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An Application of the Cooley-Tukey Algorithm to Equalization

Abstract: A new method is proposed to determine the tap settings of a delay-line filter for distortion correction of digital data after transmission by a voice line. The procedure described achieves approximately the optimal tap settings by solving a system of linear equations with a circulant matrix of coefficients. When the fast Fourier transform algorithm of Cooley and Tukey is used, the resulting quantity of computations is considerably smaller than in other known methods. Therefore, much faster equalization can be achieved.

Tapped delay-line filters are often used for the distortion correction of digital data after transmission by a voice line.[1-3]

In the present application of delay-line filters, the main problem is to find a fast procedure for the determination of the optimal tap settings. First, the principal idea of the known methods will be described.

We suppose that the message to be transmitted is a sequence of zeros (spaces) and ones (marks). Let a unit pulse be sent through the voice line, which is assumed to be linear. The sample values of this pulse will be indicated by the sequence s_1:

$$s_1 = (\cdots, 0, \cdots, 0, 1, 0, \cdots, 0, \cdots)\,. \tag{1}$$

Owing to distortions, the sample values of the pulse received are different from s_1, and are denoted by s_2:

$$s_2 = (\cdots, a_{-\nu}, \cdots, a_{-1}, a_0, a_1, \cdots, a_\nu, \cdots)\,. \tag{2}$$

Here a_0 indicates the position of the main peak.

The purpose of equalization is to regenerate sequence s_1 from s_2 as well as possible.

In the following, we assume the initial distortion D, i.e., the distortion before equalization, to be smaller than one, by which we mean

$$D = \frac{1}{a_0} \sum_{\nu=-\infty}^{\infty}{}' |a_\nu| < 1\,. \tag{3}$$

The "prime" indicates that the term $|a_0|$ has to be omitted in the sum, and we assume a_0 to be positive without loss of generality. Let T be the time between two sampling points at the receiver side. A delay-line filter with n delay elements each of delay T, and $n + 1$ tap amplifications x_k, $k = 0, 1, \cdots, n$, forms the following equalized sequence $\bar{s}_2$ from the input sequence s_2:

$$\bar{s}_2 = (\cdots, \bar{a}_{-\nu}, \cdots, \bar{a}_{-1}, \bar{a}_0, \bar{a}_1, \cdots, \bar{a}_\nu, \cdots) \tag{4}$$

with

$$\bar{a}_\nu = \sum_{j=0}^{n} x_j a_{\nu-m+j}\,, \tag{5}$$

and the parameter m can assume any arbitrary but fixed integer value from 0 to n. Lucky[1] has proved that the x_j in such an equalizer achieve optimal reduction of distortion if and only if the m values $\bar{a}_\nu$ preceding $\bar{a}_0$ and the $n - m$ ones immediately following it all vanish, and $\bar{a}_0 = 1$. Then the signal sampled after equalization is of the form:

$$\bar{s}_2 = (\cdots, \bar{a}_{-m-1}, \underbrace{0, \cdots, 0}_{m \text{ zeros}}, 1, \underbrace{0, \cdots, 0}_{n-m \text{ zeros}}, \bar{a}_{n-m+1}, \cdots)\,. \tag{4'}$$

The $\bar{a}_\nu$, $\nu < -m$ or $\nu > (n - m)$ are certain new echoes, and the distortion after equalization, which is defined by

$$D(m) = \frac{1}{\bar{a}_0} \sum_{\nu=-\infty}^{\infty}{}' |\bar{a}_\nu| \tag{6}$$

becomes

$$D(m) = \sum_{\nu=-m-1}^{-\infty} |\bar{a}_\nu| + \sum_{\nu=n-m+1}^{\infty} |\bar{a}_\nu|\,. \tag{6'}$$

Since m can only assume a finite number of values, it is theoretically very simple to find the minimum of $D(m)$ over m. In practical realizations of equalizers, a fixed value for

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m is implemented and no optimization with respect to this parameter is performed. Therefore, in the sequel, we, too, assume m, $0 \leq m \leq n$, to be arbitrary but fixed.

It follows from (5) that the tap settings x_k forming the equalized signal (4′) are the solution of the following system of linear equations:

$$\mathbf{Ax} = \mathbf{e}_m\,, \tag{7}$$

with

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ a_{-1} & a_0 & \cdots & a_{n-1} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{-n+1} & \cdots & a_0 & a_1 \\ a_{-n} & \cdots & a_{-1} & a_0 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix}, \quad \mathbf{e}_m = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \\ \end{matrix}\right\} m \text{ zeros} \\ \\ \left.\begin{matrix} \\ \\ \\ \\ \\ \end{matrix}\right\} n-m \text{ zeros} \end{matrix}$$

Since (3) is provided, the determinant of the matrix $\mathbf{A}$ does not vanish, and (7) therefore has a unique solution. In implemented equalizers, this system of equations is solved on a special purpose computer by an iterative method.[1]

The new procedure

In the following, the equalization is not based on a single pulse response but on the response of a periodic sequence of unit pulses, the distance between two succeeding pulses being $(n+1)T$. This sequence sent will be called s_3:

$$s_3 = (\cdots, 1, 0, \cdots, 0, 1, 0, \cdots, 0, 1, 0, \cdots)\,. \tag{8}$$

The sampled data received also form a periodic sequence with the same period

$$s_4 = (\cdots, b_0, b_1, \cdots, b_n, b_0, \cdots)\,. \tag{9}$$

b_0 again indicates the position of the main peak.

The b_ν can be expressed by the sample values a_ν of the single pulse response:

$$b_\nu = \sum_{l=-\infty}^{\infty} a_{\nu+l(n+1)}\,. \tag{10}$$

From (3) and (10) follows:

$$b_0 > \sum_{\nu=1}^{n} |b_\nu|\,. \tag{11}$$

It can now be seen that the tap settings y_k, $k = 0, \cdots, n$, which regenerate the sequence s_3 from s_4 are the solution to the following system of linear equations:

$$\mathbf{By} = \mathbf{e}_m \tag{12}$$

with

$$\mathbf{B} = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} & b_n \\ b_n & b_0 & \cdots & b_{n-2} & b_{n-1} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ b_2 & b_3 & \cdots & b_0 & b_1 \\ b_1 & b_2 & \cdots & b_n & b_0 \end{bmatrix},$$

$\mathbf{y}$ being the column vector of the tap settings, and $\mathbf{e}_m$ having the same meaning as in (7); m is again an arbitrary but fixed integer between 0 and n.

A matrix $\mathbf{B}$ of this special structure is called *circulant.* The eigenvalues λ_μ of this matrix are known to be[4]

$$\lambda_\mu = \sum_{l=0}^{n} b_l W^{-l\mu}\,, \tag{13}$$

with

$$W = \exp\left(\frac{2\pi i}{n+1}\right)$$

being an $(n+1)$ th root of unity.

Because of (11), all λ_μ are different from zero and (12) is uniquely solvable. Furthermore, the solution of (12) can be represented as:[4]

$$y_j = \frac{1}{n+1} \sum_{\mu=0}^{n} \frac{W^{(m-j)\mu}}{\lambda_\mu}\,. \tag{14}$$

For completeness, the simple proof will be given. Since (14) defines y_j even for all integer subscripts j, and $y_j = y_{j+n+1}$ holds, the sample values after equalization $\bar{b}_\nu$ are given by

$$\bar{b}_\nu = \sum_{k=0}^{n} b_k y_{\nu+k}\,. \tag{15}$$

Combining (13), (14) and (15), and changing the order of summation, results in

$$\bar{b}_\nu = \frac{1}{n+1} \sum_{\mu=0}^{n} W^{(m-\nu)\mu} = \begin{cases} 1 & \text{for } \nu = m \\ 0 & \text{otherwise}\,. \end{cases}$$

The tap settings given by (14) are real, since they are the solution of (12) with real coefficients and a real right-hand side.

Comparison of results and computing time

The difference between the tap settings $\mathbf{x}$ and $\mathbf{y}$ resulting from the old and the new methods, respectively, can be written in the form:

$$\mathbf{x} - \mathbf{y} = (\mathbf{A}^{-1} - \mathbf{B}^{-1})\mathbf{e}_m . \tag{16}$$

Because of (3), $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ have the converging series expansions

$$\mathbf{A}^{-1} = \frac{1}{a_0} \sum_{\nu=0}^{\infty} \left(\mathbf{I} - \frac{1}{a_0}\mathbf{A}\right)^{\nu}$$

and (17)

$$\mathbf{B}^{-1} = \frac{1}{b_0} \sum_{\nu=0}^{\infty} \left(\mathbf{I} - \frac{1}{b_0}\mathbf{B}\right)^{\nu},$$

and we get the following first-order approximation for $\mathbf{x} - \mathbf{y}$:

$$\mathbf{x} - \mathbf{y} \approx \left(\frac{1}{b_0{}^2}\mathbf{B} - \frac{1}{a_0{}^2}\mathbf{A}\right)\mathbf{e}_m + 2\left(\frac{1}{a_0} - \frac{1}{b_0}\right)\mathbf{e}_m . \tag{18}$$

This shows that the difference between $\mathbf{x}$ and $\mathbf{y}$ is small if the difference between the matrices $\mathbf{A}$ and $\mathbf{B}$ is small; the latter will clearly be fulfilled for small initial distortions. Then both methods yield approximately the same results.

It follows further from (7), (10) and (12) that the tap settings $\mathbf{x}$ are also a solution to the system of equations:

$$\mathbf{B}\mathbf{x} = \mathbf{e}_m + \Delta\mathbf{b} , \tag{19}$$

where the ν^{th} component Δb_ν of the column vector $\Delta\mathbf{b}$ is given by

$$\Delta b_\nu = \sum_{l=-\infty}^{l=\infty}{}' \, \bar{a}_{\nu+l(n+1)} ; \qquad \nu = -m, \cdots, 0, \cdots, n-m , \tag{20}$$

and the term with $l = 0$ has to be omitted in the sum. The $\bar{a}_\nu$ are the distortions after equalization defined by (5), which are small if the equalizer with tap settings $\mathbf{x}$ works well. Then the components of the vector $\Delta\mathbf{b}$ are also small. Since the right-hand sides of (12) and (19) differ just by $\Delta\mathbf{b}$, we can conclude that the difference $\mathbf{x} - \mathbf{y}$ also decreases with decreasing distortion after equalization using tap settings $\mathbf{x}$. In other words, the better the conventional method works, the better are the results of the new method. In the limit as n tends to infinity, both methods yield ideal equalization.

Some results of a simulation are shown in the following Tables. Two limiting cases have been considered. Table 1 corresponds to a transmission channel with rectangular amplitude characteristics and a sinusoidal phase characteristic, such that a large number (more than 100) of relatively small echoes occur. For a more complete description of the channel model used, see Ref. 5. Table 2 gives the results in the case where the distorted signal has only two echoes on either side. Realistic channels lie in between these limiting cases. We can see that the new method gives only slightly worse results than the conventional one, and both methods work even for initial distortion larger than one.

Table 1 Comparison of the two methods for input with large number of echoes.

Initial distortion	*Distortion after equalization*			
	Conventional method		*New method*	
	5 taps	*9 taps*	*5 taps*	*9 taps*
1.66	0.86	0.64	0.92	0.67
0.93	0.55	0.49	0.63	0.56
0.88	0.74	0.59	0.84	0.65
0.66	0.38	0.31	0.44	0.34
0.28	0.09	0.08	0.11	0.09

Table 2 Comparison of the two methods for input with two echoes.

Initial distortion	*Distortion after equalization*			
	Conventional method		*New method*	
	5 taps	*9 taps*	*5 taps*	*9 taps*
0.6	0.043	0.003	0.079	0.006
0.6	0.074	0.008	0.112	0.012
0.8	0.068	0.011	0.154	0.019
0.8	0.246	0.062	0.323	0.083
1.2	0.208	0.045	0.341	0.074

The necessary amount of computations for both methods will now be compared. For solving the system of equations (7), the fastest method would be Gaussian elimination. This requires, setting $N = n + 1$, N real divisions, $N(N^2/3 + N - 1/3)$ real multiplications and $N(N^2/3 + N/2 - 5/6)$ real additions; i.e., the number of multiplications and additions is proportional to N^3. (See Ref. 6.)

To solve the system of equations (12) with the circulant matrix of coefficients, the eigenvalues λ_μ given by (13) must first be calculated. The formula for the $\lambda\mu$ has exactly the same form as a discrete Fourier transform. To calculate the eigenvalues, the fast Fourier transform algorithm of Cooley and Tukey[7] can be applied, requiring a number of complex multiplications and additions, both being proportional to $N \log N$. Next, in order to evaluate (14), this algorithm can be easily modified, and the number of necessary multiplications and additions is still proportional to $N \log N$. If $N = 2^r$ is chosen, a power-of-two algorithm can be used[8] and the computing time for the new method is proportional to $r \cdot 2^r$ compared to 2^{3r} for the old method, i.e., our method saves a considerable amount of computation. As an example, the computing times required by the old and the new methods for calculation of the tap settings of an equalizer with 26 delay elements ($n = 26$, $N = 27$) would be 0.725 sec and 0.101 sec, respectively, on an IBM System/360 Model 40 computer.

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References

1. R. W. Lucky, "Automatic equalization for digital communication," *Bell System Tech. J.* **44,** 547 (1965).
2. M. A. Rappeport, "Automatic equalization of data transmission facility distortion using transversal equalizers," *IEEE Transactions on Comm. Technology* **12,** 65 (1964).
3. K. E. Schreiner et al., "Automatic distortion correction for efficient pulse transmission," *IBM J. Res. Develop.* **9,** 20 (1965).
4. M. Marcus, "Basic theorems in matrix theory," *National Bureau of Standards, Appl. Math. Ser.* **57,** 9 (1960).
5. R. A. Gibby, "An evaluation of AM data system performance by computer simulation," *Bell System Tech. J.* **39,** 675 (1960).
6. L. Fox, *An Introduction to Numerical Linear Algebra*, Clarendon Press, Oxford, 1964, p. 176.
7. J. W. Cooley and J. W. Tukey, "An algorithm for the machine calculation of complex Fourier series," *Math. of Comp.* **19,** 297 (1965).
8. J. W. Cooley, P. A. W. Lewis and P. D. Welch, "The fast Fourier transform algorithm and its applications," *IBM Research Report RC 1743* (1967).

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