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On the Equation $i = i_0 [\exp \alpha (v - Ri) - 1]$

In general, the current-voltage characteristic of a nonlinear resistive element can be represented by an exponential function:

$$i = i_0(e^{\alpha v} - 1),$$

where i is the current through the nonlinear resistor, v is the voltage across the resistor, and i_0 and α are constants. In a circuit containing a linear resistor R in series with the nonlinear resistor, the characteristic becomes

$$i = i_0[e^{\alpha(v - Ri)} - 1], \quad (1)$$

where v is a series voltage source.

E. M. Rutz-Philipp¹ and others have analyzed the effect of interference phenomena in circuits described by Eq. (1). In this communication we seek to represent i as a power series in v , where the coefficients of the power series are the derivatives of i with respect to v , evaluated at $v = 0$.

Orloff² has obtained the first six derivatives directly. However, by discovering that i'' can be written as a polynomial in i' we are able to determine all higher derivatives as polynomials in i' whose coefficients satisfy a simple recursion.

First, using $i' = di/dv$, $i'' = d^2i/dv^2$, we differentiate (1) twice with respect to v , to find, successively

$$i' = i_0\alpha(1 - Ri')e^{\alpha(v - Ri)} = \alpha(1 - Ri')(i + i_0) \quad (2)$$

$$i'' = -\alpha Ri''(i + i_0) + \alpha(1 - Ri')i'. \quad (3)$$

Next, we eliminate $(i + i_0)$ in (3), using (2), to find

$$i'' = -\frac{Ri''i'}{(1 - Ri')} + \alpha(1 - Ri')i'$$

or solving for i'' ,

$$i'' = \alpha(1 - Ri')^2 i'. \quad (4)$$

That is, i'' is a polynomial in i' . This fortuitous relationship will permit us to obtain all higher derivatives through this polynomial.

Let the k th derivative of i with respect to v be represented as a function H_k of i' . Thus, by definition,

$$i' \equiv H_1(i') = i', \quad (5)$$

and rewriting (4)

$$i'' = H_2(i') = \alpha(1 - Ri')^2 i'. \quad (6)$$

Now, the next derivative $H_3(i')$ will be the derivative of the polynomial $H_2(i')$ with respect to its argument i' , denoted $H_2'(i')$, multiplied by the derivative of the argument i' with respect to v ; but this latter factor is i'' , or simply $H_2(i')$. That is,

$$H_3(i') = H_2'(i')H_2(i').$$

Now, since $H_2(i')$ is a polynomial, so is $H_2'(i')$ and therefore $H_3(i')$. Continuing in this way, it is apparent that

$$H_{k+1}(i') = H_k'(i')H_k(i'), \quad k = 1, 2, \dots \quad (7)$$

and each $H_k(i')$ is a polynomial in i' .

Since the $H_k(z)$, $z = i'$, are polynomials it is possible to organize the computation of their coefficients in a simple way, by means of (7). $H_2(z)$ is of degree 3, and $H_k'(z)$ is of degree 1 less than $H_k(z)$, so it is clear that their product in (7), $H_{k+1}(z)$, will be of degree 2 greater than that of $H_k(z)$. Thus, recalling that the degree of $H_1(z)$ is 1, we find the degree of $H_k(z)$ to be $(2k - 1)$. Let, then, the coefficients of $H_k(z)$ be of the form

$$H_k(z) = \frac{\alpha^{k-1}}{R} \sum_{j=0}^{2k-1} a_{k,j}(Rz)^j, \quad (8)$$

where α , R are constants given above, and the $a_{k,j}$ are new constants to be determined. The particular form of (8) is motivated by Eq. (4) where i' (or z) is grouped naturally with the constant factor R .

By direct calculation, then, using (7) and (4), we find

$$\begin{aligned} H_{k+1}(z) &= \left[\frac{\alpha^{k-1}}{R} \sum_{j=0}^{2k-1} ja_{k,j}(Rz)^{j-1}R \right] [\alpha(1 - Rz)^2 z] \\ &= \frac{\alpha^k}{R} \sum_{j=0}^{2k-1} [ja_{k,j}(Rz)^j - 2ja_{k,j}(Rz)^{j+1} \\ &\quad + ja_{k,j}(Rz)^{j+2}]. \end{aligned}$$

Next, we sum the three terms separately, and replace the variable of summation j by $(j - 1)$ and $(j - 2)$ in the second

and third terms, respectively, to find

$$H_{k+1}(z) = \frac{\alpha^k}{R} \left[\sum_{j=0}^{2k-1} ja_{k,j}(Rz)^j + \sum_{j=1}^{2k} -2(j-1) \right. \\ \left. \times a_{k,j-1}(Rz)^j + \sum_{j=2}^{2k+1} (j-2)a_{k,j-2}(Rz)^j \right].$$

If we define $a_{k,j} = 0$ for $j < 0$ and $j > 2k - 1$, we can rewrite these three sums as one,

$$H_{k+1}(z) = \frac{\alpha^k}{R} \sum_{j=0}^{2k+1} [ja_{k,j} - 2(j-1)a_{k,j-1} \\ + (j-2)a_{k,j-2}](Rz)^j.$$

Now, comparing this last expression with (8) we see that the exponent of α and the limits of summation are as given in (8) for the index $(k+1)$, and that the coefficients $a_{k+1,j}$ are displayed as

$$a_{k+1,j} = ja_{k,j} - 2(j-1)a_{k,j-1} + (j-2)a_{k,j-2}. \quad (9)$$

The coefficients of polynomials H_k , $k = 2, 3, \dots$ can now be computed recursively by (9), using as initial values

$$a_{1,1} = 1, a_{1,j} = 0 \quad \text{if } j \neq 1. \quad (10)$$

Finally, in order to evaluate the derivatives, the polynomials H_k must be evaluated at i' when $v = 0$. Referring to (2), i' is determined by i , and, in turn, referring to (1), i is determined by v ; at $v = 0$, $i = 0$, and then

$$z = i' = \alpha i_0 / (1 + R\alpha i_0). \quad (11)$$

Thus, the power series expansion of i about $v = 0$ is given by

$$\sum_{k=1}^{\infty} \frac{1}{k!} H_k(z) v^k, \quad (12)$$

where the $H_k(z)$ are evaluated by Eqs. (8) through (11).

As noted, a power series expansion for i has been developed previously² in which the derivatives are represented as functions of i rather than i' and the first six are obtained by successive differentiation. In this case the derivatives are now rational rather than polynomial in their argument. In fact, solving for i' in (2) and substituting this in (8) the derivatives are

$$H_k \left[\frac{\alpha(i + i_0)}{1 + R\alpha(i + i_0)} \right] = \frac{\alpha^{k-1}}{R} \\ \times \sum_{j=0}^{2k-1} a_{k,j} \left[\frac{R\alpha(i + i_0)}{1 + R\alpha(i + i_0)} \right]^j.$$

References

1. E. M. Rutz-Philipp, "Power Conversion in Nonlinear Resistive Elements Related to Interference Phenomena," IBM Journal 11, 544-552 (1967), this issue.
2. L. M. Orloff, "Intermodulation Products in Crystal Mixers," M.S. Thesis, Polytechnic Institute of Brooklyn, June, 1963.

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