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Two Remarks on the Reconstruction of Sampled Non-Bandlimited Functions

This communication considers the error in the reconstruction of a deterministic, non-bandlimited real function from its sampled values. We have two objectives in mind. First, we show that in a certain sense, to be made precise in the sequel, the error in the reconstruction of $f(t)$ from its sampled values $f(n\pi/\Omega)$ by $\sin x/x$ interpolation is small provided the portion of the amplitude spectrum lying outside $[-\Omega, \Omega]$ is small. Our second objective is to show the analogous situation need not hold for the energy spectrum. Indeed, we shall exhibit a function with an arbitrarily small amount of its energy outside the frequency band $|w| \leq \Omega$ for which the $\sin x/x$ interpolation series

$$\sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right) \frac{\sin(\Omega t - n\pi)}{(\Omega t - n\pi)} \tag{1}$$

fails to converge either pointwise or in the mean square sense.

The first objective is accomplished by use of a suitable combination of the techniques employed in two recent papers by Stickler[1] and Brown,[2] who considered the case where $f(t)$ is the Fourier transform of a Lebesgue integrable function. Brown also considered the case of bandpass sampling. We note, however, that Stickler did not attempt to establish the convergence of the interpolation series formed from the sampled values, and Brown invoked in the course of his proof (see his Lemma 3) the incorrect result that the Lebesgue integrability of the function implies the integrability of its square. A simple counterexample is furnished by $f(t) = t^{-1/2}$, $(0 < t \leq 1)$.

Specifically, in this paper we establish the following:

Theorem: Let $f(t)$ be the Fourier-Stieltjes transform of a function of bounded variation

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-iwt)\, dG(w) \tag{2}$$

where $G(w)$ is continuous at $w = \pm(2k+1)\Omega$, $k = 0, 1, 2, \cdots$ and normalized to be right continuous everywhere. Define $f_a(t)$ to be the "bandlimited reconstruction" of $f(t)$ given by

$$f_a(t) = \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right) \frac{\sin(\Omega t - n\pi)}{(\Omega t - n\pi)}\ ; \tag{3}$$

then the series defining $f_a(t)$ converges for all values of t and the reconstruction error $e(t) = f(t) - f_a(t)$ is bounded by

$$|e(t)| \leqq \frac{1}{\pi}\left(\int_{\Omega}^{\infty} |dG(w)| + \int_{-\infty}^{-\Omega} |dG(w)|\right). \tag{4}$$

Proof:

$$\sum_{-M}^{N} f\left(\frac{n\pi}{\Omega}\right) \frac{\sin(\Omega t - n\pi)}{(\Omega t - n\pi)}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\sum_{-M}^{N} \exp\left(-\frac{iwn\pi}{\Omega}\right)\frac{\sin(\Omega t - n\pi)}{(\Omega t - n\pi)}\right\} dG(w)\ .$$

The expression inside the braces is a partial sum $S_{NM}(w, t)$ of the Fourier series of the periodic continuation of the function defined in the interval $-\Omega \leq w \leq \Omega$ by $\exp(-iwt)$. Since $\exp(-iwt)$ is of bounded variation in the open interval $|w| < \Omega$ the partial sums of its Fourier series are uniformly bounded in N and M for fixed t. (See e.g., Titchmarsh,[3] p. 408). $G(w)$, being of bounded variation, is expressible as the difference $G_1(w) - G_2(w)$ of two monotone non-decreasing functions which, by the assumed continuity of $G(w)$ at $w = \pm(2k+1)\Omega$, can be chosen to be continuous at the same points. The partial sums $S_{NM}(w, t)$ are then bounded almost everywhere with respect to the Lebesgue-Stieltjes measures induced by $G_1(w)$ and $G_2(w)$. The Lebesgue dominated convergence theorem can then be invoked to yield

$$\sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right) \frac{\sin(\Omega t - n\pi)}{(\Omega t - n\pi)}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{N,M\to\infty} S_{NM}(w, t)\, dG(w)$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(w, t)\, dG(w)\ ,$$

where $S(w, t)$ is the periodic extension of $\exp(iwt)$. The convergence of the series defining $f_a(t)$ having been established, we can proceed analogously as in Stickler[1] to obtain

$$e(t) = f(t) - f_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-iwt) - S(w, t)\; dG(w)\,,$$

$$|e(t)| \leqq \frac{1}{\pi}\left(\int_{\Omega}^{\infty} |dG(w)| + \int_{-\infty}^{-\Omega} |dG(w)|\right). \tag{5}$$

The result of Stickler and Brown appears as the

Corollary: If $G(w)$ defined in (2) is absolutely continuous, $dG(w) = F(w)\,dw$ and $f(t)$ is real then

$$|e(t)| \leqq \frac{2}{\pi}\int_{\Omega}^{\infty} |F(w)|dw\,. \tag{6}$$

We remark that the continuity of $G(w)$ at $w = \pm(2k + 1)\Omega$ is not a superfluous assumption since, in the presence of discontinuities at these points, the series for $f_a(t)$ may diverge. In other words, divergence may occur when there are line components in the frequency of $f(t)$ at odd harmonics of the cutoff frequency. Indeed, define $G(w)$ by

$$G(w) = \begin{cases} 0, & w < (2k_0 + 1)\Omega \\ 1, & w \geq (2k_0 + 1)\Omega\,. \end{cases}$$

Then an easy calculation shows

$$f_a(t) = \sum_{-\infty}^{\infty} \frac{\sin(\Omega t)}{2\pi} \frac{1}{(\Omega t - n\pi)}\,,$$

which is obviously divergent.

We now turn to the second objective of this note, the construction of a function $f(t)$ with an arbitrarily small amount of its energy outside the frequency band $|w| \leq \Omega$ for which the interpolation series diverges. The required function is constructed as follows:

$$f(t) = f_1(t) + f_2(t)\,, \tag{7}$$

where

$$f_1(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} \exp(-iwt)\; F_1(w)dw$$

$$f_2(t) = \begin{cases} +1\,, & \\ \dfrac{(2k+1)\pi}{\Omega} - \dfrac{d_k}{2} < t < \dfrac{(2k+1)\pi}{\Omega} + \dfrac{d_k}{2}\,, & \\ -1\,, & \dfrac{2k\pi}{\Omega} - \dfrac{d_k}{2} < t < \dfrac{2k\pi}{\Omega} + \dfrac{d_k}{2}\,, \\ & k = 1, 2, \cdots \\ 0\,, & \text{elsewhere}\,, \end{cases}$$

and $\sum_{0}^{\infty} u_k < \epsilon$.

A trivial calculation shows

$$\int_{-\infty}^{\infty} f_2(t)^2\; dt < \epsilon \tag{8}$$

and an application of Parseval's theorem, recalling $f_1(t)$ is strictly bandlimited shows

$$\frac{1}{2\pi}\int_{\Omega}^{\infty} |\hat{f}(w)^2|dw + \frac{1}{2\pi}\int_{-\infty}^{-\Omega} |\hat{f}(w)^2|dw < \epsilon\,,$$

where $\hat{f}(w)$ is the Fourier transform, in the mean square sense, of $f(t)$. Thus, the portion of the energy of $f(t)$ outside the frequency band $[-\Omega, \Omega]$ is less than ϵ in magnitude.

Now $f(t)$ can be written in the form

$$f(t) = \sum_{-\infty}^{\infty}\left[f\left(\frac{n\pi}{\Omega}\right) - f_2\left(\frac{n\pi}{\Omega}\right)\right]\frac{\sin(\Omega t - n\pi)}{(\Omega t - n\pi)} + f_2(t)\,, \tag{9}$$

the convergence of the series above being assured by our Theorem. The convergence of this series implies that the series

$$\sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right)\frac{\sin(\Omega t - n\pi)}{(\Omega t - n\pi)}\,,$$

$$\sum_{-\infty}^{\infty} f_2\left(\frac{n\pi}{\Omega}\right)\frac{\sin(\Omega t - n\pi)}{(\Omega t - n\pi)} \tag{10}$$

either both converge or both diverge. Clearly the second series diverges pointwise since it reduces to

$$-\sin(\Omega t)\sum_{0}^{\infty}\frac{1}{(\Omega t - n\pi)}\,.$$

Finally, the divergence of the series in the mean square sense follows from

$$\int_{-\infty}^{\infty}\left[\sum_{-N}^{N}\frac{\sin(\Omega t - n\pi)}{(\Omega t - n\pi)} f_2\left(\frac{n\pi}{\Omega}\right)\right]^2 dt$$

$$= \frac{\pi}{\Omega}\sum_{0}^{N} f_2^{\,2}\left(\frac{n\pi}{\Omega}\right) = (N+1)\frac{\pi}{\Omega}\,, \tag{11}$$

the last step arising from the orthogonality of the sequence $\sin(\Omega t - n\pi)/(\Omega t - n\pi)$ on $-\infty < t < \infty$.

References

1. D. C. Stickler, "An Upper Bound on Aliasing Error," Letter, *Proc. IEEE* **54**, No. 3, 418–9 (1967).
2. J. L. Brown, "On the Error in Reconstructing a Non-Bandlimited Function by Means of the Bandpass Sampling Theorem," *J. Math Anal. and Appl.* **18**, 75–84 (1967).
3. E. Titchmarsh, *Theory of Functions*, 2nd ed., Oxford Union Press, London.

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