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Similar Motion of Two-Degree-of-Freedom Nonlinear Vibrating Systems with Nonsymmetric Springs*

Abstract: If similar motion occurs, it is demonstrated by a straight line (linear trajectory) in the configuration space whose orthogonal coordinates are the displacements of the masses in the vibrating system. Since the systems considered are conservative, all solutions of the equations of motion must satisfy the Principle of Least Action and its Euler-Lagrange equation. The solution of this equation defines a trajectory in configuration space, thus reducing the problem to one of geometry.

If the trajectory is linear, the Euler-Lagrange equation assumes a simple form. The form is further simplified if the coordinates of the configuration space are rotated and translated so that one axis coincides with the linear trajectory. Hence, if linear trajectories exist, it is necessary that the equation can be so simplified; it is sufficient that the rotation and translation be real.

One application of this analysis is shown for the case of a system whose anchor springs are air bearings, as used in a disk store.

Introduction

In applications dealing with linear systems, the concept of "free vibration in normal modes" is well defined and fully understood. The meaning of this phrase is not at all clear, however, when applied to nonlinear systems. Rosenberg and others[1–4] studied the normal-mode vibrations of certain nonlinear systems having multiple degrees of freedom. Their studies were restricted to systems whose springs resist (or aid) a prescribed deflection in tension or compression to the same degree and whose potential functions are negative definite. They have shown that vibration modes can exist where the displacements of the masses are linearly related. In addition they have shown that the equations of motion can be uncoupled and hence solved.

The purpose of this paper is to develop a systematic procedure to determine the conditions necessary for a spring-mass system to have free vibration with the motion of the masses linearly related. The restrictions imposed in previous work[1–4] do not apply.

We shall consider a two-degree-of-freedom conservative and scleronomous system as shown in Fig. 1. The masses are considered as mass points, and the springs are massless one-dimensional devices that change length under the action of a force.

There exists for this system an energy integral

$$T(m_1, m_2, \dot{u}_1, \dot{u}_2) - U(u_1, u_2) = h \text{ (a constant)}, \quad (1)$$

where T is the kinetic energy and U is a potential function.

* Based on author's Ph.D. thesis, University of California, 1964. (See Ref. 18.)

Application of Hamilton's principle to this system yields the equations of motion:

$$m_i \frac{d^2 u_i}{dt^2} = m_i \ddot{u}_i = \frac{\partial U}{\partial u_i} = U_{u_i}(u_1, u_2) \,. \quad (2)$$

It is convenient to transform to a new pseudosystem by means of the transformation

$$z_i = \sqrt{m_i}\, u_i \,. \quad (3)$$

With this transformation the energy integral (1) becomes

$$T(\dot{z}_1, \dot{z}_2) - U(m_1, m_2, z_1, z_2) = h \,. \quad (1a)$$

For this pseudosystem Hamilton's principle yields the equations of motion

$$\frac{d^2 z_i}{dt^2} = \ddot{z}_i = \frac{\partial U}{\partial z_i} = U_{z_i}(z_1, z_2) \,. \quad (2a)$$

The motion of the system can be represented by a curve in the configuration space (the 2-space whose orthogonal coordinates are the two displacements). This curve, defined by the equation

$$z_2 = z_2(z_1) \,, \quad (4)$$

is a trajectory of the system. For different initial conditions, the trajectories are different, and hence Eq. (4) is different.

Linear trajectories

In this paper we seek the most general potential function that will, with the proper initial conditions, yield an Eq.

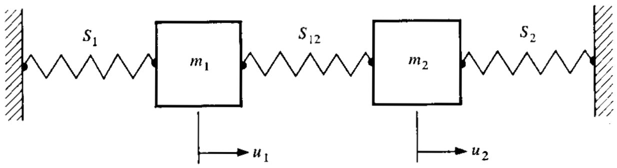
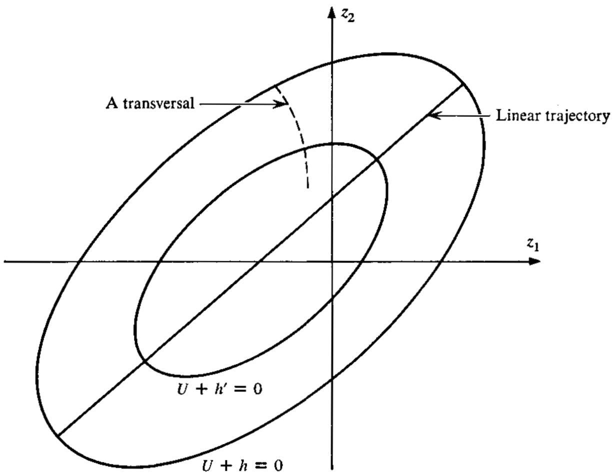


Figure 1 A conservative system having two degrees of freedom (see text).

Figure 2 Configuration space discussed in Result 1.



(4) that is linear. In other words, it is desired to find systems that can have at least one linear trajectory. In order to find these systems, we first of all enumerate three important results determined by Rosenberg.[4,5] These results are of considerable help in visualizing the configuration space and help to form the concept on which this work is based.

Result 1: Every straight line in the z_1z_2 configuration space which intersects all equipotential curves $U + h' = 0$ orthogonally, with $0 < h' \leq h$, is a linear trajectory. Conversely, every linear trajectory intersects all equipotential curves orthogonally. If the linear trajectory passes through the origin of the configuration space, it is a linear normal mode. (See Fig. 2.)

Result 2: Every trajectory of the pseudosystem which actually reaches the bounding curve $U + h = 0$ intersects it orthogonally.

Result 3: The transversal passing through any point in the configuration space is in the direction of the force acting on the unit mass at that point, i.e., at any point the force on the unit mass is in a direction normal to the potential curve.

Conventionally, the equations of motion (2) or (2a) are used for the study of vibrating systems. In this work we are interested in a trajectory in the configuration space. To approach the problem from this point of view it is convenient to consider the Principle of Least Action

$$\int_{s_a}^{s_b} \sqrt{U + h}\, ds = \text{minimum}\,. \tag{5}$$

Here s is the arc length along the path or trajectory of the motion. The Euler-Lagrange equation obtained from Eq. (5) is

$$2(U + h)z_2'' + (1 + [z_2']^2)(z_2' U_{z_1} - U_{z_2}) = 0\,. \tag{6}$$

The primes indicate differentiation with respect to z_1.

If a system is such that, with the proper initial conditions a linear trajectory can exist, it will appear as a straight line somewhere in the configuration space z_1, z_2 of the pseudosystem. Using a translation and a rotation, it is possible to move the coordinates to any desired position in the space they span. It is, therefore, possible to place a coordinate system (here taken as the x_1, x_2 system) so that one axis coincides with the linear trajectory if it exists. The Euler-Lagrange equation obtained from (5) will have exactly the same form for this coordinate system as it did in the original coordinate system.

$$2(U + h)x_2'' + (1 + [x_2']^2)(x_2' U_{x_1} - U_{x_2}) = 0. \tag{6a}$$

Here the primes indicate differentiation with respect to x_1.

Let us now assume the new coordinate system x_1, x_2 is located and orientated so the x_1 axis coincides with the linear trajectory if it exists. The linear trajectory is represented by the equation $x_2 \equiv 0$.

It follows that along the trajectory $x_2 \equiv x_2' \equiv x_2'' \equiv 0$.

Equation (6a) is valid throughout the domain, but along the trajectory it takes on the particularly simple form

$$U_{x_2}(x_1, 0) \equiv 0\,. \tag{7}$$

If a system is one that can possess a motion representable by a linear trajectory, it is necessary that Eq. (7) be satisfied. Sufficiency is satisfied if the rotation and translation used are real.

Let us now identify the systems that can meet this requirement. Since the transformation (3) to the pseudosystem presents no problem, it will be assumed in the following that we have the potential function of the pseudosystem.

The rotation and translation placing the coordinate system with the x_1 axis along the trajectory (if it exists) can be expressed by the matrix equation

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{8}$$

The rotation matrix is a nonsingular orthogonal matrix and satisfies

$$\nu_{1j}\ \nu_{1k} + \nu_{2j}\ \nu_{2k} = \delta_{jk} \qquad j, k = 1, 2\,, \tag{9}$$

where δ_{jk} is Kronecker delta. From Eq. (8), along the linear trajectory where $x_2 \equiv 0$, we can deduce

$$z_2 = \frac{\nu_{21}}{\nu_{11}} z_1 - \frac{\nu_{21}}{\nu_{11}} b_1 + b_2\,. \tag{10}$$

From Eq. (10) it is apparent that the slope of the trajectory, or modal constant, is

$$c_2 = \frac{\nu_{21}}{\nu_{11}} . \tag{11}$$

There are two ways to approach the matter of determining what systems satisfy Eq. (7) with real rotations and translations in Eq. (8). One approach is to integrate Eq. (7) and use the inverse of Eq. (8) to return to the coordinate system of the pseudosystem. The other approach is to consider different classes of systems separately and use the following procedure:

(1) Assume the linear trajectory exists.
(2) Substitute Eq. (8) in the potential function.
(3) Perform the operations indicated by Eq. (7).
(4) Solve for the rotation matrix and translation matrix.

Because of its generality, the first approach is cumbersome to use; consequently, the second will be used in this paper.

Some general systems

If springs whose forces are everywhere analytic in the deflections are considered, one general form of a potential function of the pseudosystem is

$$U(z_1, z_2) = -\sum_{m=0}^{r} \left\{ \frac{a_1^{(m)}}{m+1} z_1^{m+1} + \frac{a_2^{(m)}}{m+1} z_2^{m+1} + \frac{a_{12}^{(m)}}{m+1} (z_1 - \mu_{12} z_2)^{m+1} \right\} . \tag{12}$$

Here $\mu_{12}^2 = m_1/m_2$ is the mass ratio, $a_i{}^{(m)}$ and $a_{12}^{(m)}$ are coefficients that may have any value including zero, and r is some number, the maximum value of m. Considering now a rotation and translation as given by Eq. (8) and differentiation as given by Eq. (7), we have ($b_i' = b_i/\nu_{11}$)

$$\begin{aligned} U_{x_2}(x_1, 0) = -\sum_{m=0}^{r} \nu_{11}^{m+1} \frac{\nu_{22}}{\nu_{11}} \Big\{ & -a_1^{(m)} c_2 (x_1 + b_1')^m \\ & + a_2^{(m)} (c_2 x_1 + b_2')^m \\ & - a_{12}^{(m)} (c_2 + \mu_{12}) ([1 - \mu_{12} c_2] x_1 \\ & + [b_1' - \mu_{12} b_2'])^m \Big\} \equiv 0 . \end{aligned} \tag{13}$$

The satisfaction of (13) is necessary for the existence of linear trajectories. It is sufficient that c_2, b_1, and b_2 be real.

It is apparent that there are different powers K of x_1, $K = 0, 1, \cdots, r$ in Eq. (13). The coefficient of each must vanish in order that $U_{x_2}(x_1, 0) \equiv 0$. Hence, we have $r + 1$ equations from (13), and there are three unknowns. In general, only a favorable combination of coefficients will allow a solution.

Based on the above discussion the following result is stated:

Result 4: A two-degree-of-freedom spring-mass system whose potential function is given by (12) has a linear trajectory provided *real* c_2 and b_i can be found that satisfy the $r + 1$ equations obtained from Eq. (13).

Considering linear normal modes, $b_i = 0$ and Eq. (13) take

$$\begin{aligned} U_{x_2}(x_1, 0) = -\sum_{m=0}^{r} \nu_{11}^{m+1} \frac{\nu_{22}}{\nu_{11}} \Big\{ & -a_1^{(m)} c_2 + a_2^{(m)} c_2^m \\ & - a_{12}^{(m)} (c_2 + \mu_{12}) (1 - \mu_{12} c_2)^m \Big\} x_1^m \equiv 0 . \end{aligned} \tag{14}$$

This can be satisfied regardless of the values assumed by m under certain conditions.

Result 5: Given a two-degree-of-freedom spring-mass system whose potential function is given by (12),

(1) The system has a linear normal mode with $c_2 = \mu_{21}$ if $a_1^{(m)} = \mu_{21}^{m-1} a_2^{(m)}$.
(2) The system has a linear normal mode with $c_2 = -\mu_{12}$ if $a_1^{(m)} = (-\mu_{12})^{m-1} a_2^{(m)}$.
(3) If the masses are equal and the anchor springs behave as though m is always odd, then both the above can be satisfied.

(See Appendix V of Ref. 18 for the method of solution when the exponents are not odd but behave as though odd.)

The homogeneous case is defined as that where m assumes a single particular value throughout the potential function. If linear normal modes of homogeneous systems are sought, a single equation is obtained from Eq. (14):

$$f(c_2) = -a_1^{(m)} c_2 + a_2^{(m)} c_2^m - a_{12}^{(m)} (c_2 + \mu_{12}) (1 - \mu_{12} c_2)^m = 0 . \tag{15}$$

Expanding the term with the exponent, Equation (15) can be written

$$f(c_2) = a_{12}^{(m)} \mu_{12}^m c_2^{m+1} + p_m c_2^m + \cdots + p_1 c_2 - a_{12}^{(m)} \mu_{12} = 0 . \tag{16}$$

If m is an odd number (or if the springs produce forces that are odd functions of the displacement regardless of the value of m), there will always be at least one positive and one negative real root to Eq. (16). This is easily proved as follows:

If $c_2 = 0$, then $f(0) < 0$ (assuming $a_{12}{}^{(m)} > 0$; reverse reasoning can be used if $a_{12}^{(m)} < 0$).

If $c_2 = a$ or $c_2 = -b$, where a and b are sufficiently large, the first term of (16) will become predominant, and we have $f(a) > 0$ and $f(-b) > 0$.

From the continuity of $f(c_2)$, it follows that there must be a real root $0 < c_2 < a$ and a real root $-b < c_2 < 0$.

From this observation follows the theorem:

A homogeneous two-degree-of-freedom system with a potential function of the form of (12) and whose springs produce forces that are odd functions of their displacement always *has at least one linear normal mode of positive slope (in-phase mode or i-mode) and at least one linear normal mode of negative slope (out-of-phase mode or o-mode).*

• *Systems with first- and second-degree spring-force terms*
As previously mentioned, it is apparent that there are different powers K of x_1, $K = 0, 1, \cdots, r$ in Eq. (13). The coefficient of each must vanish in order that $U_{x_2}(x_1, 0) \equiv 0$. It follows that there are $r + 1$ equations from (13), and there are three unknowns c_2, b_1', and b_2'. It is logical to investigate the case where $r = 2$, since this produces the same number of equations as unknowns. One of these equations (the vanishing of the coefficients of x_1^2) is identical to that obtained when seeking linear normal modes for a system that is homogeneous of degree 2. We conclude then that any linear trajectory for such a system must be parallel to the linear normal modes of the second-degree homogeneous system. The remaining two equations provide restrictions on the coefficients that must be satisfied for the linear trajectories to exist.

• *Systems with piecewise linear springs*
A frequently investigated form of nonlinear spring mass system is that where the spring is piecewise linear.[6] An example of this is shown in Fig. 3. Here, different springs are applied, depending upon the direction of the motion; hence, a force vs deflection curve with a corner is created as shown. If one considers a two-degree-of-freedom system with piecewise linear positive springs (springs that resist stretching or compressing), the forces of the pseudosystem are as shown in Fig. 4.

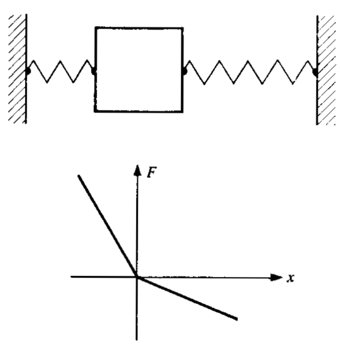


Figure 3 System having piecewise linear spring, in which different springs are applied, depending on direction of motion.

The force-displacement functions for anchor spring 1 can be written as follows:

$$F_1 = -mz_1 + (m - s)b \qquad z_1 \le b$$
$$F_1 = -sz_1 \qquad b \le z_1 \le a$$
$$F_1 = -rz_1 + (r - s)a \qquad z_1 \ge a\,.$$

Similar expressions can be written for the other two springs.

The potential function for the pseudosystem that has these forces can be written

$$U(z_1, z_2) = -a_1^{(0)} z_1 - \frac{a_1^{(1)}}{2} z_1^2 - a_2^{(0)} z_2 - \frac{a_2^{(1)}}{2} z_2^2 - a_{12}^{(0)}(z_1 - \mu_{12} z_2) - \frac{a_{12}^{(1)}}{2}(z_1 - \mu_{12} z_2)^2\,. \tag{17}$$

In this case the coefficients $a_1^{(0)}$, $a_1^{(1)}$, etc., take on different values, depending upon what zone of the appropriate force-deflection curve is applicable at any given instant. For example, if $z_1 \ge a$, then $a_1^{(1)} = r$ and $a_1^{(0)} = (s - r)a$.

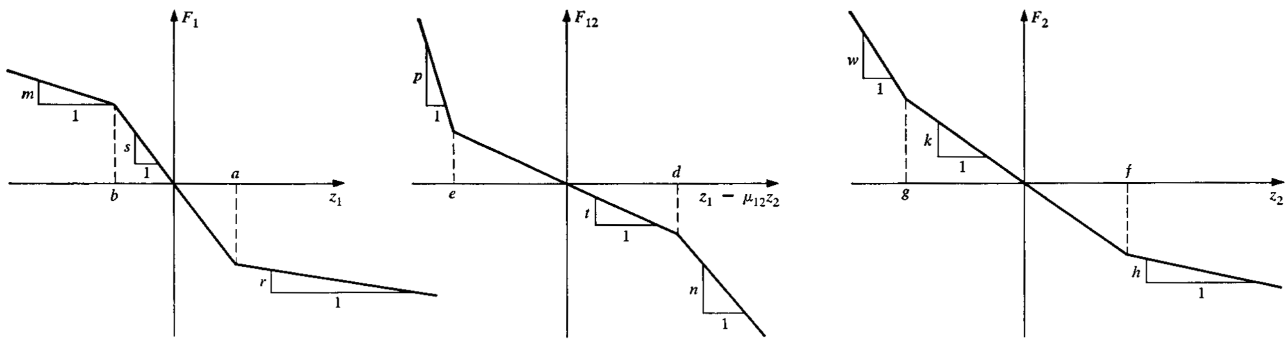
As in the previous cases, it is necessary that Eq. (7) be satisfied along any linear trajectory that exists. Substitution of the transformation (8) in Eq. (17) and differentiation produces Eq. (13) with m having the values 0 and 1.

If Eq. (13) is to vanish identically, it is clear that two equations must be satisfied (the coefficient of x_1 must vanish, and the constant terms must vanish).

Using these two equations, the slope of the linear trajectory and a relationship between b_1 and b_2 can be found. The relationship between b_1 and b_2 is, as inspection will reveal, a linear relationship $b_2 = c_2 b_1 + A$.

As previously stated, all the coefficients assume different values in different ranges of the independent variables. The equations must both be satisfied everywhere on the trajectory. If the springs are like the one shown in Fig. 3, or if at any point along the trajectory all springs are operating in the center range shown in Fig. 4, at least one set of

Figure 4 Forces in system having two degrees of freedom and incorporating piecewise linear positive springs.



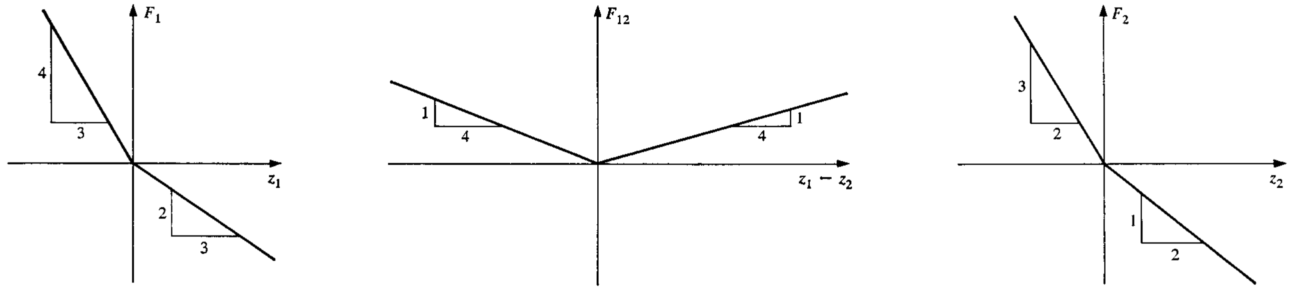


Figure 5 Spring forces in a piecewise linear system where the two masses are equal.

$a_1^{(0)} = a_2^{(0)} = a_{12}^{(0)} = 0$, and it can easily be shown that $A = 0$. If $A = 0$, we may choose $b_1 = b_2 = 0$ since the line along which b_1 and b_2 are located passes through the origin. Under these conditions the problem reduces to one of finding linear normal modes since the linear trajectories are given by $z_2 = c_2 z_1 + A$. The following work is confined to the search for linear normal modes for these systems. The procedure is the same but algebraically difficult if all linear trajectories are considered. We have, then, two equations with only one unknown (vanishing of coefficients of x_1 and vanishing of constant terms). One of these can be considered an equation for the modal constant c_2 and the other a restriction on the coefficients:

$$x_1\colon a_2^{(1)} c_2 - a_1^{(1)} c_2 - a_{12}^{(1)} (c_2 + \mu_{12})(1 - \mu_{12} c_2) = 0 \tag{18}$$

$$\text{const: } a_2^{(0)} - c_2 a_1^{(0)} - a_{12}^{(0)} (c_2 + \mu_{12}) = 0\,. \tag{19}$$

These equations must remain invariant regardless of the zone of operation of the springs at any given instant. It is convenient to write Eq. (18) in a different form to see the consequence of this requirement.

$$c_2^2 + \mu_{21} \left\{ \frac{a_2^{(1)} - a_1^{(1)}}{a_{12}^{(1)}} - (1 - \mu_{12}^2) \right\} c_2 - 1 = 0\,. \tag{20}$$

It is apparent, then, that the ratio $(a_2^{(1)} - a_1^{(1)})/a_{12}^{(1)}$ must not change all along the linear normal mode even though the three coefficients are different in different zones of the deflection. If we refer to Fig. 4 and consider values of c_2 in three ranges, this requirement demands that the slopes of the force-deflection curves be related as follows:

(1) If $c_2 > \mu_{21}$,

$$\frac{a_2^{(1)} - a_1^{(1)}}{a_{12}^{(1)}} = \frac{h - r}{p} = \frac{k - s}{t} = \frac{w - m}{n} \tag{21}$$

(2) If $0 < c_2 < \mu_{21}$,

$$\frac{a_2^{(1)} - a_1^{(1)}}{a_{12}^{(1)}} = \frac{h - r}{n} = \frac{k - s}{t} = \frac{w - m}{p} \tag{22}$$

(3) If $c_2 < 0$,

$$\frac{a_2^{(1)} - a_1^{(1)}}{a_{12}^{(1)}} = \frac{w - r}{n} = \frac{k - s}{t} = \frac{h - m}{p}\,. \tag{23}$$

The requirement that $(a_2^{(1)} - a_1^{(1)})/a^{(1)}$ remain constant along the modal line also requires that more than one spring be deflected to a corner if any are. Considering now the case where all three springs have corners in their respective force-deflection curves, the following requirements are deduced by demanding that the deflections be such that corners on all force-displacement curves be reached simultaneously.

(1) If $c_2 > \mu_{21}$,

$$\begin{aligned} a - e &= \mu_{12} f \\ b - d &= \mu_{12} g \\ c_2 \quad &= f/a = g/b \end{aligned} \tag{24}$$

(2) If $0 < c_2 < \mu_{21}$,

$$\begin{aligned} a - d &= \mu_{12} f \\ b - e &= \mu_{12} g \\ c_2 \quad &= f/a = g/b \end{aligned} \tag{25}$$

(3) If $c_2 < 0$,

$$\begin{aligned} a - d &= \mu_{12} g \\ b - e &= \mu_{12} f \\ c_2 \quad &= g/a = f/b\,. \end{aligned} \tag{26}$$

Equation (19) remains to be satisfied. (It is also easy to show that the value of c_2 given by (24), (25), or (26) satisfies Eq. (20).) From the above discussion we state the following result:

Result 6: A two-degree-of-freedom system, whose springs have continuous piecewise linear force-deflection curves, as shown in Fig. 4, has a linear normal mode providing either (24) and (21) or (25) and (22) or (26) and (23) are satisfied in addition to Eq. (19). The slope of the modal line is as given by (24), (25), or (26).

From the previous discussion one might be tempted to conclude that systems with piecewise linear springs cannot have more than one modal line. This is not true; in fact, it is possible to have three modal lines, one with $c_2 > \mu_{21}$, one with $0 < c_2 < \mu_{21}$, and one with $c_2 < 0$.

Since Eqs. (21), (22), and (23) all have one common term, it follows that the best we can hope for is to obtain the same value of $(a_2^{(1)} - a_1^{(1)})/a_{12}^{(1)}$ from each and hence obtain two values of c_2 from Eq. (20). The restrictions (24), (25), and (26) appear to further reduce this to one possibility. There is, however, one way to circumvent this difficulty. The restrictions given by (24), (25), and (26) or the equations from which they are derived, are all satisfied if there is a single corner allowed for each spring and that is at the origin. Under these conditions, $a = b = d = e = f = g = 0$. In effect, this system satisfies the restriction (19) by putting $a_1^{(0)} = a_2^{(0)} = a_{12}^{(0)} = 0$ in all "zones of operation."

By allowing a single corner at the point of zero deflection of each spring, the requirement that the ratio $(a_2^{(1)} - a_1^{(1)})/a_{12}^{(1)}$ be constant along a modal line again produces Eqs. (21), (22), and (23), except the common term is now missing; so it may be possible to satisfy these and have the ratio be $(a_2^{(1)} - a_1^{(1)})/a_{12}^{(1)}$ different in each case.

Equations (21), (22), and (23) are all satisfied if $n = -p$ and $h - r = m - w$. If this is the case, there will be at least two, and perhaps three, different values of $(a_2^{(1)} - a_1^{(1)})/a_{12}^{(1)}$ obtained from the three equations.

Example 1

Consider now a piecewise linear system with the two masses equal. The spring forces of the system are as shown in Fig. 5. Referring to Fig. 4 for comparison, the constants for this system are:

$$s_1:\quad m = \frac{4}{3},\ r = \frac{2}{3} \qquad s_{12}:\quad p = \frac{1}{4},\ n = -\frac{1}{4}$$

$$s_2:\quad w = \frac{3}{2},\ h = \frac{1}{2}\,.$$

For this system, the masses are equal, and Eq. (21) is valid for $c_2 > 1$. This equation yields

$$\frac{a_2^{(1)} - a_1^{(1)}}{a_{12}^{(1)}} = \frac{h-r}{p} = \frac{w-m}{n} = -\frac{2}{3}\,.$$

Using this value of $(a_2^{(1)} - a_1^{(1)})/a_{12}^{(1)}$, Eq. (20) yields $c_2 = -0.721, 1.388$.

Of the two values, only $c_2 = 1.388$ is greater than one, and hence it is the only one of the two that is valid due to the restriction on (21). Similarly, considering Eqs. (22) and (23), we obtain other valid values of c_2 i.e., $c_2' = 0.721$ and $c_2'' = -0.28$.

This example, therefore, satisfies all three of the equations, (21), (22), and (23), and yields a different value of the ratio $(a_2^{(1)} - a_1^{(1)})/a_{12}^{(1)}$ in each case. With each value the quadratic equation (20) produces a modal constant that is in the restricted range related to each of these equations. This system has piecewise linear springs and has 2 degrees of freedom and there are 3 modal lines. Figure 6 shows the configuration space with equipotential contours and the modal lines for this case, as found by direct solution of the differential equations on an analog computer.

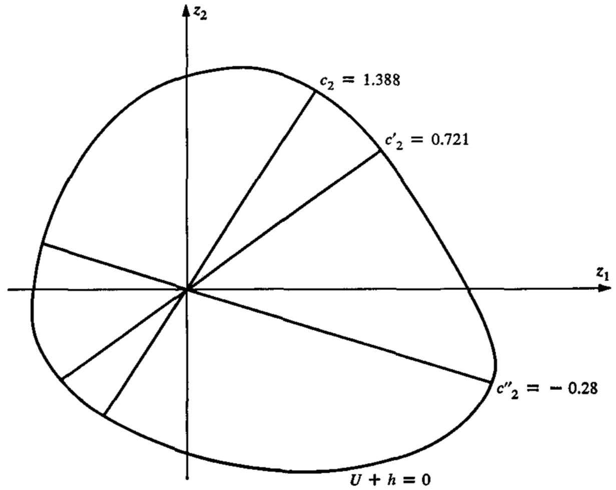


Figure 6 Configuration space with equipotential contours and modal lines for system having piecewise linear springs and two degrees of freedom.

- *Systems with other piecewise springs*

The techniques used in the study of systems with piecewise linear springs can be used for systems where the force-deflection curves of the springs are piecewise combinations of any functions. The functions, for example, may be such that the forces are proportional to the displacement raised to some exponent in one range of displacement and some other exponent in another range.

Potential functions of other forms

This discussion and examples to this point have dealt with systems whose potential function for the pseudosystem can be written in the form of Eq. (12). Even though some of the results presented up to this point are restricted to systems of this form, the general techniques used are valid for other systems. For example, there are no restrictions on the form of the potential function as long as it satisfies Eq. (7).

A system whose anchor springs are air bearings as described by Haughton[7] and whose coupling spring is linear is an example of a system with a potential function that is quite different from those previously considered. This is a system that is realistic; it occurs in actual operation in some large capacity magnetic memory devices.

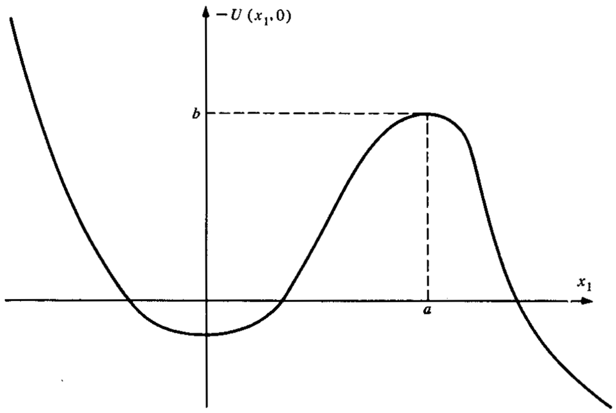


Figure 7 Plot for determining whether or not trajectory is bounded. (See explanation of Eq. (29) in text.)

For this system, the potential function of the pseudosystem can be written

$$U(z_1, z_2) = -\frac{1}{ma_0}\left\{F_0\left[\frac{1}{(n-1)(1+z_1)^{n-1}} + \frac{1}{(n-1)(1-z_2)^{n-1}} + z_1 - z_2\right] + \frac{Aa_0}{2}(z_1 - z_2)^2\right\}. \quad (27)$$

Here, F_0 = equilibrium load in the coupling spring in lbs.
A = coupling spring rate in lbs/in.
m = slider mass in lb-sec^2/in.
a_0 = equilibrium spacing of air bearing in in.
$z_i = u_i/a_0$.

Using the techniques described above in this paper, it can be shown that the linear normal mode $z_2 = -z_1$ exists for this system.

Stability

In order to discuss stability even briefly, it is necessary that a clear definition of stability be established. A trajectory, linear or not, in two-dimensional configuration space can be represented by Eq. (4).

Suppose that $z_2^*(z_1)$ is a trajectory, where the $z_i(t)$ satisfy the equations (2a) of motion. $z_2^*(z_1)$ is said to be orbitally stable if for any $0 < \epsilon \ll 1$ which satisfies

$$|z_2(z_1) - z_2^*(z_1)| < \epsilon\,,$$

there exists a $0 < \delta(\epsilon) \leq \epsilon$ such that

$$|z_2(z_1^0) - z_2^*(z_1^0)| < \delta(\epsilon)\,.$$

Here z_1^0 represents the maximum displacement. In other words, the solution is orbitally stable if the motion in the configuration space is contained in an ϵ tube about the trajectory $z_2^*(z_1)$.

In this work a solution is considered stable if it is orbitally stable and z_2 is bounded. It may seem superfluous to say z_2 is bounded, but potential functions that are not negative definite have been allowed, and it is possible to have orbitally stable systems with unbounded deflections. If z_2 is bounded, then the length of the trajectory is bounded.

Considering the above discussion, it is quite natural to investigate stability along the trajectory to see if z_2 is bounded and normal to the trajectory to see if the trajectory is orbitally stable.[11] The previous work supplies a good basis from which to study stability in this manner since the transformations found using Eqs. (8) and (7) place the x_1 axis along the linear trajectory.

The trajectory is given by $x_2 \equiv 0$, in this coordinate system, so the coordinates of a point that neighbors the trajectory in configuration space are given as (x_1, η_2), where η_2 represents a "small" value in the x_2 direction. In this neighborhood the potential function can be expanded in the power series

$$U(x_1, \eta_2) = U(x_1, 0) + \eta_2 U_{x_2}(x_1, 0) + \frac{\eta_2^2}{2!} U_{x_2x_2}(x_1, 0) + \cdots \quad (28)$$

The existence of a linear trajectory requires the satisfaction of Eq. (7); therefore, the second term of the potential function (28) vanishes and, to first order in the variable η_2, the following equations of motion can be written:

$$\ddot{x}_1 = U_{x_1}(x_1, 0) \quad (29)$$

$$\ddot{\eta}_2 = \eta_2 U_{x_2x_2}(x_1, 0)\,. \quad (30)$$

The solution of Eq. (29) indicates directly whether or not the trajectory is bounded. Since the potential function is the negative of the potential energy, it is easy to tell if the trajectory is bounded by inspecting a plot showing the negative of the potential function along the x_1 axis. Figure 7 shows a possible graph of this type. As long as the initial conditions are such that additional energy must be supplied to escape the concave upward section of the energy curve, the trajectory is bounded. For example, in Fig. 7, if the initial energy level is less than b and the initial displacement is less than a, then the motion will be such that x_1 is contained by the energy well. However, if the initial displacement is greater than a, the deflection will increase indefinitely since the system will seek the lowest energy level. If the initial displacement is negative and the energy level is greater than b, the motion will be such that $x_1 = a$ will be exceeded and also go on to unbounded deflections.

The solution of Eq. (30) determines if the η_2 variable stays within the ϵ tube about the trajectory.

Example 2

As a final example, we consider a system of considerable interest since the system has one stable linear normal mode,

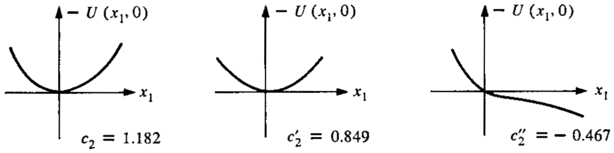
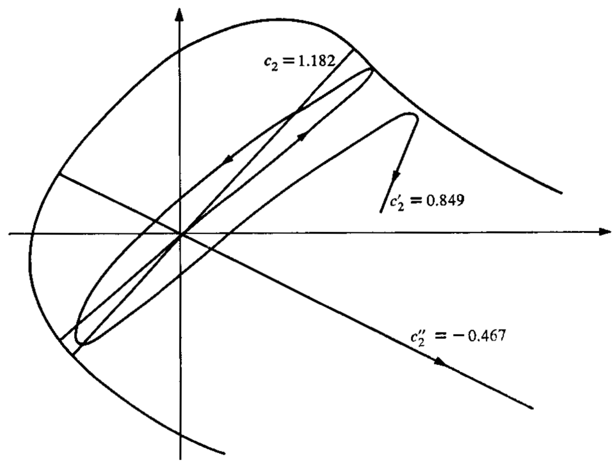


Figure 8 Energy contours for three modes in the system of Example 2.

Figure 9 Energy contours and modal lines in system of Example 2.



one where the deflections become unbounded along the modal line, and one that is unstable normal to the modal line. Consider a system with piecewise linear springs that is the same as Example 1 except the coupling spring is twice as stiff.

For this system, Eqs. (21), (22) and (23) yield

$$c_2 > 1: \qquad \frac{w - m}{n} = \frac{h - r}{p} = -\frac{1}{3}$$

$$0 < c_2 < 1: \qquad \frac{h - r}{n} = \frac{w - m}{p} = \frac{1}{3}$$

$$c_2 < 0: \qquad \frac{h - m}{p} = \frac{w - r}{n} = -\frac{5}{3}\,.$$

In each case the number determined above is substituted into Eq. (20), and it is solved for c_2. In this case each quadratic equation yields a valid constant, and there are three modal constants:

$$c_2 = 1.182, \qquad c_2{'} = 0.849, \qquad c_2{''} = -0.467\,.$$

Figure 8 shows these energy contours as calculated using Eq. (17). It is apparent, since the system will seek the position of minimum potential energy, that the modal line with $c_2{''} = -0.467$ will have unbounded deflections and hence is unstable along the modal line.

To study stability normal to the modal line, Eq. (30) must be investigated. For the piecewise linear case, as we have here, $U_{x_2x_2}(X_1, 0)$ is piecewise a constant. For this example, the motion with $C_2{'} = 0.849$ is unstable normal to the modal line since $U_{x_2x_2}(X_1, 0) > 0$ in part of the range of motion. For the case $C_2 = 1.182$ a careful study is required (beyond the scope of this paper) since $U_{x_2x_2}(X_1, 0)$ assumes two different values, one when $X_1 > 0$ and another when $X_1 < 0$ and both are negative, hinting possible stability. Figure 9 shows the energy contours and modal lines for this system as plotted by analog computer (Note the modal line $C_2 = 1.182$ is stable).

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