cz(t) = -J[D(y)(t-\tau) - C(z)(t-\tau)], \quad (28)$$

where

$$D = \begin{pmatrix} -Z & 0 \\ 0 & Z \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The transfer matrix is

$$W(i\omega) = \begin{pmatrix} -Z_0(i\omega) & 0 \\ 0 & Z_1(i\omega) \end{pmatrix}, \quad (29)$$

where Z_0 and Z_1 are the measured impedances at the ports.

Applying Theorem 1, the characteristic function is

$$\begin{aligned}\Delta(s) &= |sI_n - A|\,|D + CF|^{-1} \times |D + CW + Je^{-sT}(D - CW)| \\ &= |sI - A_0|\,|sI - A_1| \\ &\quad \times \left|\begin{pmatrix} -Z & 0 \\ 0 & Z \end{pmatrix} + \begin{pmatrix} -f_0 & 0 \\ 0 & f_1 \end{pmatrix}\right|^{-1} \\ &\quad \times \begin{vmatrix} -(Z_0 + Z) & -e^{-s\tau}(Z_1 - Z) \\ e^{-s\tau}(Z_0 - Z) & Z_1 + Z \end{vmatrix} \\ &= |sI - A_0|\,|sI - A_1|(Z + f_0)^{-1}(Z + f_1)^{-1} \\ &\quad \times (Z_0 + Z)(Z_1 + Z)(1 - \Gamma_0\Gamma_1 e^{-2s\tau}), \quad (30)\end{aligned}$$

where $\Gamma_0 = (Z_0 - Z)/(Z_0 + Z)$ and $\Gamma_1 = (Z_1 - Z)/(Z_1 + Z)$ are the reflection coefficients at the ports. Thus in applying Theorem 2, if we assume that each terminating circuit is open-circuit stable, we need to test the function

$$g(s) = (Z_0 + Z)(Z_1 + Z)(1 - \Gamma_0\Gamma_1 e^{-2s\tau}) \quad (31)$$

for its change in argument on the contour $\mathcal{C}$. If we require that the first two factors of $g(s)$ have no zeros or poles for $\mathrm{Re}(s) \geqslant 0$ and that the third factor have positive real part, then we can interpret these conditions: That each terminating circuit when loaded with Z should be stable and that the product of the reflection coefficients, $\Gamma_0\Gamma_1$, should be less than one in absolute value.

The case where $Z_j(\pm i\omega) \to 0$ as $\omega \to \infty$ and $\mathrm{Re}[Z_j(\pm i\omega)] > 0$, $j = 0, 1$, is a borderline example since this would mean that the zeros $\{s_k\}$ of $g(s)$, while satisfying $\mathrm{Re}(s_k) < 0$, would accumulate at $\pm i\infty$. This is just the situation in which it is impossible to say anything about stability.

If we let $\tau \to 0$ in the above example, then

$$h(s) = |sI - A_0|\,|sI - A_1|(Z + f_0)^{-1}(Z + f_1)^{-1} \times (2Z)(Z_0 + Z_1). \quad (32)$$

The stability criterion would be that the sum of the input impedances, $Z_0 + Z_1$, should have no roots for $\mathrm{Re}(s) \geqslant 0$. This says that any lumped network that can be separated into two one-ports is stable if each port is open-circuit stable and if the sum of their input impedances satisfies the Nyquist criterion.

In the next example we consider a network which has the following property of evenness: Form the graph obtained by replacing lumped networks by nodes and transmission lines by branches; if this graph has any loops, the number of branches in any loop should be even. Then the matrix $W(i\omega)$ can be put in the form

$$W = \begin{pmatrix} -G_0 & 0 \\ 0 & G_1 \end{pmatrix}, \quad (33)$$

where G_0 and G_1 are $m \times m$ matrices (m is the number of transmission lines). Also assume that all the inputs y are voltages and all the outputs z are currents. Then Eqs. (8) can be written with the matrices

$$D = I_{2m}, \quad C = \begin{pmatrix} -Z & 0 \\ 0 & Z \end{pmatrix}, \quad \text{and} \quad J = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}.$$

Combining these matrices, we obtain

$$\hat{h}(s) = |sI_n - A|\hat{g}(s), \quad (34)$$

$$\begin{aligned}\hat{g}(s) &= |D + CW - Je^{-sT}(D - CW)| \\ &= \begin{vmatrix} I_m + ZG_0 & -e^{-sT_1}(I_m - ZG_1) \\ -e^{-sT_1}(I_m - ZG_0) & I_m + ZG_1 \end{vmatrix} \\ &= |I_m + ZG_0|\,|I_m + ZG_1|\,|I_m - e^{-sT_1} \\ &\quad \times (I_m - ZG_1)(I_m + ZG_1)^{-1}e^{-sT_1} \\ &\quad \times (I_m - ZG_0)(I_m - ZG_1)^{-1}|, \quad (35)\end{aligned}$$

where $T_1 = \mathrm{diag}(\tau_1, \cdots, \tau_m)$.

Note that this function only requires evaluating determinants of order m and not of order $2m$. If G_0 and G_1 are diagonal, then the last determinantal factor in Eq. (35) reduces to

$$|\cdots| = \prod_{k=1}^{m} (1 - e^{-2s\tau_k}\Gamma_{0k}\Gamma_{1k}),$$

where Γ_{0k} and Γ_{1k} are the reflection coefficients at the beginning and end of the kth transmission line, respectively.

Discussion

There are two considerations that are pertinent to the practicality of the criteria given in Theorems 1 and 2. First, it could be that the exact equations describing the lumped part of the network are not easily determined, i.e., the matrices A, B, E, and F are not easily found. For example, the lumped networks could contain transistors or could be integrated circuits where many difficult measurements and approximations would have to be made to obtain a good lumped-parameter model. Since the stability criterion in Theorem 2 is based only on external measurements, i.e., input-output measurements, it is possible that it would be considerably easier to make these measurements if the network exists physically, rather than to construct a model and compute determinants using A, B, E, and F. On the other hand, the scattering matrix (or transfer matrix) must be measured at a number of frequencies and with sufficient accuracy. If the network consists of many copies of the same circuit interconnected by transmission lines, then these measurements would not be difficult.

Second, having the scattering matrix obtained either from measurements or from A, B, E, and F, the time required to compute the determinant $|I_{2m} - Je^{-sT}W(i\omega)|$ should be considered. This is the determinant of a $2m \times 2m$ matrix where m is the number of transmission lines in the network. The number of operations required to compute the determinant of an $n \times n$ complex matrix, using the method of Gaussian elimination, is $\frac{1}{2}n(n-1)$ complex divisions, $\frac{1}{3}n(n^2+2)$ complex multiplications, and $\frac{1}{3}n(n^2-1)$ complex additions. Again this must be done at a number of frequencies.

There have recently been developed some methods for solving systems of linear equations, $Ax = b$, where the matrix of coefficients, A, is a sparse matrix, i.e., one with many zeros. These methods can also be used for computing determinants since they use Gaussian elimination. Thus it may be advantageous to compute the determinant of the large $(n + 2m)$-dimensional system $h(i\omega)$ rather than the smaller $2m$-dimensional system $g(i\omega)$ since the latter involves $(sI - A)^{-1}$ which is not necessarily sparse.

Appendix 1 — Theorems

We consider a system of the form

$$\dot{x}(t) = Ax(t) + B(y)(t-\tau)\,, \tag{A1}$$

$$y(t) = C'x(t) + D(y)(t-\tau)\,, \tag{A2}$$

where $\tau = (\tau_1, \cdots, \tau_m, \tau_1, \cdots, \tau_m)'$, $\tau_1 > \tau_2 \cdots > \tau_m > 0$. The coefficients are constant matrices; A is $n \times n$, B and C are $n \times 2m$, and D is $2m \times 2m$. The notation $(y)(t-\tau)$ is explained following Eq. (8). Note that if it were possible to solve for $y(t-\tau)$ from (A1), then the system could be reduced by eliminating y. For example, if the left inverse of B exists, then

$$y(t-\tau) = B_l^{-1}[\dot{x}(t) - Ax(t)]$$

and the system (A1), (A2) becomes

$$\dot{x}(t) - Ax(t) - B(C'x + DB_l^{-1}(\dot{x} - Ax))\,(t-\tau) = 0\,. \tag{A3}$$

In general (A3) is a system of difference-differential equations of neutral type because of the presence of the $DB_l^{-1}(\dot{x})(t-\tau)$ term. If $D = 0$, then we get the retarded type.

From this point on in the appendix we make extensive use of the book by Bellman and Cooke,[4] and the proofs will be completed only to the point where the remaining part of the proof follows essentially that of the corresponding theorem in Bellman and Cooke.

- *Asymptotic behavior of characteristic roots*

We first prove that the roots of the characteristic equation for (A1), (A2) have no advanced chains, i.e., there is no sequence of roots $\{s_j\}$ such that $\mathrm{Re}(s_j) \to \infty$ as $j \to \infty$. The characteristic equation corresponding to (A1), (A2) is

$$\Delta(s) \equiv \begin{vmatrix} sI_n - A & -Be^{-sT} \\ -C' & I_{2m} - De^{-sT} \end{vmatrix} = 0\,. \tag{A4}$$

Theorem A1

The roots of (A4) satisfy the condition $\mathrm{Re}(s) \leqslant c$ for some $c < \infty$.

Proof of Theorem A1

The form of $\Delta(s)$ is

$$\Delta(s) = \sum_{j=0}^{n} p_j(s)e^{-\beta_j s} = e^{-\beta_0 s}\sum_{j=0}^{n} p_j(s)e^{(\beta_0-\beta_j)s} \equiv e^{-\beta_0 s}\sum_{j=0}^{n} p_j(s)e^{\alpha_j s}\,,$$

where $0 = \beta_n < \beta_{n-1} < \cdots < \beta_0$ and $p_j(s)$ is a polynomial in s of degree m_j. We use the results in Ref. 4, pages 410 to 416, to determine the asymptotic behavior of the roots of $\Delta(s) = 0$ for large $|s|$. The theory states that the roots are asymptotically located in the regions defined by

$$|\mathrm{Re}\,(s + \mu_r \log s)| \leqslant c_1\,, \tag{A5}$$

where μ_r is the slope of a line segment obtained as follows: Plot the points $P_j = (\alpha_j, m_j)$ in a Cartesian plane; the line segments L_r of the upper boundary of the convex hull of these points have slope μ_r.

From (A5) it follows that if all the μ_r are nonnegative, then all the roots lie in a left half-plane. Thus it is sufficient to show that $m_n \geqslant m_j$ or that the polynomial $p_n(s)$ has the largest degree. For this we expand $\Delta(s)$ using the Laplace expansion formula:

$$\Delta(s) = |sI_n - A|\,|I_{2m} - De^{-sT}| + \cdots\,,$$

where the unwritten terms have polynomial coefficients of degree $\leqslant n-1$. Clearly $p_n(s) = s^n + \mathcal{O}(s^{n-1})$ and hence $m_n = n \geqslant m_j$. Since it is possible that $m_j = n, j \neq n$, there are, in general, neutral, as well as retarded, root chains of $\Delta(s)$.

- *Existence and uniqueness*

Theorem A2

Consider the system (A1), (A2) where the initial conditions $y(t) = g(t) \in C[-\tau, 0]$* and $x(0) = x_0$ are given such that $g(0) = C'x_0 + D(g)(-\tau)$. Then there exist unique solutions $x(t) \in C_1[0, \infty)$ and $y(t) \in C[0, \infty)$.

Proof of Theorem A2

We can demonstrate existence by standard continuation arguments. Thus for $0 \leqslant t \leqslant \tau_m$ (τ_m is the shortest delay > 0),

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} B(g)(s - \tau)ds \tag{A6}$$

and

$$y(t) = C'x(t) + D(g)(t - \tau) . \tag{A7}$$

Clearly x and y are in $C[0, \tau_m)$ since $x(0^+) = x_0$ and $y(0^+) = C'x_0 + D(g)(-\tau) = g(0)$. The right-hand side of (A6) is obviously continuously differentiable; hence $x \in C_1[0, \tau_m)$. Since x and y are defined for $[0, \tau_m]$ and $[-\tau, \tau_m]$, respectively, we can continue the solution via the formulas

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} B(y)(s - \tau)ds , \tag{A8}$$

$$y(t) = C'x(t) + D(y)(t - \tau) , \tag{A9}$$

for $\tau_m \leqslant t \leqslant 2\tau_m$. By a straightforward calculation we find $x(\tau_m^-) = x(\tau_m^+)$ and $y(\tau_m^-) = y(\tau_m^+)$. Again the right-hand side of (A8) is differentiable and, in taking its derivative, we see that $\dot{x}(\tau_m^-) = \dot{x}(\tau_m^+)$. Clearly this process can be continued indefinitely and hence $x \in C_1[0, \infty)$ and $y \in C[0, \infty)$. Uniqueness is obvious.

A more general theorem is established if the requirement $g(0) = C'x_0 + D(g)(-\tau)$ is not made. Then only piecewise continuity is obtained for y and $\dot{x}$.

- *A priori estimates*

We shall obtain some growth estimates for $\|y(t)\|$ and $\|x(t)\|$ in order to be able to take Laplace transforms. For a vector the notation $\|x\|$ means $(x, x)^{\frac{1}{2}}$.

Theorem A3

The solutions $x(t)$, $y(t)$ of Eqs. (A8) and (A9) satisfy the relations

$$\|x(t)\| \leqslant k_1 e^{\alpha t} \quad \text{and} \quad \|y(t)\| \leqslant k_2 e^{\alpha t} , \tag{A10}$$

where k_1, k_2, and α are constants depending on the system and on the initial data.

* This means that $g_j \in C[-\tau_j, 0]$. The notation $C[\alpha, \beta]$ means the set of continuous functions defined on the closed interval $[\alpha, \beta]$. Similarly $C[\alpha, \beta)$ means the set of functions with continuous first derivatives defined on the half-closed interval $[\alpha, \beta)$.

Proof of Theorem A3

The proof, which follows along the lines suggested in Bellman and Cooke,[4] Section 6.5, Exercise 5, is left to the reader.

- *Representation of the solution by Laplace transform*

Since it has been shown that $x(t)$ and $y(t)$ are exponentially bounded, we can take the Laplace transforms of Eqs. (A1) and (A2):

$$\int_0^\infty \dot{x}(t)e^{-st}dt = \int_0^\infty Ax(t)e^{-st}dt + \int_0^\infty B(y)(t - \tau)e^{-st}dt ,$$

$$\int_0^\infty y(t)e^{-st}dt = \int_0^\infty C'x(t)e^{-st}dt + \int_0^\infty D(y)(t - \tau)e^{-st}dt ,$$

where $\mathrm{Re}(s) > \alpha$. By straightforward calculations we obtain

$$-x_0 + s\tilde{x}(s) = A\tilde{x}(s) + Be^{-sT}[\tilde{y}(s) + Y(s)] ,$$

$$\tilde{y}(s) = C'\tilde{x}(s) + De^{-sT}[\tilde{y}(s) + Y(s)] ,$$

where the tilde indicates Laplace transform and

$$Y(s) = [Y_1(s), \cdots, Y_{2m}(s)]' ,$$

$$Y_j(s) = \int_{-\tau_j}^0 g_j(t)e^{-st}dt .$$

Thus

$$H(s)\begin{bmatrix} \tilde{x}(s) \\ \tilde{y}(s) \end{bmatrix} = \begin{bmatrix} x_0 + Be^{-sT}Y(s) \\ De^{-sT}Y(s) \end{bmatrix},$$

where

$$H(s) = \begin{pmatrix} sI_n - A & -Be^{-sT} \\ -C' & I_{2m} - De^{-sT} \end{pmatrix} .$$

By using the result of Theorem A1, one can show that $H^{-1}(s)$ exists for $\mathrm{Re}(s)$ large enough and hence we can solve for $[\tilde{x}(s), \tilde{y}(s)]'$ and take the inverse Laplace transform to obtain

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \int_{\mathfrak{C}} H^{-1}(s)\begin{bmatrix} x_0 + Be^{-sT}Y(s) \\ De^{-sT}Y(s) \end{bmatrix}e^{st}ds . \tag{A11}$$

Here $\mathfrak{C}$ denotes the contour shown in Fig. A1.

- *Stability*

Using (A11) we can now obtain the required stability theorem.

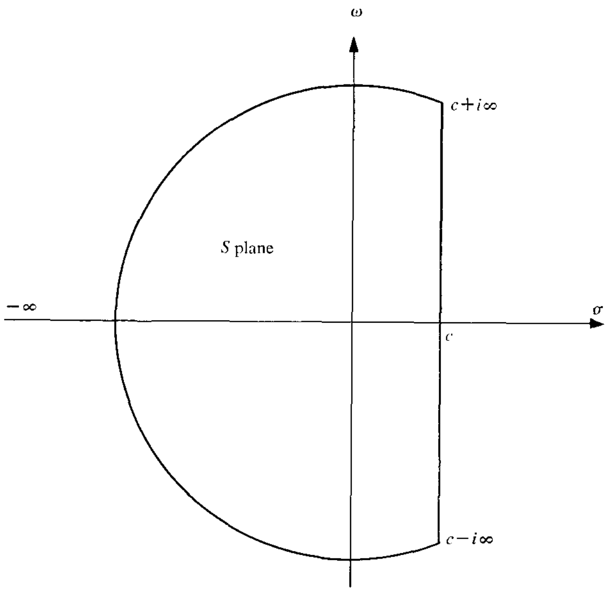

Figure A1 The contour of integration for Eq. (A11).

Theorem A4
If the roots of $\Delta(s) = 0$ satisfy $\mathrm{Re}(s_j) \leqslant \delta$, then the solutions $x(t)$, $y(t)$ of (A1), (A2) satisfy

$$[\|x(t)\|^2 + \|y(t)\|^2]^{1/2} = \mathcal{O}(e^{\delta' t}) \quad \text{as} \quad t \to \infty \ ,$$

where $\delta' > \delta$.

Proof of Theorem A4
Since $H^{-1}(s)$ has no poles where $\mathrm{Re}(s) > \delta$, we can alter the contour of integration (where c is replaced by $\delta' > \delta$) without changing the value of the integral. To obtain the result of the Theorem we rely on Theorems 12.19 and 12.20 in Ref. 4. The proof of Theorem A4 is the same except for the assumption that $\det A_0 \neq 0$, which is not needed in view of our results in Theorems A1 to A3. It should be noted especially that Theorem A1 gives the following result which is needed in proving Theorem 12.19:

$$\Delta(s) = s^n | I_{2m} - De^{-sT}| + \mathcal{O}(|s|^{n-1}), \ |\mathrm{Im}\,(s)| \to \infty \ .$$

Appendix 2 — Measurement of the scattering matrix

To measure $W(i\omega)$ for $-\infty < \omega < \infty$ we form the network shown in Fig. 2. Then let $v_{ij} = \epsilon\delta_{jk} \cos \omega t$, $1 \leqslant k \leqslant 2m$, $1 \leqslant j \leqslant 2m$, and measure v_l, $1 \leqslant l \leqslant 2m$, after the network has settled down to steady state. We obtain

$$v_l = a_{lk} \cos (\omega t + \theta_{lk})$$

and, using the definition of v_{rl}, we have

$$v_{rl} = 2v_l - v_{il} = 2a_{lk} \cos (\omega t + \theta_{lk}) - \epsilon\delta_{lk} \cos \omega t \ .$$

This we express as

$$v_{rl} = \rho_{lk} e^{i\omega t} + \bar{\rho}_{lk} e^{-i\omega t} \ , \tag{A12}$$

where the bar indicates complex conjugate and

$$\rho_{lk} = a_{lk} e^{i\theta_{lk}} - \tfrac{1}{2}\epsilon\delta_{lk} \ .$$

On the other hand, from the differential equations assuming steady-state conditions, i.e.,

$$x = x_0 e^{i\omega t} + \bar{x}_0 e^{-i\omega t}, \quad z = z_0 e^{i\omega t} + \bar{z}_0 e^{-i\omega t},$$

we have

$$\begin{aligned} i\omega x_0 e^{i\omega t} - i\omega \bar{x}_0 e^{-i\omega t} &= A(x_0 e^{i\omega t} + \bar{x}_0 e^{-i\omega t}) \\ &\quad + B(y_0 e^{i\omega t} + \bar{y}_0 e^{-i\omega t}) \ , \end{aligned}$$

$$\begin{aligned} z_0 e^{i\omega t} + \bar{z}_0 e^{-i\omega t} &= E'(x_0 e^{i\omega t} + \bar{x}_0 e^{-i\omega t}) \\ &\quad + F(y_0 e^{i\omega t} + \bar{y}_0 e^{-i\omega t}) \ , \end{aligned}$$

where $y_{0l} = \frac{1}{2}\epsilon\delta_{lk}$, $1 \leqslant l \leqslant 2m$.

Equating coefficients of $e^{i\omega t}$ and $e^{-i\omega t}$ and eliminating x_0 we obtain

$$\begin{aligned} z_0 e^{i\omega t} &= [E'(i\omega I_n - A)^{-1} B + F] y_0 e^{i\omega t} \\ &= W(i\omega) y_0 e^{i\omega t} \ . \end{aligned}$$

Then

$$\begin{aligned} z(t) &= z_0 e^{i\omega t} + \bar{z}_0 e^{-i\omega t} \\ &= v_r(t) = W(i\omega) y_0 e^{i\omega t} + W(-i\omega) y_0 e^{-i\omega t} \end{aligned}$$

or

$$v_r(t) = \tfrac{1}{2}\epsilon[W(i\omega)e^{i\omega t} + W(-i\omega)e^{-i\omega t}] e_k \ , \tag{A13}$$

where e_k is the vector with $e_{kl} = \delta_{kl}$. The result of equating (A12) and (A13) is

$$[W(i\omega)]_{lk} = \frac{2}{\epsilon}\rho_{lk} = \frac{2}{\epsilon} a_{lk} e^{i\theta_{lk}} - \delta_{lk} \ .$$

Appendix 3 — Characteristic equation for general input-output

We have the equations

$$\dot{x} = \hat{A}x + \hat{B}(y)(t - \tau) \ ,$$

$$z = \hat{E}'x + \hat{F}y \ ,$$

$$\begin{aligned} &(\alpha - K\gamma)y(t) + (\beta - K\delta)z(t) \\ &\quad = J[(\alpha + K\gamma)(y)(t - \tau) + (\beta + K\delta)(z)(t - \tau)] \ . \end{aligned}$$

If we take the Laplace transforms and eliminate z, we obtain

$$s\tilde{x} = \hat{A}\tilde{x} + \hat{B}e^{-sT}\tilde{y} \ ,$$

$$\begin{aligned} &(\alpha - K\gamma)\tilde{y} + (\beta - K\delta)(\hat{E}'\tilde{x} + \hat{F}\tilde{y}) \\ &\quad = J(\alpha + K\gamma)e^{-sT}\tilde{y} \\ &\quad + J(\beta + K\delta)e^{-sT}(\hat{E}'\tilde{x} + \hat{F}\tilde{y}) \ . \end{aligned}$$

Thus we arrive at the characteristic equation

$$\Delta(s) \equiv \begin{vmatrix} sI_n - \hat{A} & -\hat{B}e^{-sT} \\ (\beta - K\delta)\hat{E}' - J(\beta + K\delta)e^{-sT}\hat{E}' & \alpha - K\gamma + (\beta - K\delta)\hat{F} - J(\alpha + K\gamma)e^{-sT} - J(\beta + K\delta)e^{-sT}\hat{F} \end{vmatrix} = 0.$$

Expansion of the determinant yields

$$\begin{aligned} \Delta(s) = |sI_n - \hat{A}|\,|\alpha - K\gamma + (\beta - K\delta)\hat{F} \\ - J(\alpha + K\gamma)e^{-sT} - J(\beta + K\delta)e^{-sT}\hat{F} \\ + [\beta - K\delta - J(\beta + K\delta)e^{-sT}]\hat{E}' \\ \times (sI_n - \hat{A})^{-1}\hat{B}e^{-sT}| \\ = |sI_n - \hat{A}|\,|\alpha - K\gamma - J(\alpha + K\gamma)e^{-sT} \\ + [\beta - K\delta - J(\beta + K\delta)]e^{-sT}\hat{W}|, \end{aligned}$$

where $\hat{W} = \hat{E}'(sI_n - \hat{A})^{-1}\hat{B} + \hat{F}$.

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