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Time-Optimal Control of a Moving-Coil Linear Actuator

Abstract: The time-optimal control problem for a moving-coil linear actuator has been worked out by means of functional analysis and a related graphical procedure requiring only data from an impulse response. On the basis of experience with a test model, there is good correlation between the theoretical and the experimental methods. While the latter is accurate over only very short distances, the usefulness of the technique can be extended as needed by operating the system in piecewise linear fashion. The nonlinearity of the coil inductance can be handled under computer control by including in the program the inductances for successive segments of travel.

Introduction

In engineering applications where mechanisms must operate repetitively at high speed and with great precision, time-optimal control is often essential. This is the case with the moving-coil linear actuator built and tested in the course of the work reported here. In general, time-optimal control requires finding that input function, with suitable constraints, which causes a dynamic system to pass from some initial state to a desired final state in minimum time. In this case, the change of state involves a travel range of one inch, and the load is moved in 10-mil steps. As is typical, the magnitude of the input (specifically, the voltage to be applied to the coil) is bounded, and the control is bang-bang.

As Friedland[1] points out, two related factors stimulate, in a profitable way, attention to problems of optimum control: ". . . the increasing demand for systems of high performance, and . . . the availability of computing machinery which makes feasible the calculations attendant on the design of an optimum control system." Since the time-optimal problem can be approached in a number of ways, and since each approach involves special mathematical problems, we are particularly interested in methods whose accompanying problems are amenable to computer solution. Such amenability is not typical of some widely known approaches described in the literature.[2–6] Pontriagin's maximum principle, for example, usually entails a two-point boundary value problem not readily handled by computer methods, and the same thing is true of variational methods.

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In addition, these approaches (and other familiar ones such as dynamic programming and gradient methods) do not in general offer a straightforward procedure for determining the optimal control explicitly. In the Pontriagin formulation, for instance, the initial values of the adjoint variables which determine the switching function are not known a priori, and often must be found by trial and error.

The approach taken in this paper is an impulse response method. This method has the advantage that the mathematical problem is well adapted to experimental and computer solution. Such an approach is especially suited to the practical application of interest to us here—finding out how to achieve time-optimal control in a moving-coil linear actuator—for two reasons:

First, it affords a straightforward graphical procedure for determining the times at which the control is reversed, i.e., the switching times (when the characteristic roots of the system are nonpositive real numbers and the control is bang-bang, which is the case with the moving-coil linear actuator).

Second, system parameters need not be determined accurately, since an impulse response, determined experimentally, satisfies this approach. That is, the optimal control can be calculated without explicit knowledge of the system dynamics, since the experimentally determined impulse response can be fed directly into a computer. This offers a simple means of updating the control for changes in system parameters due to some characteristic changes in the components.

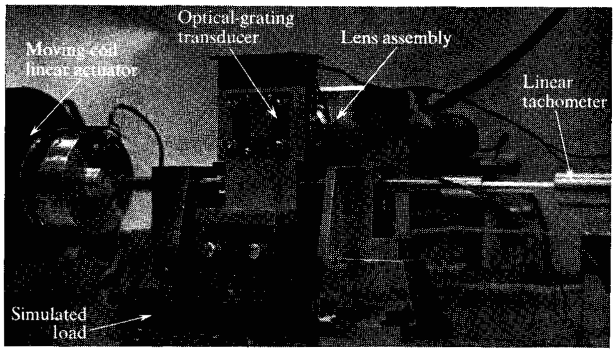


Figure 1 Test model of the moving-coil linear actuator system.

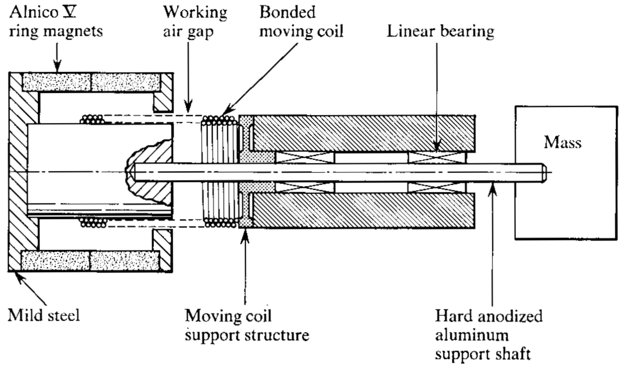


Figure 2 Actuator assembly.

For the moving-coil linear actuator system, stiction and coulomb friction effects are present, but properly chosen initial conditions can remove these nonlinearities in the mathematical formulation of the time-optimal control problem. In a real-time simulation of the system these effects must, of course, be included.

This paper includes, first, a physical description of the moving-coil linear actuator, with a brief qualitative discussion of its dynamic characteristics. Next, the mathematical representation of the system is derived as a set of third-order ordinary differential equations. The following section formulates the time-optimal control problem and gives the form of the control and a graphical procedure for finding the switching times. Finally, the theoretical basis is discussed for experimentally determining the impulse response, and the results obtained with the test model of the moving-coil linear actuator system are presented. From these data the switching times are calculated and compared with those found by adjusting the control of the test model until the optimal control was achieved.

System description

The system to be analyzed (Fig. 1) consists of a moving-coil linear actuator, a simulated load attached to an optical-grating transducer, a linear tachometer, a light source, and a lens assembly. The linear actuator, the simulated load, and the linear tachometer are connected by a common hardened shaft (nonmagnetic stainless steel) mounted in linear bearings. The load glides horizontally on hardened guide rails, with three roller bearings to minimize friction. The range of travel is one inch and the band is moved in 10-mil steps. Since the center of mass of the assembly is near the centerline of the shaft, the driving force acts on the center of mass of the total system. This prevents an induced moment, which would tend to cause oscillations in the system.

The actuator assembly (Fig. 2) includes Alnico V ring magnets, grain-oriented axially. The moving coil itself is made of two layers of No. 24 Formvar* magnet wire, bifilar wound for low inductance. The coil is supported at one end by an aluminum structure which transmits the generated force to the moving part of the actuator. The measured flux density in the working air gap was 8,200 gauss; the coils had a measured resistance of 1.55 ohms and inductance of 3.4 mH at 100 Hz.

The force generated by a current-carrying conductor in a magnetic field is given by: $F = 5.63\,(10^{-7})\,BDi$, where F is the force in pounds, B is the flux density in gauss, D is the length of the conductor in inches, and i is the current in amperes. The selected length of the conductor produces a force of approximately 2 pounds per ampere.

The optical position transducer, a glass grating with narrow parallel lines, alternately opaque and transparent, is attached to the load. The transducer is positioned between a light source and a lens system which magnifies (10×) the light that reaches two phototransistors. The servomechanism moves the load and the transducer until each phototransistor is aligned half on an opaque line and half on a transparent line. Since the two phototransistors then see the same level of "grey," the push-pull system is nulled and the system reaches the desired state.

A tachometer attached to the opposite side of the load mass provides a velocity feedback signal for the actuator control system. The control electronics generate the control signals to the actuator and provide negative feedback proportional to position and velocity errors, for control after the bang-bang control mode.

Since the complete assembly is rigid and the force applied passes through its center of mass, the assembly can, for practical purposes, be treated as a particle with one degree of freedom. The applied force is assumed to be proportional to the current in the conductor. It is assumed that dry, or coulomb, friction is the type generated.

* Shawinigan Products Corporation.

The dynamics of the moving coil are difficult to describe since, over a long stroke, the effective inductance varies with the coil position relative to the permanent magnet. Over a short distance, the dynamic characteristics can be considered linear. If the total travel of the coil is desired, the windings of the coil can be arranged to compensate for the variations in inductance. In the case described here, eddy currents have little effect on the moving coil dynamics, and therefore are neglected in this study.

State equations and boundary conditions

Neglecting the inductive nonlinearity over a short distance, the system can be treated as a linear, lumped-parameter system, as shown in Fig. 3. This assumes that the back emf caused by the motion of the coil relative to the permanent magnet is proportional to the coil velocity. Let $X_1 = X$, $X_2 = \dot{X}$, $X_3 = i$ (where the dot notation indicates differentiation with respect to time). The state equations then become

$$\begin{aligned} \dot{X}_1 &= X_2\,, \\ \dot{X}_2 &= \frac{\alpha_1}{m}X_3 - \frac{\mu}{m}f(X_2)\,, \\ \dot{X}_3 &= -\frac{R}{L}X_3 - \frac{\alpha_2}{L}X_2 + \frac{1}{L}e(t)\,, \end{aligned} \tag{1}$$

where

$$f(X_2) = \operatorname{sgn} X_2\,, \qquad X_2 \neq 0\,,$$

$$\text{and } -(1+\epsilon) \leqslant f(X_2) \leqslant +(1+\epsilon)\,, \quad X_2 = 0 \quad (\epsilon > 0)\,. \tag{2}$$

The boundary conditions for the time-optimal control problem of (1) are

$$\left.\begin{aligned} &X_1 = X_2 = X_3 = 0 && \text{at } t = 0 \\ &\text{and} \\ &X_1 = X_f,\ X_2 = X_3 = 0 && \text{at } t = T\,, \end{aligned}\right\} \tag{3}$$

where the final time, T, is not specified. For $0^+ \leq t \leq T^-$, X_2 is positive. The nonlinear friction factor $f(X_2)$ may be eliminated from (1) if the initial time is taken as the instant of impending motion, i.e., when the current in the coil causes a force on the actuator that just overcomes stiction. When this is done, (1) and (3) become

$$\begin{aligned} \dot{X}_1 &= X_2\,, \\ \dot{X}_2 &= \frac{\alpha_1}{m}X_3 - \frac{\mu}{m}\,, \\ \dot{X}_3 &= \frac{-R}{L}X_3 - \frac{\alpha_2}{L}X_2 + \frac{1}{L}e\,(t)\,, \end{aligned} \tag{4}$$

and then

$$X_1 = X_2 = 0^+\,, \qquad X_3 = \frac{\mu(1+\epsilon)}{\alpha_1} \quad \text{at } t = 0$$

and (5)

$$X_1 = X_f\,, \qquad X_2 = X_3 = 0 \qquad \text{at } t = T\,.$$

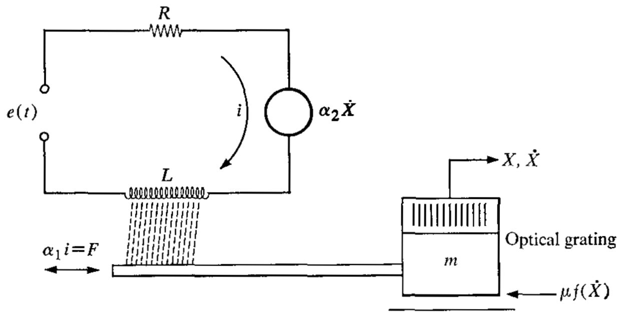


Figure 3 Moving-coil actuator as a linear system with lumped parameters.

$e(t)$ = applied voltage	L = inductance
$f(\dot{X})$ = stiction/coulomb friction characteristic	R = resistance
μ = coefficient of coulomb friction	m = load mass, including coil and shaft
i = current	X = load position
	α_1, α_2 = electromechanical coupling constants

Time-optimal control problem

We now turn to the statement and solution of the time-optimal control problem. Following the formulation given by Kranc and Sarachik,[6] we write (4) in vector form as

$$\dot{\mathbf{X}}(t) = \mathbf{A}\mathbf{X}(t) + \mathbf{B}\mathbf{u}(t)\,, \tag{6}$$

where

$$\mathbf{X}(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \alpha_1/m \\ 0 & -\alpha_2/L & -R/L \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{u}(t) = \begin{bmatrix} -\mu/m \\ (1/L)\ e(t) \end{bmatrix}$$

The solution of (6) is

$$\mathbf{X}(t) = \boldsymbol{\phi}(t)\ \mathbf{X}(0) + \int_0^t \boldsymbol{\phi}(t-\xi)\ \mathbf{B}\mathbf{u}(\xi)d\xi\,, \tag{7}$$

where $\boldsymbol{\phi}(t)$ is solution matrix and

$$\mathbf{X}(0) = \begin{bmatrix} X_1(0) \\ X_2(0) \\ X_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mu(1+\epsilon)/\alpha_1 \end{bmatrix}. \tag{8}$$

If we solve (6) with $\mathbf{u} = 0$ such that $\boldsymbol{\phi}(0)$ is the identity matrix, the components of $\boldsymbol{\phi}(t)$ are thus found to be

$$\phi_{11}(t) = 1,$$

$$\phi_{12}(t) = \frac{1}{r_2 - r_1}\left[\frac{r_2}{r_1}(1 - e^{-r_1 t}) - \frac{r_1}{r_2}(1 - e^{-r_2 t})\right],$$

$$\phi_{13}(t) = \frac{\alpha_1}{m}\left(\frac{1}{r_2 - r_1}\right) \times \left[\frac{1}{r_1}(1 - e^{-r_1 t}) - \frac{1}{r_2}(1 - e^{-r_2 t})\right],$$

$$\phi_{21}(t) = 0,$$

$$\phi_{22}(t) = \frac{1}{r_2 - r_1}\left[r_2 e^{-r_1 t} - r_1 e^{-r_2 t}\right],$$

$$\phi_{23}(t) = \frac{\alpha_1}{m}\left(\frac{1}{r_2 - r_1}\right)(e^{-r_1 t} - e^{-r_2 t}),$$

$$\phi_{31}(t) = 0,$$

$$\phi_{32}(t) = \frac{m}{\alpha_1}\left(\frac{r_1 r_2}{r_2 - r_1}\right)(e^{-r_2 t} - e^{-r_1 t}),$$

$$\phi_{33}(t) = \frac{1}{r_2 - r_1}(r_2 e^{-r_2 t} - r_1 e^{-r_1 t}), \tag{9}$$

where the characteristic roots of the system are 0, r_1, r_2 and

$$\begin{matrix} r_1 \\ r_2 \end{matrix} = \frac{R}{2L} \mp \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{\alpha_1 \alpha_2}{mL}}. \tag{10}$$

Let

$$\mathbf{u}_1 = \frac{-\mu}{m}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{11}$$

$$\mathbf{u}_2 = \begin{pmatrix} 0 \\ i \end{pmatrix}, \tag{12}$$

$$\mathbf{H}(t - \xi) = \boldsymbol{\phi}(t - \xi)\mathbf{B}\mathbf{u}_2, \tag{13}$$

and

$$u(t) = \frac{1}{L}e(t). \tag{14}$$

Substituting into (7), we obtain

$$\mathbf{X}(t) = \boldsymbol{\phi}(t)\mathbf{X}(0) + \int_0^t \boldsymbol{\phi}(t - \xi)\mathbf{B}\mathbf{u}_1 d\xi + \int_0^t \mathbf{H}(t - \xi)u(\xi)d\xi. \tag{15}$$

When we define

$$\mathbf{y}(t) \triangleq \mathbf{X}(t) - \boldsymbol{\phi}(t)\mathbf{X}(0) - \int_0^t \boldsymbol{\phi}(t - \xi)\mathbf{B}\mathbf{u}_1 d\xi, \tag{16}$$

then (15) becomes

$$\mathbf{y}(t) = \int_0^t \mathbf{H}(t - \xi)u(\xi)d\xi = \int_0^t \begin{bmatrix} \phi_{13}(t - \xi) \\ \phi_{23}(t - \xi) \\ \phi_{33}(t - \xi) \end{bmatrix} u(\xi)d\xi. \tag{17}$$

The initial and final values of $\mathbf{y}(t)$ are found from (5), (8), (9) and (16) to be

$$\mathbf{y}(0) = \mathbf{0}, \quad \text{and} \tag{18}$$

$$\mathbf{y}(T) \equiv \mathbf{y}_d = \begin{bmatrix} X_f \\ 0 \\ 0 \end{bmatrix} - \frac{\mu(1 + \epsilon)}{\alpha_1}\begin{bmatrix} \phi_{13}(T) \\ \phi_{23}(T) \\ \phi_{33}(T) \end{bmatrix} + \frac{\mu}{m}\int_0^T \begin{bmatrix} \phi_{12}(T - \xi) \\ \phi_{22}(T - \xi) \\ \phi_{32}(T - \xi) \end{bmatrix} d\xi. \tag{19}$$

Notice that $\mathbf{y}_d$ depends on the final time, T.

Continuing with Kranc's functional analysis approach (see Appendix), we can write the required constraint on the optimal control $u_0(t)$ as

$$|u_0(t)| \leqslant \frac{E}{L}, \tag{20}$$

where E is the maximum voltage.

Clearly, the voltage applied to the coil in the actuator must be bounded.

For the constraint given by (20), we obtain

$$u_0(t) = \frac{E}{L}\operatorname{sgn}[k(T_0 - t)], \tag{21}$$

or

$$e_0(t) = E \operatorname{sgn}[k(T_0 - t)]. \tag{22}$$

where T_0 is the final time obtained with the time-optimal control function, $u_0(t)$.

Equation (22) gives the voltage to be applied to the coil to achieve the desired time-optimal control. The control is bang-bang with the switching points corresponding to the zeros of the function $k(T_0 - t)$. With a, b, and c as constants,

$$k(T_0 - t) = a + be^{-r_1(T_0 - t)} + ce^{-r_2(T_0 - t)}, \quad t \leqslant T_0. \tag{23}$$

Therefore, if r_1 and r_2 are positive real numbers, $k(T_0 - t)$ can have at most two zeros. This implies that the control is bang-bang with no more than two reversals on the interval $0 < t < T_0$.

To avoid the complexity of an analytical expression for the time-optimal control, we will solve for the switching times, t_1 and t_2, and the final time, T_0, numerically. Consider (17), whose scalar components are

$$y_j(T) = \int_0^T \phi_{j3}\,(T - \xi)u(\xi)d\xi\,, \qquad j = 1, 2, 3\,. \tag{24}$$

Now take

$$u(\xi) = \frac{E}{L}\{1 - 2\,[U(\xi - t_1) - U(\xi - t_2)]\}\,, \tag{25}$$

where t_1 and t_2 are the unknown switching times to be found such that $0 < t_1 < t_2 \leqslant T$, and where $U(\tau)$ is defined as the unit step function; i.e.,

$$U(\tau) = \begin{cases} 1\,, & \tau \geqslant 0 \\ 0\,, & \tau < 0 \end{cases}. \tag{26}$$

Let $\Phi_{jk}(t)$ denote the antiderivative of $\phi_{jk}(t)$, $[j, k = 1, 2, 3]$, i.e.,

$$\frac{d\Phi_{jk}(t)}{dt} = \phi_{jk}(t)\,.$$

Substituting (25) into (24) and integrating, we obtain

$$\Phi_{j3}(T - t_1) - \Phi_{j3}(T - t_2) = \frac{1}{2}\left[\Phi_{j3}(T) - \Phi_{j3}(0) - \frac{L}{E}y_j(T)\right], \qquad j = 1, 2, 3\,. \tag{27}$$

Equation (27) shows that for any given T the right-hand side of the equation is known. Let this quantity be $\psi_j(T)$. If we denote $T - t_i$ by τ_i, $i = 1, 2$, then (27) becomes

$$\Phi_{j3}(\tau_1) - \Phi_{j3}(\tau_2) = \psi_j(T)\,, \qquad j = 1, 2, 3\,. \tag{28}$$

The simultaneous solution of (28) yields the three unknowns τ_1, τ_2, and T, and a straightforward graphical method can solve for these. Note, in (19), that only $y_1(T)$ depends on the desired final position X_f; $y_2(T)$ and $y_3(T)$ do not. Substituting $y_1(T)$ from (19) into the first of (27) or (28) yields

$$X_f = \frac{E}{L}\left\{\Phi_{13}(T) - \Phi_{13}(0) - 2\,[\Phi_{13}(\tau_1) - \Phi_{13}(\tau_2)]\right\} + \frac{\mu}{m}\,[\Phi_{12}(T) - \Phi_{12}(0)] - \frac{\mu(1+\epsilon)}{\alpha_1}\,\phi_{13}(T)\,. \tag{29}$$

The graphical solution, which is facilitated by drawing curves of $\Phi_{j3}(\tau)$ beforehand, can be carried out in the following manner:

(a) Choose a value of T.

(b) Compute $\psi_j(T)$, $j = 2, 3$, and plot curves of τ_2 and τ_1 which satisfy the second and third parts of (28). The intersection of these two curves gives τ_1 and τ_2 for the value of T chosen, and $\tau_1 > \tau_2$.

(c) With these values (τ_1, τ_2, and T) calculate X_f by (29).

(d) Repeat this procedure to obtain curves of X_f vs T, t_1, and t_2. The final and switching times can be obtained for any desired final position, X_f, from these curves.

Since we have taken the initial time ($t = 0$) to be the instant of impending motion, we must make a corresponding adjustment here. To the final and switching times found above, we must add the time required for the moving coil current to reach the value $\mu(1 + \epsilon)/\alpha_1$, given in (5). This occurs in the shortest time when the voltage applied to the coil is maximum. The time correction is obtained from (1), with $X_2 = 0$ and $e(t) = E$:

$$\Delta t = -\frac{L}{R}\ln\left[1 - \frac{R\mu(1+\epsilon)}{\alpha_1 E}\right]. \tag{30}$$

For completeness, the functions $\Phi_{jk}(\tau)$ appearing in (28) and (29) are given below:

$$\Phi_{13}(\tau) = \frac{\alpha_1}{m}\left[\frac{1}{r_2 - r_1}\right] \times \left[\frac{1}{r_1}\left(\tau + \frac{e^{-r_1\tau}}{r_1}\right) - \frac{1}{r_2}\left(\tau + \frac{e^{-r_2\tau}}{r_2}\right)\right], \tag{31}$$

$$\Phi_{23}(\tau) = \frac{\alpha_1}{m}\left(\frac{1}{r_2 - r_1}\right)\left(\frac{1}{r_2}e^{-r_2\tau} - \frac{1}{r_1}e^{-r_1\tau}\right), \tag{32}$$

$$\Phi_{33}(\tau) = \frac{1}{r_2 - r_1}\,(e^{-r_1\tau} - e^{-r_2\tau})\,, \tag{33}$$

$$\Phi_{12}(\tau) = \frac{1}{r_2 - r_1}\left[\frac{r_2}{r_1}\left(\tau + \frac{e^{-r_1\tau}}{r_1}\right) - \frac{r_1}{r_2}\left(\tau + \frac{e^{-r_2\tau}}{r_2}\right)\right]. \tag{34}$$

Theoretical basis for impulse response method

The method described thus far can provide time-optimal control of the moving-coil linear actuator if the system parameters used in the mathematical model are accurately known. Such knowledge is difficult to obtain for the effective inductance and resistance of the coil, and the frictional coefficients. But an alternative is suggested by the fact that a linear, time-invariant system can be characterized by its response to a unit-impulse function. If the system is designed to make stiction and coulomb friction negligible, we need only measure and record the impulse response. In such a case the output is given by

$$X(t) = \int_0^t g(t - \xi)e(\xi)d\xi\,, \tag{35}$$

where the impulse response is denoted by $g(t)$.

When we apply a step voltage to the coil and measure the signal output of the tachometer, this signal (modified by

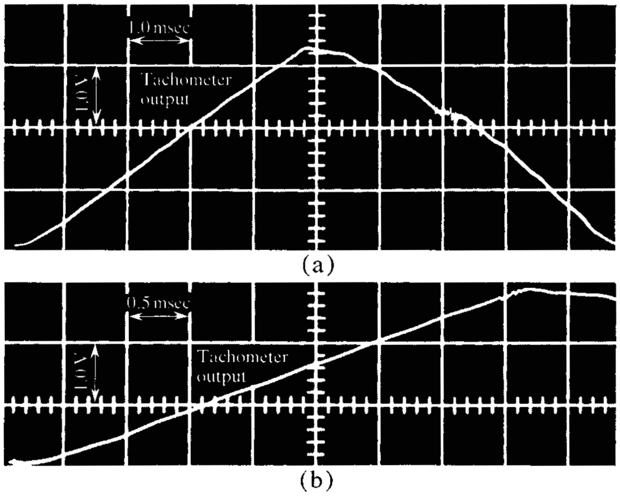


Figure 4 Impulse response measured as input data for calculation of switching times. The curve in (b) is a horizontal expansion of (a).

appropriate scale factors) corresponds to the impulse response of the system. The impulse response thus obtained for the test model is shown in Fig. 4. The tachometer conversion factor was 5.1 ips/volt, and 18 volts was applied to the coil.

We can compute the switching times for optimal control on the basis of the function $g(t)$ given in Fig. 4. (These conditions are assumed: $g(0) = g'(0) = 0$ and $\dot{X}(T) = \ddot{X}(T) = 0$.) We differentiate both sides of (35) successively with respect to time. Substituting,

$$e(t) = E\{1 - 2[U(t - t_1) - U(t - t_2)]\}, \qquad 0 < t_1 < t_2 < T, \tag{36}$$

for the control, and integrating between 0 and T, we obtain the following equations:

$$X_f = X(T) = E\left\{\int_0^T g(\xi)d\xi - 2\left[\int_0^{T-t_1} g(\xi)d\xi - \int_0^{T-t_2} g(\xi)d\xi\right]\right\}, \tag{37}$$

$$g(T - t_1) - g(T - t_2) = \tfrac{1}{2}g(T), \tag{38}$$

$$g'(T - t_1) - g'(T - t_2) = \tfrac{1}{2}g'(T). \tag{39}$$

Since (37) to (39) are in the form of (28), the graphical procedure applicable there can be applied again at this point. The integrals and derivatives of $g(t)$ in (37) and (39) can be conveniently computed from the graph of $g(t)$.

Since the inductance of the coil varies with the position of the actuator, this method is accurate only over a short distance of $\sim 0.050''$. But if we consider the total travel as consisting of linear segments of constant inductance, i.e., if we treat the system as piecewise linear, we can obtain a set of impulse responses for these segments and thus cover the whole range of travel for purposes of computation.*

Calculated results vs measured results

We have shown that the impulse response of the system is simple to obtain experimentally when friction is negligible and that, once obtained, it can be substituted into the appropriate equations and the switching times can be calculated. It remains then to determine the accuracy of this approach by comparing the calculated switching and final times with those found by trial-and-error adjustment of the moving-coil test model. Such a comparison can be made on the basis of Fig. 5, which shows the output from the phototransistor (optical positioning transducer). Movement from track to track is shown for three different positions on the recording surface. For the same distance moved (final position $X_T = 10$ mils), there is relatively good agreement:

	Times (milliseconds)	
Track No. 10	*Calculated*	*Measured*
Switching time t_1	1.72	2.00
Switching time t_2	3.63	3.80
Final time T	3.98	4.00

The discrepancies may be due in part to these circumstances:

(a) The step input voltage to the coil varied somewhat, the amplitude decreasing with time. (The tachometer output signal thus varied slightly from the true value.)

(b) The impulse response curve was differentiated graphically to compute the switching times, and some error is introduced when the amplitudes of points are read from the curve.

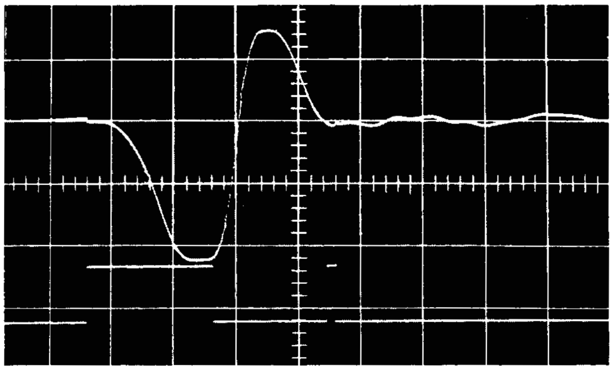
We can estimate the effect of the voltage variations on the impulse response in this way: Consider the input

$$e(t) = a - bt, \qquad t \geq 0;\ a, b > 0,$$

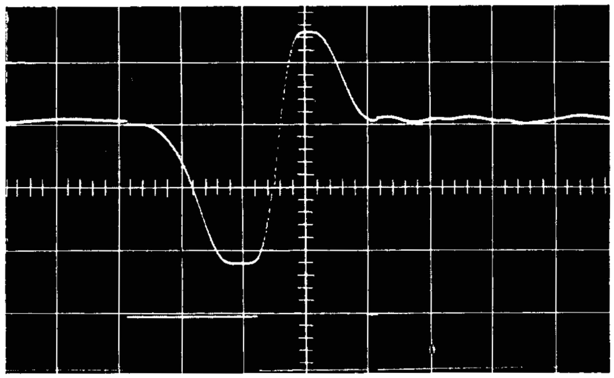
where b is small. Then the impulse response is given by

$$g(t) = \frac{1}{a}\dot{X}_1(t) + \frac{b}{a^2}X(t) + \frac{1}{a}\left(\frac{b}{a}\right)^2\int_0^t X(\xi)e^{(b/a)(t-\xi)}\,d\xi. \tag{40}$$

* This statement is valid only if the system is operated in a track-to-track fashion; e.g., it cannot be applied to compute switching times for a step of, say, 100 mils.

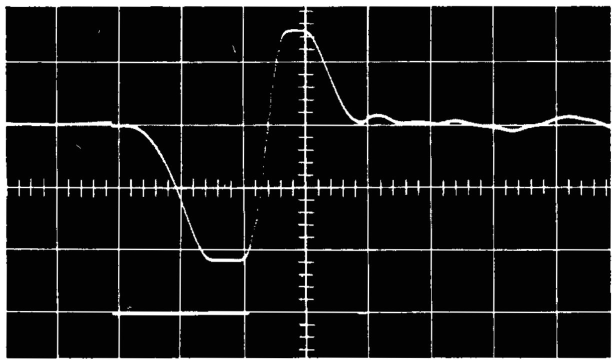


(a)



(b)

Figure 5 Output voltage of optical positioning transducer, showing actual switching times. The ordinate scale is voltage output and the abscissa is time, 1.0 msec per large division.



(c)

Observed results are:

	(a) Track 10	(b) Track 45	(c) Track 92
t_2	3.8 msec	3.9 msec	3.9 msec
t_1	2.0/1.8	2.1/1.8	2.2/1.7
T	4.0	4.0	4.1

Therefore, as a first-order correction we will take

$$g(t) \approx \frac{1}{a} \dot{X}(t) + \frac{b}{a^2} X(t)$$

or, if $v(t)$ is the signal out of the tachometer,

$$g(t) \approx \frac{1}{a} v(t) + \frac{b}{a^2} \int_0^t v(\tau) d\tau \, .$$

From the specific data shown in Fig. 4, we have

$$v(4) \approx 2.7$$

and

$$\int_0^4 v(\tau) d\tau = 5.0 \cdot$$

Therefore, if $b > 0.06$, the error may be significant.

Conclusion

Starting from the functional analysis approach of Kranc and Sarachik, we have worked out one practical way of achieving time-optimal control of a moving-coil linear actuator. To circumvent the difficulties caused by the lack of precise information about system parameters, we varied the approach to show how the results can be conveniently obtained by an impulse response method. When we compare the calculated results and the results obtained with a test model of the actuator, we find good correlation, which tends to support this as a sound approach.

To implement the method in hardware over the whole range of travel for such mechanisms will require further attention to the problems raised by the nonlinearity of the coil inductance. Since such an actuator is usually accessed under computer control, the system could be operated piecewise linearly, with the values of the coil inductances over the segments of travel programmed into the instructions.

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Appendix:

Time-optimal control problem as described by functional analysis

The time optimal control problem can be stated as follows: For the linear, time-invariant system with response given by Eq. (17), determine a control $U_0(t)$ which causes the system to pass from the given initial state $y(0)$ to the desired final state $\mathbf{y}_d$ in minimum time, T_0. In addition, the norm of $U_0(t)$, defined by

$$\|U\|_p \triangleq \left[\int_0^T |U_0(t)|^p \, dt\right]^{1/p}, \tag{41}$$

is to be bounded, i.e.,

$$\|U\|_p \leq N, \tag{42}$$

where N is a given number. By assigning different values to the parameter p in (41), we vary the nature of the constraint. When $p = \infty$, the norm of $U_0(t)$ is the maximum value it can take in the interval $(0, T)$ whenever the function is piecewise continuous. Clearly, the voltage applied to the coil must be bounded, i.e.,

$$|e(t)| \leq E, \tag{43}$$

where E is known, so that the p-infinity norm corresponds to a realistic constraint on the control. The required constraint on the control $U_0(t)$ thus becomes

$$\|U\|_\infty = |U(t)| \leq E/L. \tag{44}$$

The optimal control, $U_0(t)$, for the problem stated above is given by:

$$U_0(t) = N^{\left(\frac{p}{p-1}\right)} |k(T_0 - t)|^{\left(\frac{1}{p-1}\right)} \operatorname{sgn}[k(T_0 - t)], \tag{45}$$

where

$$k(T - t) = \sum_{j=1}^{3} \lambda_j \, \phi_{j3}(T - t). \tag{46}$$

The ϕ_{jk} are elements of the fundamental matrix, and the λ_j are constants as yet undetermined. For the constraint given by equation (43), we obtain

$$U_0(t) = E/L \operatorname{sgn}[k(T - t)], \tag{47}$$

or

$$e_0(t) = E \operatorname{sgn}[k(T - t)]. \tag{48}$$

Equation (48) gives the voltage that is to be applied to the coil to achieve the desired time-optimal control.

An explicit analytical expression for the optimal control may be difficult, if not impossible, to obtain because determining the constants λ_j and T_0 analytically is such a complex problem.

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