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Implicit Implementation of the Weighted Backward Euler Formula

Abstract: This communication describes how the weighted backward Euler formula, as applied to analyze electromechanical systems, can be implicitly implemented by replacing capacitors and inductors by resistors and voltage or current sources, respectively, and by replacing the driving functions by their first differences. This replaces the set of differential equations, which describes the capacitive voltages and the inductive currents, by a set of algebraic first difference equations.

It has been shown[1] that the set of first order differential equations in the form:

$$\frac{dF}{dt} = -BF + DE \tag{1}$$

can be accurately solved for the vector F, independently of the integration step size, using a weighted backward Euler formula. An orders of magnitude increase in speed can be accomplished, with comparable accuracy, by using the weighted backward Euler formula instead of the fourth order Runge-Kutta formula. It has also been shown[2] that Eq. (1) describes the state vector for electromechanical systems. Hence the weighted backward Euler formula is of great usefulness, especially in systems that contain relatively small time constants. It will be demonstrated now how the weight factors can be related to the different system constituents and how to obtain a set of algebraic equations that form an implicit application of the weighted backward Euler formula to obtain the solution for the vector F in Eq. (1).

Application of the weighted backward Euler formula to electromechanical systems consisting of two-part components

Following Wirth[2] it can be shown that the state vector of an electromechanical system has the form:

$$\begin{bmatrix} \dfrac{dX_b}{dt} \\ \dfrac{dY_c}{dt} \end{bmatrix} = -\begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} X_b \\ Y_c \end{bmatrix} + \begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} E_b \\ I_c \end{bmatrix}, \tag{2}$$

where X_b and Y_c are the across and through variables of the capacitive branches and inductive links, respectively, and E_b and I_c are the across and through drivers included as branches and links, respectively.

This can be written in the compact form

$$\frac{dF}{dt} = -Z^{-1}BF + Z^{-1}DE. \tag{3}$$

Applying the weighted backward Euler formula to Eq. (3) yields the following recursion equation:

$$(U + H\Delta t Z^{-1}B)F_{n+1} = F_n + \Delta t Z^{-1}DE_{n+1}, \tag{4}$$

where the Z and H matrices are assumed to be piecewise constant.

We approximate the driving function by a set of step functions; the response of the system is the summation of the responses due to each such step. Thus, taking the first difference of (4) yields

$$(U + H\Delta t Z^{-1}B)\Delta F_{n+1} = \Delta F_n + \Delta t Z^{-1}D\Delta E_{n+1}, \tag{5}$$

where

$$\Delta X_{n+1} = X_{n+1} - X_n. \tag{6}$$

Consider

$$H\Delta t Z^{-1} = \begin{bmatrix} H_{11} & 0 \\ 0 & H_{22} \end{bmatrix} \Delta t \begin{bmatrix} C^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix}; \tag{7}$$

H_{11} and H_{22} are diagonal submatrices and since C^{-1}, L^{-1} represent one-port capacitors and inductors, they are also diagonal. Thus

$$H\Delta t Z^{-1} = \Delta t \begin{bmatrix} H_{11}C^{-1} & 0 \\ 0 & H_{22}L^{-1} \end{bmatrix} = \begin{bmatrix} \Delta t H_{11}C^{-1} & 0 \\ 0 & \Delta t H_{22}L^{-1} \end{bmatrix}. \tag{8}$$

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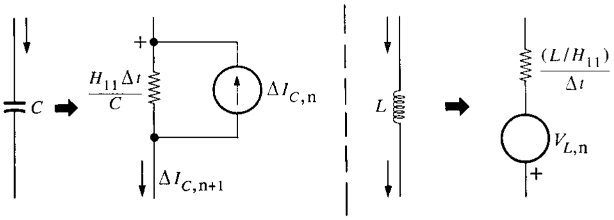

Figure 1 Equivalent representation of capacitors and inductors.

Consequently, the weight factors can serve as modification factors, which change the values of the capacitive and inductive components, so that the final answer is accurate. Since H is dimensionless and $\Delta t Z^{-1}$ has the dimensions of ohms, it is possible to replace the capacitive and inductive components by equivalent resistances. Further, Eq. (5) implies a driving function ΔF_n in connection with each equivalent ohmic representation of an inductive or capacitive component. Equation (5) also states that the $(n + 1)$th difference is a function of the nth difference of the state vector and the $(n + 1)$th difference of the driving function. It also states that once the weight factors have been applied to modify the values of the capacitive and inductive components, the portion of the $(n + 1)$th difference of the state vector F that is due to the $(n + 1)$th difference of the driving function has to be amplified by the same weight factor. Figure 1 suggests one possible way of specifying the equivalent representation of the capacitive and inductive components. One should follow the following steps if the solution for Eq. (1) is to be obtained implicitly, using difference equations:

(a) Choose a solution step Δt such that the driving function is adequately described;

(b) calculate the weight factors[1];

(c) replace each capacitor and each inductor by its equivalent representation (Fig. 1);

(d) replace all the driving functions by first difference generators[1];

(e) for the nth step, calculate the capacitive currents and inductive voltages due to the first difference generator at time $t_0 + n\Delta t$ and amplify them by the inverse of the weight factor; and

(f) calculate the capacitive voltages and inductive currents due to the incremental capacitive voltages and inductive currents at time $t_0 + (n - 1)\Delta t$. Steps (e) and (f) can be accomplished by using the Gauss-Jordan elimination technique.[3]

(g) Then the incremental capacitive voltages and inductive currents, ΔF_n, at $t_0 + n\Delta t$ are equal to the summation of the results of (e) and (f).

(h) The value of the state vector at the nth step is given by

$$F_n = \sum_{i=1}^{n} \Delta F_i \,. \tag{9}$$

Conclusions

It is possible not only to increase the accuracy of the backward Euler formula by the introduction of a set of weight factors, but also to obtain these accurate results implicitly, using the equivalent representation of Fig. 1 and following the steps (a) through (h).

References

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