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Synthesis of Transfer Functions in a Prescribed Frequency Band

Abstract: A signal-processing system for synthesizing complicated transfer functions in a prescribed frequency band is described. The system consists of a multiplier followed by a closed loop containing a delay line and phase modulator in series. One of the inputs to the multiplier is the signal to be filtered. The second input to the multiplier is periodic, the period being equal to the loop delay. One cycle of the periodic waveform is identical to the real part of the transfer function over a prescribed frequency band of width equal to the reciprocal of the loop delay. The imaginary component of the transfer function is the negative of the Hilbert transform of the real part, as in all physically realizable filters. Two applications of the system are discussed. It is shown how a continuously variable delay line and chirp filter can be synthesized using these techniques.

One of the perplexing problems that continually confront the communications or signal-processing engineer is the synthesis of filters having complicated transfer functions. This paper describes a system for electronically synthesizing certain types of filters. The system behaves as a narrowband, linear filter but has two input signals. One of the inputs, $x(t)$, is the narrowband signal to be filtered. The second is a periodic waveform, $w(t)$, each cycle of which represents the real part of the desired transfer function over a particular frequency band. A schematic of the system is shown in Fig. 1. The product of the two inputs, $x(t)$ and $w(t)$, enters a unity-gain recirculating loop containing a delay line and frequency shifter. The combination of a delay line and frequency shifter in a recirculating loop can also be operated as a spectral analyzer. Operation of the loop in this fashion has received considerable attention, and is usually called a "coherent memory filter."[1–4]

The delay line delays the product for a time T, and the frequency shifter increases the frequency by an amount $1/T$. The frequency shifter can be regarded somewhat more precisely as a phase modulator, where the phase is increased by an amount $2\pi t/T$. A bandpass filter is connected to the loop. The output of the filter $y(t)$ is the system output.

Assume the following four conditions:

1. $x(t)$ is limited to the band $[k/T, (k+1)/T]$, where k is a positive integer.
2. $w(t) = w(t + nT)$, where n is an integer. That is, $w(t)$ is periodic with period T.
3. The highest frequency component in $w(t)$ is less than k/T, and $w(t)$ has zero mean value.
4. The bandpass filter has unity transfer in the band $[k/T, (k+1)/T]$, and zero elsewhere.

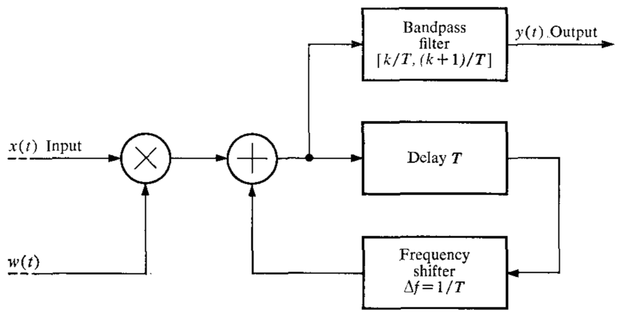


Figure 1 Filter system diagram.

If these conditions are satisfied, then the Fourier transform of the output $y(t)$ is

$$Y(f) = \tfrac{1}{2}\, X(f)[w(fT^2) - j\hat{w}(fT^2)]\,, \tag{1}$$

where
$Y(f)$ = Fourier transform of $y(t)$,
$X(f)$ = Fourier transform of $x(t)$,
$w(fT^2)$ = $w(t)$ evaluated at $t = fT^2$,
$\hat{w}(fT^2)$ = $w(t)$ advanced 90° in phase and evaluated when $t = fT^2$. In other words, $\hat{w}(t)$ is the Hilbert transform of $w(t)$.[5]

Therefore the transfer function of the filter is

$$W(f) = \frac{Y(f)}{X(f)} = \tfrac{1}{2}\,[w(fT^2) - j\hat{w}(fT^2)]\,. \tag{2}$$

Equations (1) and (2) are valid for any functions $x(t)$ and $w(t)$ which satisfy the four stated conditions; a rigorous proof of this is contained in the Appendix. However, considering the special case of a sinusoidal input will provide

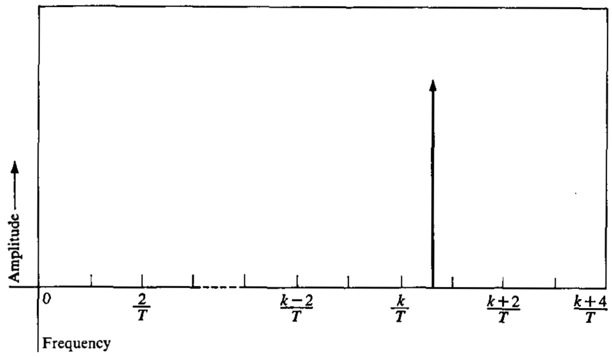


Figure 2 Spectrum of input $x(t) = B \exp(j\theta)$

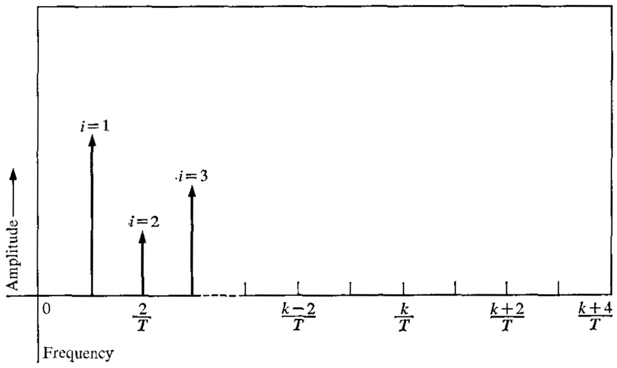


Figure 3 Spectrum of periodic waveform $w(t) = A_i \exp(j\phi_i)$, $i = 1, 2, 3.$

an understanding of the system operation that may be difficult to obtain from the proof for the general case.

Let the input $x(t)$ be

$$x(t) = B \cos(2\pi f t + \theta), \tag{3}$$

where B is the input signal amplitude, f is the input frequency, and θ is the input phase. The spectrum of the input is illustrated in Fig. 2. As has already been pointed out, the input frequency must lie in the band $[k/T, (k+1)/T]$. Because $w(t)$ is periodic with period T, $w(t)$ can be expanded into a Fourier series with fundamental frequency $1/T$. Let $w(t)$ be

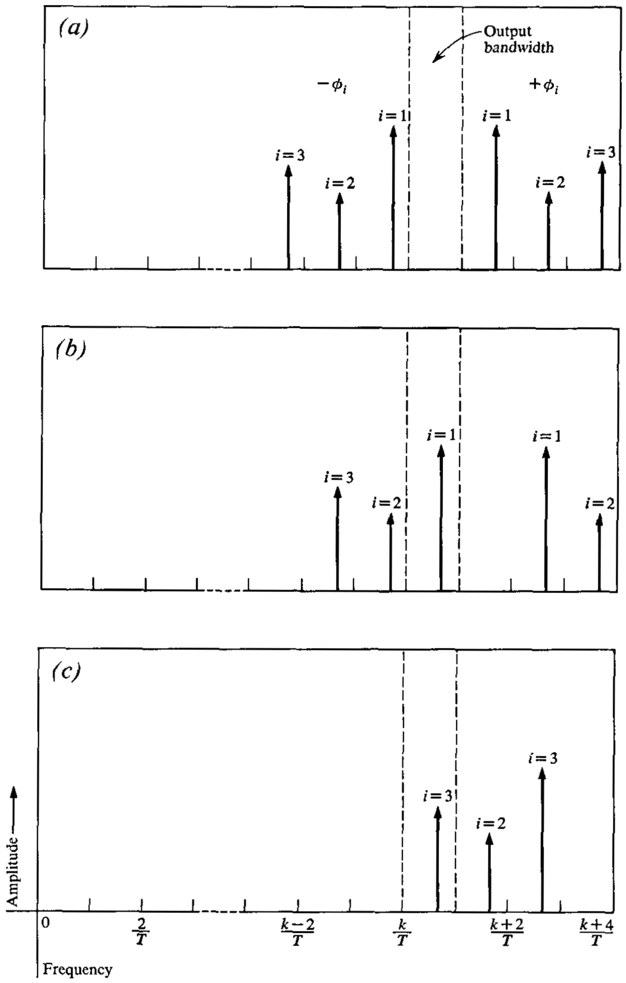
$$w(t) = \sum_{n=1}^{k} A_n \cos[(2\pi nt/T) + \phi_n], \tag{4}$$

where A_n and ϕ_n are respectively the amplitude and phase of the n^{th} harmonic. The spectrum of $w(t)$ is illustrated in Fig. 3. The product $x(t)w(t)$ is

$$\begin{aligned} x(t)w(t) &= \frac{B}{2} \sum_{n=1}^{k} A_n \cos[2\pi f t + \theta + (2\pi nt/T) + \phi_n] \\ &+ \frac{B}{2} \sum_{n=1}^{k} A_n \cos[2\pi f t + \theta - (2\pi nt/T) - \phi_n]. \end{aligned} \tag{5}$$

The spectrum of $x(t)w(t)$ is illustrated in Fig. 4a. This signal is always present at the input to the bandpass filter. In passing through the loop $x(t)w(t)$ is retarded in phase by an amount $2\pi fT$, and the frequency is increased by an amount $1/T$. Therefore, the spectrum of the input to the bandpass filter after one circulation in the loop is as shown in Fig. 4b. When $x(t)w(t)$ passes through the loop a second time, it is again retarded in phase and shifted in frequency. This yields a total phase shift of $(-4\pi fT)$ and a frequency shift of $2/T$. This process continues so that after i circulations the phase shift is $(-2i\pi fT)$ and the frequency shift is i/T. The spectrum of the input to the bandpass filter after

Figure 4 Spectra of signals in the loop: $\frac{1}{2} BA_i \exp[j(\theta \pm \phi_i - 2\pi nfT)]$, where n is the number of loop circulations; (a) $n = 0$, (b) $n = 1$, (c) $n = 3$. The actual spectrum appearing in the loop after a given number of circulations is the superposition of the individual spectra generated during each of the completed loop circulations.



three circulations is shown in Fig. 4c. It should be noted that the spectrum appearing in the loop is the superposition of all the individual spectra generated during each preceding circulation of the loop.

The bandpass filter will pass energy in the band $[k/T, (k+1)/T]$ only. Furthermore, after sufficient time, energy that has been around the loop once, twice, three times, etc., will exist at the input to the output bandpass filter. Adding together all terms in the band $[k/T, (k+1)/T]$ yields the spectrum of the output of the bandpass filter:

$$\begin{aligned} Y(f) &= (BA_1/2)\exp[j(\theta-\phi_1-2\pi fT)] \\ &\quad + (BA_2/2)\exp[j(\theta-\phi_2-4\pi fT)] \\ &\quad + (BA_3/2)\exp[j(\theta-\phi_3-6\pi fT)] + \cdots \\ &= [X(f)/2]\sum_{n=1}^{k} A_n \exp[-j(2\pi nfT+\phi_n)] \\ &= [X(f)/2]\sum_{n=1}^{k} A_n \cos(2\pi nfT+\phi_n) \\ &\quad - j[X(f)/2]\sum_{n=1}^{k} A_n \sin(2\pi nfT+\phi_n) \\ &= [X(f)/2][w(fT^2) - j\hat{w}(fT^2)]\,. \end{aligned} \tag{6}$$

This last expression constitutes a proof of Eq. (1) for the special case of a sinusoidal input.

Several points are worth mentioning at this time. First, it is not essential that $w(t)$ have zero mean value. This restriction was invoked purely for notational simplicity. If $w(t)$ has a nonzero mean value, the transfer function cannot be described in terms of the modulating waveform $w(t)$, but must be expressed in terms of the spectral components of $w(t)$. For modulating waveforms containing dc components the transfer function is

$$W(f) = \frac{1}{2}\sum_{n=1}^{k} A_n \exp[-j(2\pi nfT+\phi_n)] + A_0\,, \tag{7}$$

where A_0 is the dc value of $w(t)$. Second, because $w(t)$ is periodic with period T, the transfer function of the filter $W(f)$ is one cycle of a periodic function in frequency with period $1/T$. This can be seen by substituting $f = n/T + f'$ in Eq. (6). The "phase" or position in the spectrum of the periodic transfer function is determined by the phase of the local oscillator inherent in the phase modulator. Third, the value of the real part of the transfer function at harmonics of $1/T$ (i.e., zero phase) is determined at the instant that the phase shift introduced by the phase modulator is an integral multiple of 2π. Therefore, to properly position the transfer function in the frequency domain, the periodic waveform $w(t)$ must be synchronized with the phase modulator.

The power transfer function of the filter is proportional to the square of the instantaneous envelope of $w(t)$. That is,

$$|W(f)|^2 = \tfrac{1}{4}[w^2(fT^2) + \hat{w}^2(fT^2)]\,. \tag{8}$$

Therefore, if the phase of the filter is not important, only the envelope of $w(t)$ need be specified. However, $w(t)$ must still be periodic with period T.

At first glance, Eq. (6) may seem to impose undue phase limitations on the proposed filter because the imaginary part of the transfer function is completely determined by the real part. This is indeed a phase limitation, but it must be remembered that the imaginary part of the transfer function of any physically realizable filter is the negative of the Hilbert transform of the real part. Therefore, the conclusion is that any physically realizable filter can be synthesized with this technique. Usually the desired filter is specified in terms of the magnitude and phase versus frequency; but, using elementary complex algebra, one can easily obtain the real part of the transfer function from these two relationships.

In this explanation of the system operation, the "sum" frequency components resulting from the multiplication process were neglected. By using a negative frequency shift in the loop, the system can be designed to use the "sum" components instead of the "difference" frequency components, but a high-pass filter would have to be inserted in the loop to eliminate the accumulating "difference" components. In this case, the transfer function of the filter can be shown to be

$$W(f) = w(-fT^2) + j\hat{w}(-fT^2)\,. \tag{9}$$

Equation (9) shows that if $w(t)$ is an even function, then the transfer function in the "sum" mode is the complex conjugate of the transfer function in the "difference" mode.

It is worthwhile to consider the accuracy with which one can approximate a desired transfer function. The actual transfer function can be represented as a Fourier series with N terms where N must be less than k. Therefore, a particular transfer function in the band (f_1, f_2) can be approximated with an accuracy no better than a Fourier series of N terms where N is the highest integer in the ratio $f_1/(f_2-f_1)$. In other words, high-frequency, narrowband filters are theoretically approximated more accurately.

Applications

- *Delay synthesis*

A delay line with delay τ is characterized by the fact that

$$\frac{d\phi(f)}{df} = -2\pi\tau \tag{10}$$

and

$$|W(f)|^2 = \text{constant}\,.$$

Integrating Eq. (10) yields

$$\phi(f) = -2\pi\tau f + \phi_0\,, \tag{11}$$

where ϕ_0 is an arbitrary constant phase shift. A transfer function that satisfies the requirements is

$$W(f) = \tfrac{1}{2} \exp [-j(2\pi f\tau - \phi_0)] , \quad (12)$$

where it is understood that $W(f)$ is periodic with period $1/T$, so that Eq. (12) represents the response in the band $[k/T, (k+1)/T]$ only.

The real part of the transfer function is

$$\tfrac{1}{2} \cos (2\pi f\tau - \phi_0) . \quad (13)$$

Using the transformation from frequency to time, $fT^2 = t$, yields one period of the modulating waveform

$$w(t) = \cos [(2\pi\tau t/T^2) - \phi_0) . \quad (14)$$

If τ is not an integral multiple of T, then $w(t)$ will have a periodic discontinuity. This means that, theoretically, the previously stated condition 3 cannot be satisfied because $w(t)$ is not bandlimited. However, the effects of bandlimiting $w(t)$ to a large but finite number of harmonics will merely introduce linear distortion near the band edges, namely k/T and $(k+1)/T$. This effect can be tolerated by using an abundant bandwidth or, equivalently, a small T. The special case where τ is an integral multiple of the delay T results in a system that is equivalent to the frequency-shifting delay loop invented by Munster.[6] In this case, the delay line of length T is used in a simple frequency-multiplex manner to obtain delays of T, $2T$, $3T$, etc.

- *Chirp-filter synthesis*

Chirp filters are extensively used in radar and sonar matched-filter systems because of their Doppler insensitivity.[7] A chirp filter is characterized by square-law phase such that

$$\phi(f) = 2\pi\alpha f^2 + \phi_0 , \quad (15)$$

where α is a constant. The delay at any frequency is

$$\frac{d\phi(f)}{df} = 4\pi\alpha f . \quad (16)$$

A transfer function that satisfies this requirement is

$$W(f) = \tfrac{1}{2} \exp [j(2\pi\alpha f^2 + \phi_0)] . \quad (17)$$

As before, it is understood that Eq. (17) represents the response over the output band only. Using the frequency-time transformation yields

$$w(t) = \cos [(2\pi\alpha t^2/T^4) + \phi_0] . \quad (18)$$

As in the delay-synthesis application, the fact that $w(t)$ must be periodic with period T means that, if α/T^2 is not an integer, then $w(t)$ will exhibit a periodic discontinuity. In fact, because of the periodic abrupt change in the rate-of-change of the phase, the derivative of $w(t)$ will exhibit a periodic discontinuity irrespective of the value of α/T^2, and therefore $w(t)$ is not bandlimited. However, as argued in the delay-synthesis case, this effect can be tolerated by using a sufficiently large bandwidth.

Summary and conclusions

An electronic system has been described for synthesizing complicated filters in a prescribed frequency band. The key element of the system is a periodic, modulating waveform, one period of which is identical to the real part of the transfer function. In many cases this waveform may be difficult to generate, but for the important applications discussed in this paper the periodic waveform is easily produced. For other applications, digital techniques may be necessary to generate the periodic waveform.

Theoretically, the system can be used to synthesize any physically realizable narrowband filter. The technique should prove useful for matched-filter applications, delay-line synthesis, and channel equalization. The system requires accurate positive feedback, and therefore will pose some difficult analog-circuit design and maintenance problems. However, the operations could be performed digitally with considerable accuracy, thereby alleviating the design and maintenance difficulties.

Appendix

This appendix presents a proof of the system operation for the general input signal, $x(t)$. For convenience, the complex-polar, or analytic, signal representation that results from the single-sided inverse Fourier transform is used.[8] The analytic signal representation is denoted with the subscript a. For example

$$w(t) = \text{real part of } [w_a(t)] , \quad (A1)$$

and

$$w_a(t) = w(t) + j\hat{w}(t) . \quad (A2)$$

To begin the proof, define the product

$$u(t) = x(t)w(t) . \quad (A3)$$

Using conditions 1 and 3, and Bedrosian's theorem[9] for the Hilbert transform of the product of a low-pass and high-pass signal, yields

$$\begin{aligned} u_a(t) &= w(t)x(t) + jw(t)\hat{x}(t) \quad (A4) \\ &= w(t)x_a(t) . \end{aligned}$$

After one circulation, this signal is delayed T seconds and shifted in frequency by $1/T$ Hz. This yields

$$u_a(t - T) \exp (2\pi jt/T) .$$

After N circulations, the signal at the input to the bandpass filter is

$$\sum_{n=0}^{N} u_a(t - nT) \exp (2\pi jnt/T) . \quad (A5)$$

The Fourier transform of the analytic representation of the output of the bandpass filter is

$$Y_a(f) = \int_{-\infty}^{+\infty} G(f) \sum_{n=0}^{N} u_a(t - nT) \times \exp\{2\pi jt[(n/T) - f]\} dt , \quad \text{(A6)}$$

where $G(f)$ is the transfer function of the bandpass filter. Performing a time translation $t - nT \rightarrow t$ gives

$$Y_a(f) = \int_{-\infty}^{+\infty} \sum_{n=0}^{N} \exp(-2\pi jnfT) G(f) u_a(t) \times \exp[2\pi jt(n/T - f)] dt \quad \text{(A7)}$$

Because $w(t)$ is periodic (condition 2), it may be expanded into a Fourier series. This yields

$$u_a(t) = x_a(t) \sum_{-M}^{+M} A_m \exp(2\pi jmt/T) , \quad \text{(A8)}$$

where A_m is the complex amplitude of the m^{th} harmonic and, for notational simplicity, A_0 is assumed to be zero; M is the number of harmonic components in $w(t)$. Combining Eqs. (A7) and (A8):

$$Y_a(f) = \sum_{n=0}^{N} \sum_{m=-M}^{M} A_m \exp(-2\pi jnfT) G(f) \times \int_{-\infty}^{+\infty} x_a(t) \exp\{-2\pi jt[f - (m + n)/T]\} dt . \quad \text{(A9)}$$

Now, by condition 1, the Fourier transform of $x_a(t)$—namely $X_a(f)$—is nonzero only in the band $[k/T, (k+1)/T]$. Therefore, the integral in Eq. (A9) will vanish unless

$$(k/T) < [f - (m + n/T)] < [(k + 1)/T] .$$

Also by condition 4, $G(f)$ is nonzero only if

$$(k/T) < |f| < [(k + 1)/T] .$$

These two conditions ensure that $Y_a(f)$ will vanish unless

$$m = -n . \quad \text{(A10)}$$

Because n has only positive integer values, $Y_a(f)$ will vanish except for negative integer values of m. Therefore, Eq. (A9) becomes (after sufficient time to ensure that $N > M$)

$$Y_a(f) = \sum_{n=1}^{M} A_{-n} \exp(-2\pi jnfT) \times \int_{-\infty}^{+\infty} x_a(t) \exp(2\pi jft) dt \quad \text{(A11)}$$

$$= X_a(f) \sum_{n=1}^{M} A_{-n} \exp(-2\pi jnfT) . \quad \text{(A12)}$$

The transfer function of the bandpass filter has been omitted because it was assumed to be unity in the band $[k/T, (k + 1)/T]$. Because $w(t)$ is real, A_{-n} is the complex conjugate of A_n. Also recall that the spectrum of the analytic signal is twice that of the real signal for positive frequencies. Therefore

$$Y_a(f) = X_a(f) \sum_{m=1}^{M} A_m^* \exp(-2\pi jmfT) = \tfrac{1}{2} X_a(f) w_a^*(fT^2) . \quad \text{(A13)}$$

where the asterisk denotes a complex conjugate.

Equation (A13) describes a linear filter with the following transfer function:

$$W(f) = \tfrac{1}{2} [w(fT^2) - j\hat{w}(fT^2)] . \quad \text{(A14)}$$

Consequently, proof of Eq. (2) has been established for a general input and for filters that are subject to the constraints imposed by the conditions stated in the paper.

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