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Estimation of Temperature Rise in Electron Beam Heating of Thin Films

Abstract: Expressions are given for estimating the transient temperature distribution in a thin film heated by an electron beam. Graphical presentations in terms of dimensionless (reduced) variables are included to aid in specific calculations for designing an electron beam for use as a heating tool.

List of symbols

a	radius (or equivalent radius) of the heat supply (see Figs. 2 and 3)
c	heat capacity of the target
D	thickness of the film
$Ei(x)$	exponential integral $= \int_{-\infty}^{x} e^u \, du/u \qquad (x < 0)$
F	total heat flow per unit time
H	a constant related to emissivity and average temperature of the target (definition following Eq. (4))
I_0	modified Bessel function of order zero
J_0	Bessel function of order zero
J_1	Bessel function of order one
K	thermal conductivity of the target
q	heat supply per unit time per unit volume
Q	heat supply per unit time per unit area
Q_0	maximum heat supply (see Figs. 2 and 3)
r	distance from the center of the beam to a point (see Fig. 1)
R	range of electron penetration
t	time
T	temperature (in °K)
T_0	ambient temperature (in °K)
w	heat loss per unit time per unit volume
W	heat loss per unit time per unit area
Y_0	Bessel function of the second kind of order zero
Y_1	Bessel function of the second kind of order one
z	distance in the beam direction (Fig. 1)
α	thermal diffusivity of the target
γ	Euler's constant $= 0.5772 \cdots$
ρ	density of the target
ϕ	angular variable (Fig. 1)
θ	excess temperature of target over the ambient
Θ	dimensionless temperature (Section 4)
τ	dimensionless time (Section 4)
ξ	dimensionless radial distance (Section 4)
η	dimensionless radiation coefficient (Section 4)

1. Introduction

When an electron beam hits a target, part of its kinetic energy is converted into thermal energy in the target. The conversion is almost instantaneous, compared to the time scale of heat conduction,† but is by no means complete. Some of the electrons are reflected and scattered away, and some are transmitted through the target, if it consists of a very thin film. Kinetic energy is also changed into forms other than thermal, e.g., into energies of secondary electrons, bremsstrahlung, X-rays, etc. The relative importance of each of these phenomena depends on the accelerating voltage, as well as on the material of the target. When the target is heated close to its boiling point, still another mode of energy loss is caused by the scattering of electrons by the vapor molecules.

Nevertheless, the electron beam stands out as a heat source with the highest energy density yet available.[1] The ease of controlling the intensity, shape and position of the electron beam, in addition to its high energy density, makes electron beam drilling and welding highly advantageous.[1,2] The possibility of heating an extremely small area lends itself also to many applications in microelectronics.[3,4]

The temperature rise in the target depends not only on the heat capacity of the material but also on its thermal conductivity and emissivity. References 1 and 2 include some simple calculations to estimate the maximum temperature rise under steady-state conditions (long heating times) and with no radiation loss. When the heating time is short, we need to estimate the temperature rise under transient conditions, that is, the temperature as a function of time and space. Pittaway[5] has treated the case of transient temperature rise due to stationary and moving Gaussian sources on both thin films and semi-infinite targets with no

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† An estimate of the time for conversion is given in the Appendix.

radiation loss. Steady-state temperature of heating semi-infinite targets with a Gaussian source has been computed by Vine and Einstein,[6] who included the effect of beam penetration into the target. Morrison and Morgan[7] also calculated the steady-state temperature of heating thin films on highly conductive substrates assuming the beam penetrates only within the film. Wells[8] tried to estimate the heat-affected zone in a semi-infinite target. None of these calculations included radiation loss from the surface. Radiation loss was treated by Leisegang,[9] who computed the steady-state temperature of heating thin films with a uniform circular source. To solve his nonlinear differential equation, he had to make assumptions in two extreme cases and obtained approximate solutions.

In the present paper we try to calculate the transient temperature rise of heating thin films with a stationary Gaussian or uniform circular source when radiation loss from the surface is not neglected. A linearized Stefan-Boltzmann law of black-body radiation is used to render the differential equation linear. When the temperature variation is not extremely high, such an assumption is well justified. Under the same conditions, one is justified to assume that the thermal conductivity and other physical properties of the target are constants. In Section 4, formulae and graphs are given for the calculation of temperature as a function of time, space, heat supply rate, thermal conductivity, heat capacity, thickness, emissivity, etc. In Sections 5 and 6, comparisons are made to the results using other simplifying assumptions, and their relative errors are displayed against the rigorous solutions obtained in Section 4. Section 7 extends the calculation to approximate an idealized model when phase transition in the target material occurs. Finally, examples are given to show numerical calculations.

2. Assumptions

A) A schematic representation of the electron beam configuration is shown in Fig. 1. Because of the repulsive force between electrons, the size of the focus spot can never be made vanishingly small, even with aberration-free lenses.[10,11] In addition, the current density within the spot may not be uniform. If the origin of the coordinate system is chosen at the center of the spot, it is usually assumed that the current density J is independent of ϕ (i.e., axially symmetrical), but J will be a function of r, being highest at $r = 0$ and diminishing to zero at large r.

B) When the target is thick, all electrons are stopped within a region of roughly spherical shape. Its diameter is called the range of penetration R, and may be estimated from Schonland's formula,[12–14] or its modification (cf. Appendix).

Kinetic energy is converted into thermal energy in this region. Therefore, there is a distribution of heat sources of varying strength.[15–17] But when film thickness D is roughly equal to the range R, we may assume that heat is supplied uniformly along the z direction, so that the rate Q of heat supply per unit area is equal to the product of the rate q of heat supply per unit volume and the thickness D of the film. Therefore, we are assuming that q and Q are functions of r only, independent of ϕ and z (Fig. 1).

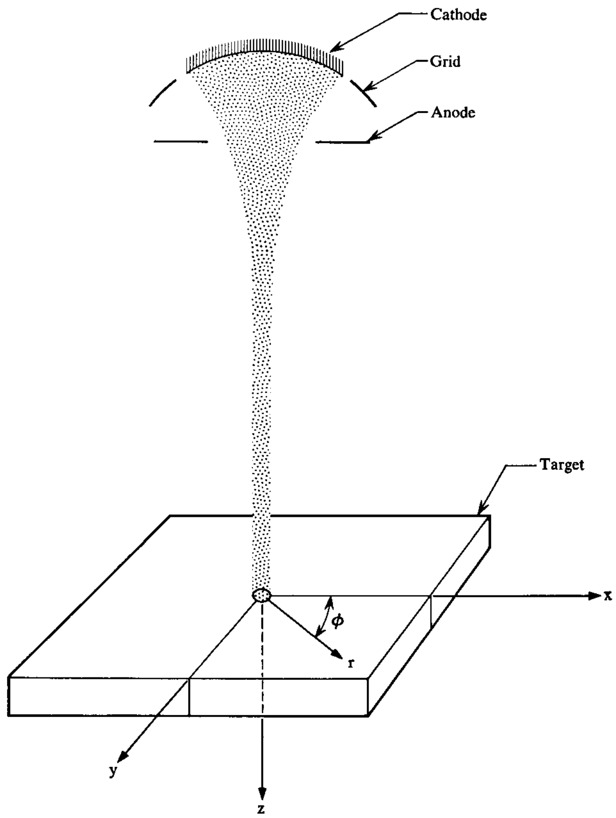
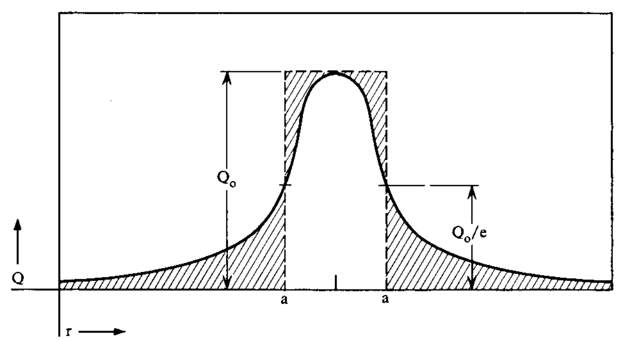


Figure 1 Schematic representation of electron beam configuration.

Figure 2 A gaussian distribution of heat supply.



In this paper we shall discuss the following two types of heat supply distributions:

$$\textit{Gaussian}: \text{(Fig. 2) } Q = Q_0 \exp(-r^2/a^2) \qquad (1)$$

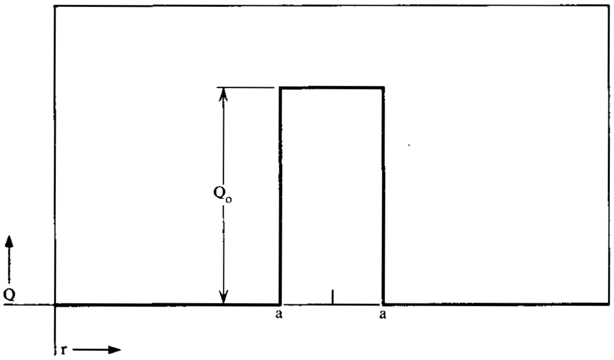


Figure 3 A uniform distribution of heat supply.

$$\textit{Uniform}: \text{(Fig. 3)}\quad Q = \begin{cases} Q_0 & 0 \le |r| < a \\ 0 & a \le |r| \end{cases} \tag{2}$$

Similarly, we shall assume any heat loss (sink) w to be independent of both ϕ and z in thin films.

C) We shall assume that the thermal conductivity K is constant, independent of both temperature T and position (r, ϕ, z). The fundamental differential equation for heat conduction[18] is

$$\rho c(\partial T/\partial t) = K\nabla^2 T + q - w, \tag{3}$$

where

T = temperature (in °K) at point (r, ϕ, z) (r, z in cm) and time t (in seconds)
ρ = density of the solid (in gm/cm^3)
c = heat capacity (in cal/gm °C)
K = thermal conductivity (cal/cm sec °C)
q = heat supply (source) per unit time per unit volume (cal/cm^3 sec) at point (r, ϕ, z) and time t
w = heat loss (sink) per unit time per unit volume (cal/cm^3 sec) at point (r, ϕ, z) and time t.

D) We shall assume that there is radiation loss from both surfaces of the film. The *linearized* Stefan-Boltzmann law of black-body radiation[19] states:

$$W = 2H\theta, \tag{4}$$

where

W = heat loss per second/cm^2
θ = $T - T_0$ = excess temperature over the ambient (°K)
H = a constant depending on the emissivity and average temperature of the film = $1.37 \times 10^{-12} E \times 4 T_{av}^3$ cal/cm^2 − °K

The factor 2 in Eq. (4) comes from the fact that there are two surfaces of radiation.

The heat loss w per unit volume may be taken as $W/D = 2H\theta/D$, where D is the thickness of the film. This assumption is justified because D is small and T is large.

E) We shall assume the initial temperature distribution is uniform and the same as ambient temperature T_0. Thus, T is also independent of ϕ and z. Equation (3) then reduces to:

$$\frac{1}{\alpha}\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{Q}{KD} - \frac{2H}{KD}\theta \tag{5}$$

where $\alpha = \dfrac{K}{\rho c}$ = thermal diffusivity (cm^2/sec).

F) We shall also assume that ρ and c are constant, independent of both temperature and position, and that the film is infinite in two dimensions.

3. General solution

Under the above assumptions, the most general solution of differential Eq. (5) is given by:[20–22]

$$\theta(r, t) = \frac{1}{2KD}\int_0^t \int_0^\infty \frac{Q(r', t')}{t - t'} \times \exp\left[-\frac{2H}{KD}\alpha(t - t') - \frac{r^2 + r'^2}{4\alpha(t - t')}\right] \times I_0\left(\frac{rr'}{2\alpha(t - t')}\right) r'dr'dt', \tag{6}$$

where I_0 is the modified Bessel function of order zero.

When Q is given as a function of r and t, Eq. (10) may be integrated (sometimes by numerical methods only) to give θ as a function of r and t. We shall discuss only the cases where Q is independent of time t and dependent on r according to Eq. (1) or (2).

4. Steady Gaussian distribution

In this section we shall assume that Q is given by Eq. (1), where

Q_0 = a constant, independent of t (and r) = maximum heat supply per unit time per unit area
a = "equivalent radius" of heat supply, i.e., the total heat supply per second over a circular area of radius a with a uniform intensity Q_0 (cf. Fig. 2) is equal to that over the whole infinite plane with a Gaussian distribution $Q_0 \exp(-r^2/a^2)$.

The mathematical formulae will appear much simpler in terms of the following dimensionless variables:

"temperature" $\Theta = 4KD\theta(r, t)/Q_0 a^2$
"radial distance" $\xi = r/a$
"time" $\tau = 4\alpha t/a^2 = 4Kt/\rho c a^2$
"radiation" $\eta = a^2 H/2KD$.

Equation (6) then becomes

$$\Theta(\xi, \tau, \eta) = \int_0^\tau \int_0^\infty \exp\left[-\eta(\tau - \tau') - \frac{\xi^2 + \xi'^2}{\tau - \tau'} - \xi'^2\right] I_0\left(\frac{2\xi\xi'}{\tau - \tau'}\right) 2\xi' d\xi' \frac{d\tau'}{\tau - \tau'}$$

Noting that the Laplace transform of $I_0(2a\sqrt{t})$ becomes $\exp[(a^2/s)]/s$, we may carry out one of the iterated integrals to obtain:

$$\Theta(\xi, \tau, \eta) = e^{\eta} \int_{1/(1+\tau)}^{1} \exp\left[-\xi^2 u - \frac{\eta}{u}\right] \frac{du}{u},$$
$$\text{where } u = 1/(\tau - \tau' + 1). \qquad (7)$$

Equation (7) gives the reduced temperature Θ as a function of ξ, τ, and η in an integral form. When one or more of these variables is given special values, a closed-form expression is sometimes possible. The following is a partial list of the integrated forms, where

$$Ei(x) = \int_{-\infty}^{x} e^{u} \frac{du}{u} \qquad (x < 0)$$

is the "exponential integral" and is tabulated in standard handbooks.[23]

The discussion of individual cases will be given following the list of equations:

$$\Theta(0, \tau, 0) = \ln(1 + \tau) \qquad (8)$$

$$\Theta(\xi, \tau, 0) = -Ei\left(-\frac{\xi^2}{1+\tau}\right) + Ei(-\xi^2) \qquad (9)$$

$$\Theta(0, \tau, \eta) = e^{\eta}|-Ei(-\eta) + Ei[-\eta(1+\tau)]| \qquad (10)$$

$$\Theta(\xi, \tau, 0) = e^{-\xi^2}\left[\tau + \frac{\xi^2 - 1}{2!}\tau^2 + \frac{\xi^4 - 4\xi^2 + 2}{3!}\tau^3 + \cdots\right]$$
$$\text{for small } \tau \qquad (11)$$

$$\Theta(\xi, \tau, 0) = \ln(1 + \tau) - \ln\xi^2 - \gamma + Ei(-\xi^2) + \left(\frac{\xi^2}{1+\tau}\right) - \frac{1}{2\cdot 2!}\left(\frac{\xi^2}{1+\tau}\right)^2 + \frac{1}{3\cdot 3!}\left(\frac{\xi^2}{1+\tau}\right)^3 - \cdots$$
$$\text{for large } \tau \qquad (12)$$

$$\Theta(0, \tau, \eta) = \tau - \frac{\eta + 1}{2!}\tau^2 + \frac{\eta^2 + 2\eta + 2}{3!}\tau^3 - \frac{\eta^3 + 3\eta^2 + 6\eta + 6}{4!}\tau^4 + \cdots$$
$$\text{for small } \tau \qquad (13)$$

$$\Theta(0, \infty, \eta) = e^{\eta}\left[-\gamma - \ln\eta + \frac{\eta}{1\cdot 1!} - \frac{\eta^2}{2\cdot 2!} - \frac{\eta^3}{3\cdot 3!}\cdots\right] \approx -\gamma - \ln\eta$$
$$\text{for small } \eta \qquad (14)$$

$$\Theta(0, \infty, \eta) = \frac{0!}{\eta} - \frac{1!}{\eta^2} + \frac{2!}{\eta^3} - \frac{3!}{\eta^4} + \cdots \approx \frac{1}{\eta}$$
$$\text{for large } \eta \qquad (15)$$

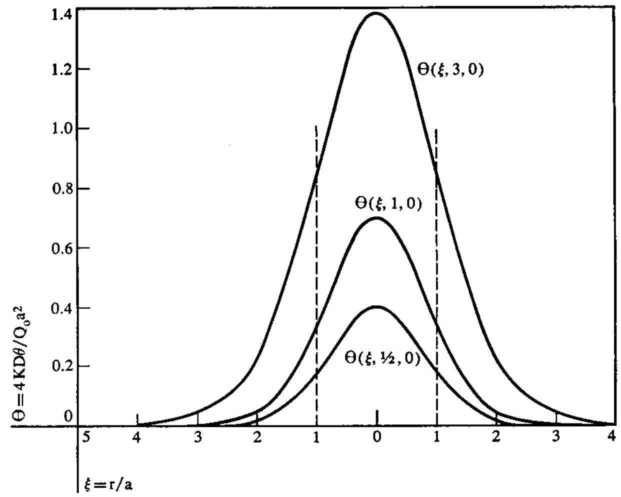
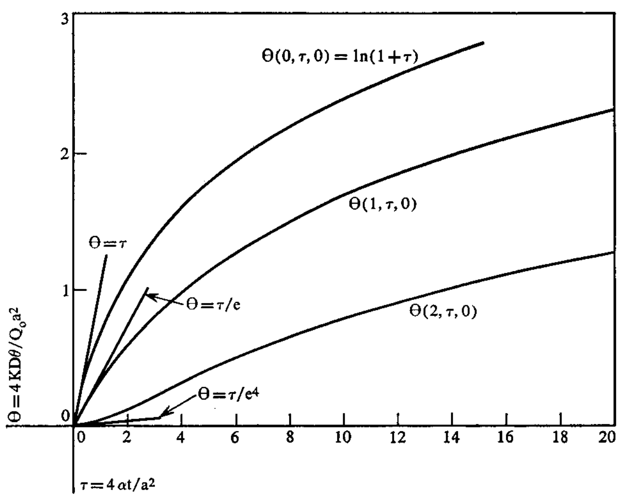


Figure 4 Temperature as a function of distance (with no radiation loss).

Figure 5 Temperature rise as a function of time (with no radiation loss).



A) Results with no radiation loss assumed

When H is negligibly small compared to other terms in Eq. (5), we may set $\eta = a^2H/2KD = 0$. Equation (7) then reduces to Eq. (9). Figure 4 shows the "temperature" distribution at different "time" τ. We see that the temperature is highest at the center and drops off quite rapidly as r increases, becoming negligible when $r > 3a$.

Figure 5 shows the temperature rise at various locations. When $\xi = 0$, Eq. (9) becomes Eq. (8). The simplicity of this equation makes it very useful in estimating temperature rise in practical design.

For small values of τ, the Taylor series expansion (11) is more helpful. When τ is very large (at very long times), Eq.

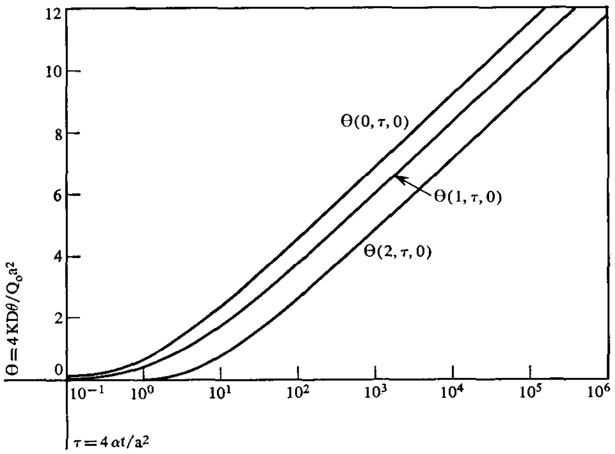


Figure 6 Temperature rise as a function of time (with no radiation loss).

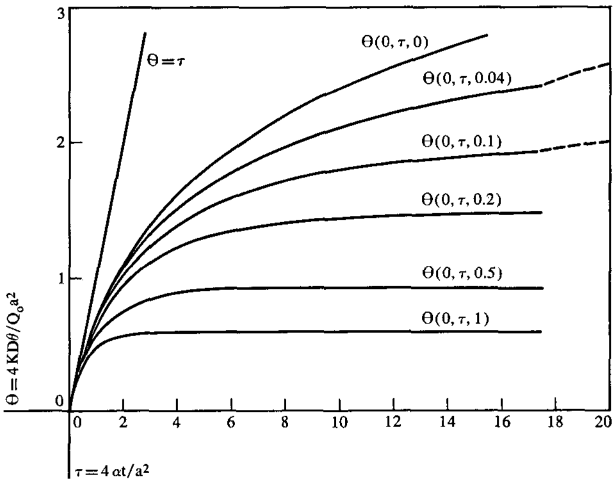


Figure 7

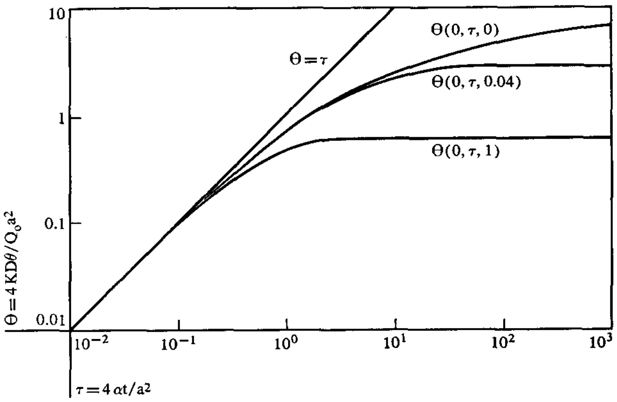


Figure 8

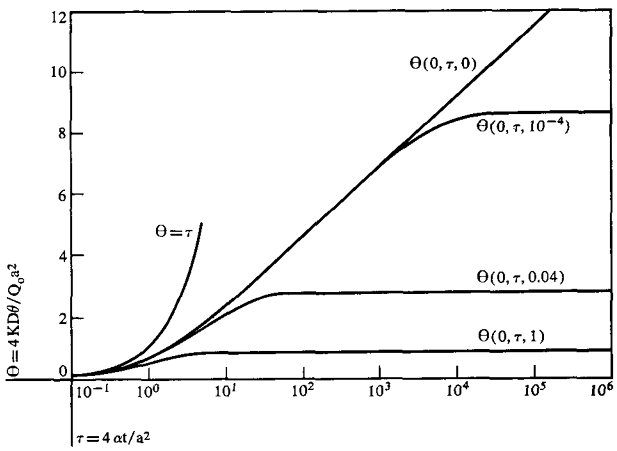


Figure 9

Figures 7, 8 and 9 Temperature rise of a function of time with different radiation loss.

(9) may be simplified with the help of a series expansion of exponential integral to give Eq. (12), which may be approximated by

$$\Theta(\xi, \tau, 0) \approx \ln\tau - \gamma - \ln\xi^2 + Ei(-\xi^2)\,, \qquad \text{for large } \tau\,,$$

where $\gamma = 0.5772 \cdots =$ Euler's constant.

The graphs are straight lines when Θ is plotted against $\ln\tau$ (Figure 6). For example, when ξ approaches zero as a limit,

$$\Theta(0, \tau, 0) \approx \ln\tau \qquad \text{for large } \tau\,. \tag{16}$$

A quantity of practical importance is the total heat flow F per second through a circular cylindrical surface. This quantity is given by:

$$F = 2\pi r D \cdot (\text{flux}) = 2\pi r D \cdot \left(-K\frac{d\Theta}{dr}\right)$$

$$= -\frac{\pi}{2}a^2 Q_0 \xi \frac{\partial\Theta}{\partial\xi}\,.$$

From Eq. (9), we find:

$$F(\xi, \tau, 0) = \pi a^2 Q_0\left[\exp\left(-\frac{\xi^2}{1+\tau}\right) - \exp(-\xi^2)\right].$$

B) Results when radiation losses are not negligible

When H is not negligible, Θ is a function of three variables as given in Eq. (7), and the relation is more complicated. To illustrate the dependence of Θ on η, let us investigate the temperature rise at the center point ($\xi = 0$), since the temperature rise at all other points will follow a similar trend. Equation (10) gives Θ in the case of $\xi = 0$. Figures 7, 8, and 9 are plots of temperature rise at various values of η on rectangular, log-log and semi-log scales, respectively. They all show the property that, when $t \longrightarrow \infty$, Θ approaches a definite value. This is the steady-state temperature and is

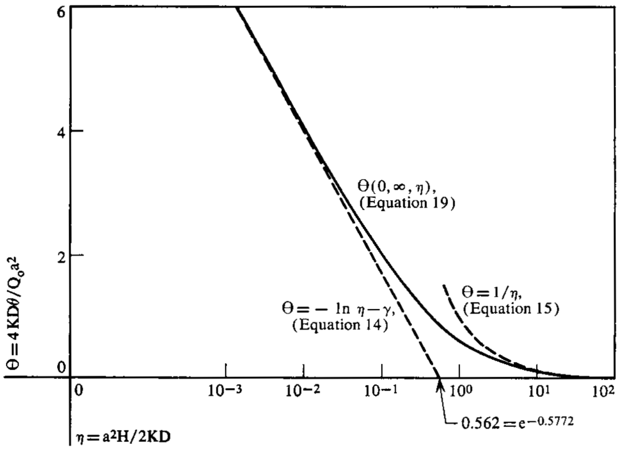


Figure 10 Steady-state temperature as a function of radiation loss.

given by the first term on the right-hand side of Eq. (10). This is the upper limit of attainable temperature by electron beam heating, no matter how long the film has been bombarded. Naturally, the smaller the radiation loss (i.e., the smaller η), the higher the steady-state temperature. Figure 10 shows Θ (0, ∞, η) as a function of η. For extreme values of η, Eqs. (14) and (15) are more useful.

When $\eta \longrightarrow 0$, there is no upper limit temperature, a fact more easily observed in Fig. 9.

When τ is small, again the Taylor series expansion (13) is more helpful. In conclusion, we observe that a typical temperature rise curve $\Theta = \Theta(0, \tau, \eta)$ may be approximated in three regions:

$\Theta \approx \tau$	for very small τ	(17)
$\Theta \approx \ln(1 + \tau)$	for intermediate τ	(18)
$\Theta \approx e^{\eta}\{-Ei(-\eta)\}$	for very large τ	(19)

We may call the first region the "heat capacity limited region," the second, "thermal conduction limited region," and the third, "radiation limited region."

5. Steady uniform distribution

Although Gaussian distribution is probably closer to reality, many engineering calculations are based on the simpler assumption of uniform circular distribution. In this section, we compare the results obtained in the last section with those when the heat supply rate $Q(r, t)$ is assumed to be steady (independent of t) and uniform over a circular area, according to Eq. (2).

Using the same dimensionless variables as before, Eq. (6) becomes

$$\Theta^0(\xi, \tau, \eta) = \int_0^{\tau}\int_0^1 \exp\left[-\eta(\tau - \tau') - \frac{\xi^2 + \xi'^2}{\tau - \tau'}\right] \times I_0\left(\frac{2\xi\xi'}{\tau - \tau'}\right)2\xi' d\xi' \frac{d\tau'}{\tau - \tau'} \tag{20}$$

where Θ^0 denotes "temperature" derived under uniform distribution assumption.

The integral

$$P(z, r) = (1 - e^{-z^2})^{-1}\int_0^z e^{-(r^2+s^2)} I_0(2rs)2sds$$

is called a P-function and has been tabulated by Masters.[24] Equation (20) may then be written

$$\Theta^0(\xi, \tau, \eta) = \int_0^{\tau} e^{-\eta(\tau-\tau')}[1 - e^{-1/(\tau-\tau')}] \times P\left(\frac{1}{\sqrt{\tau - \tau'}}, \frac{\xi}{\sqrt{\tau - \tau}}\right)d\tau' \tag{21}$$

This equation, corresponding to Eq. (7) above, gives Θ as a function of ξ, τ, η. We shall discuss two special cases.

A) At the center of beam, $r = \xi = 0$: Since the P-function is normalized such that $P(z, 0) = 1$, Eq. (21) reduces to

$$\Theta^0(0, \tau, \eta) = \int_{1/\tau}^{\infty} e^{-\eta/u}(1 - e^{-u})(du/u^2) \tag{22}$$

B) When $H = \eta = 0$ and at center:

$$\Theta^0(0, \tau, 0) = \tau(1 - e^{-1/\tau}) - Ei(-1/\tau)\,. \tag{23}$$

Equation (23) is plotted in Figs. 11 and 12 together with other results.

For short times, the asymptotic expression for an exponential integral

$$-Ei(-1/\tau) = (\tau - 1!\tau^2 + 2!\tau^2 - \cdots)e^{-1/\tau}$$

may be used and Eq. (23) becomes

$$\Theta^0(0, \tau, 0) = \tau - (1!\tau^2 - 2!\tau^2 + \cdots)e^{-1/\tau}\,. \tag{24}$$

For very short times, this equation again reduces to Eq. (17). This property is easily observed on Fig. 11.

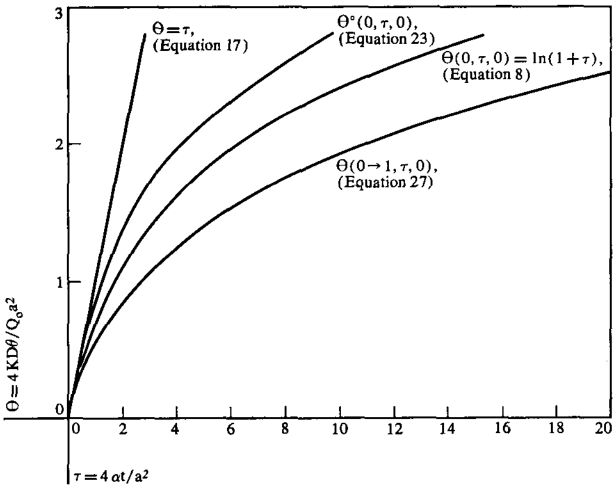
For long times, the use of a series expansion of the exponential integral results in

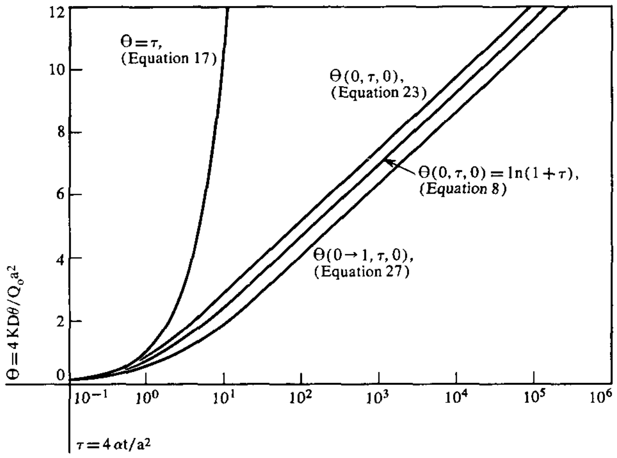
$$\Theta^0(0, \tau, 0) = \ln\tau + (1 - \gamma) + \frac{1}{1\cdot 2!\tau} - \frac{1}{2\cdot 3!\tau^2} + \frac{1}{3\cdot 4!\tau^3} - \cdots\,. \tag{25}$$

For very long times τ,

$$\Theta^0(0, \tau, 0) = \ln\tau + (1 - \gamma)\,, \tag{26}$$

which differs from the corresponding expression in Eq. (16) by only a constant number: $1 - \gamma = 0.4228 \cdots$ (Fig. 12). This means that the time required to heat a film to a temperature Θ with a uniform circular heat supply is about $e^{-0.4228} = 65\%$ of that with a Gaussian heat supply for any $\Theta > 3$.





Figures 11 and 12 Comparison of temperature rise under various assumptions (from data in Carslaw and Jaeger).

6. Comparison to the heating of underground cable

The heating of underground cable is very similar to this case, except the thermal conductivity inside $r = a$ is much greater than that of the material outside. An idealized case where α is assumed to be infinity inside $r = a$ is solved by Carslaw and Jaeger.[25] In terms of our notation, their Eq. (8) on page 344 is

$$\Theta(0 \to 1, \tau, 0) = \frac{32}{\pi^2} \int_0^\infty \times \frac{[1 - e^{-\tau u^2/4}]du}{u^3[uJ_0(u) - 2J_1(u)]^2 + u^3[uY_0(u) - 2Y_1(u)]^2} \tag{27}$$

Note in our present case, $S = \pi a^2 \rho c$, $KV/Q = \Theta/4\pi$, $\kappa t/a^2 = \tau/4$. Note also that the temperature inside the cable will be uniform at all times. That is why $\xi = 0 \to 1$ (from zero to one) is put in the left-hand side of Eq. (27). Radiation loss in this case is zero. The curve on page 343 of their book is replotted in Figs. 11 and 12 to facilitate comparison. At large τ, their curve seems to approach $\Theta \approx \ln\tau - 0.65$.

7. Phase transition

In the above discussion, we assume that all temperatures are below the melting point T_m of the film. When $\Theta > \Theta_m$ (where $\Theta_m = (4KD/Qa^2)\theta_m = (4KD/Qa^2)(T_m - T_0)$, there is a phase change. The molten phase may be assumed to be in the form of a circular disk of the same thickness D as the film and of increasing radius r_m (in cm). The latent heat of melting H_m(cal/gm) may then be regarded as a moving negative surface heat source of strength: $-H_m\rho D(dr_m/dt)$cal/cm · sec, where r_m is the radius of the phase boundary at time t. The function $Q(r, t)$ in Eq. (6) must then be replaced by

$$Q(r, t) = Q_0 e^{-r^2/a^2} - H_m \rho D \frac{dr_m(t)}{dt} \delta[r - r_m(t)]$$

where $\delta(x)$ is the Dirac delta function. Hence,

$$\int_0^\infty Q 2\pi r dr = Q_0 \pi a^2 - H_m \rho D \frac{dr_m}{dt} 2\pi r_m$$

$$= Q_0 \pi a^2 - H_m \rho D \frac{d}{dt}(\pi r_m^2) .$$

Assuming that ρ, c, K are the same for both liquid and solid phases, Eq. (6) becomes then (for $\tau > \tau_m$)

$$\Theta(\xi, \tau, \eta) = \int_0^\tau \exp\left[-\eta(\tau - \tau') - \frac{\xi^2}{\tau - \tau' + 1}\right] \times \frac{d\tau'}{\tau - \tau' + 1} - L_m \int_{\tau_m}^\tau \exp\left[-\eta(\tau - \tau') - \frac{\xi^2 + \mu(\tau')^2}{\tau - \tau'}\right] \times I_0\left(\frac{2\xi\mu(\tau')}{\tau - \tau'}\right) \frac{2\mu(\tau')\dot{\mu}(\tau')d\tau'}{\tau - \tau'} \tag{28}$$

where

$L_m = (4KDH_m/Q_0a^2c)$

H_m = latent heat of melting (cal/gm)

T_m = melting point in °K

$\mu = r_m(t)/a$

$$\dot{\mu}(\tau) = \frac{d\mu}{d\tau} = \frac{a}{4\alpha}\frac{dr_m}{dt}.$$

The quantity τ_m is the "time" τ at which the center melts, that is, τ_m is the solution of

$$e^\eta\{-Ei(-\eta) + Ei[-\eta(1 + \tau)]\} = \Theta_m \tag{29}$$

(See Eq. 10).

Equation (28) involves a function $\mu(\tau)$ which must be determined by the condition that the temperature at the boundary of the phase transition is the melting point. The leads to an integral equation

$$\Theta[\mu(\tau), \tau, \eta] = \Theta_m. \tag{30}$$

Eqs. (28) and (30) may be solved by numerical methods. Qualitatively, the temperature distribution will look something like the one shown in Fig. 13 where the dotted curve on top is the distribution if there were no melting. For a first approximation of the heating time, we may proceed as follows:

For $T < T_m$ let $\Theta = (4KD/Q_0a^2)(T - T_0)$ and use Eq. (7)

For $T > T_m$ let $\Theta = (4KD/Q_0a^2) \times [T + (H_m/c) - T_0]$ and use Eq. (7)

In other words, we regard H_m/c as a fictitious temperature rise. Such an approximation is equivalent to assuming a temperature distribution shown schematically in Fig. 14.

When the central region of the spot is heated to a sufficiently high temperature, so that the corresponding vapor pressure (and thus the rate of evaporation) becomes appreciable, a term representing another type of heat sink (latent heat of evaporation) should be added to the right-hand side of Eq. (28). However, in this case, the heat sink is strongly temperature dependent. Another complication is that when a hole is formed in the central region of the spot there is no longer any heat conduction or heat supply in that region.

8. Numerical examples

To illustrate the usefulness of the above results, let us compare the time requirements for heating thin bismuth and aluminum films to their melting points.

Assume: Beam power density Q_0 $10^6 \text{W/cm}^2 = 2.39 \times 10^5$ cal/sec·cm²
Beam radius $a = 1\mu\text{m} = 10^{-4}$ cm
Film thickness $D = 2\mu\text{m} = 2\times 10^{-4}$ cm.

Therefore, $(4D/Q_0a^2) = 0.335$ sec·cm/cal.

For bismuth:

Thermal conductivity	K	=	0.02 cal/sec cm °C
Thermal diffusivity	α	=	0.07 cm²/sec
Thermal emissivity	E	=	0.048
Melting point	T_m	=	271°C

Therefore, $\theta_m = 271 - 25 = 246°\text{C}$

$$\Theta_m = \frac{4D}{Q_0a^2} K\theta_m = 1.65$$

Since η is of the order of 10^{-4} or less, from Fig. 8, we see that Eq. (8) is applicable. Thus $1 + \tau = e^{1.65} = 5.21$, and the time

$$t = \frac{a^2}{4\alpha}\tau = 0.15\ \mu\text{sec}.$$

If the analogous calculations were made for aluminum:

Thermal conductivity	K	=	0.48 cal/sec·cm°C
Thermal diffusivity	α	=	0.86 cm²/sec
Thermal emissivity	E	=	0.11 (oxidized surface)
Melting point	T_m	=	660°C,

the time required to heat aluminum to 200°C (i.e., 175°C above ambient) would be 4.8×10^3 sec = 1 hr 20 min. Obviously, for such long time, the radiation effect η is no longer negligible. In fact, such an effect will set an upper limit of attainable temperature for aluminum, according to Fig. 8 or 9. To calculate η, we assume an average temperature of 400°K. Thus,

$$H = 1.37 \times 10^{-12} \times 4 \times (400)^3 \times 0.11 = 3.858 \times 10^{-5}$$

$$\eta = \frac{a^2H}{2KD} = \frac{10^{-8} \times 3.858 \times 10^{-5}}{2 \times 0.48 \times 2 \times 10^{-4}} = 2.0 \times 10^{-9}$$

From Fig. 10, we see that for such a small η, Eq. (14) is applicable for estimating the steady-state temperature limitation. Hence,

$$\Theta = -0.577 - 2.303 \log_{10} (2.0 \times 10^{-9}) = 19.46$$

$$\theta = \frac{Q_0a^2}{4D}\frac{\Theta}{K} = \frac{19.46}{0.335 \times 0.48} = 121$$

or

$$T = 121 + 25 = 146^\circ\text{C}.$$

Consequently, under such conditions, the aluminum film will never be heated to melting, no matter how long it is bombarded by the beam.

9. Conclusions

Transient temperature rise has been calculated for a thin film heated by a stationary Gaussian or uniform circular source when radiation loss from the surfaces is present. Graphical presentations in terms of dimensionless (reduced) variables are included to aid in specific calculations for designing. The linearized Stefan-Boltzmann law of black body radiation is employed, which is justified when the temperature variation is not extremely high. It is found that the formulas for calculating temperature rise can be reduced to very simple formulas in each one of the following three regions:

(1) the heat-capacity-limited region (for very small τ, Eq. (17)),

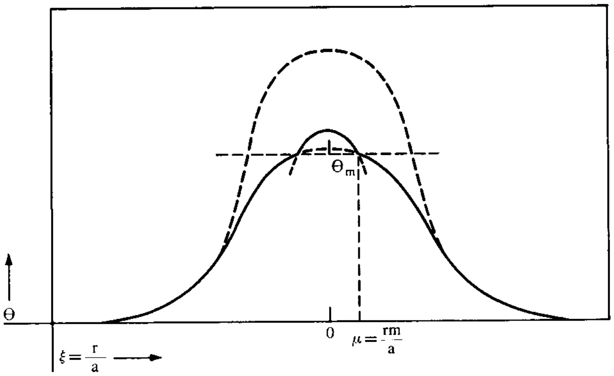


Figure 13 Temperature distribution above melting point. (see text.)

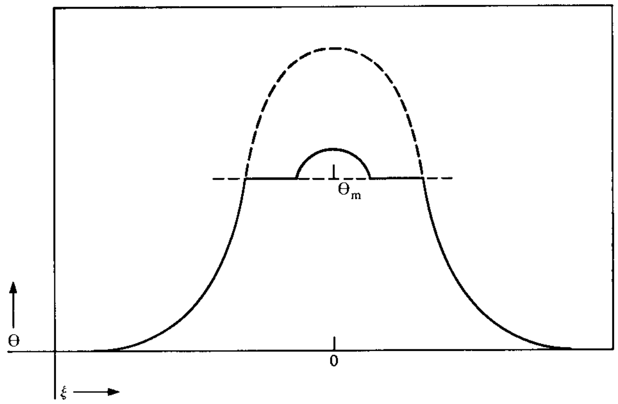


Figure 14 Simplified temperature distribution above melting point.

(2) the thermal-conduction-limited region (for intermediate τ, Eq. (18)),

(3) the radiation-limited-region (for very large τ, Eq. (19)).

10. Appendix

- *Estimation of dwelling time*

The most probable kinetic energy E of an electron beam after penetrating a distance z into the target may be expressed as

$$E^n = E_0^n - bz\,,$$

where

E_0 = initial kinetic energy of the electrons,

b = a constant, proportional to the density of the target,

n = another constant.

When $n = 2$, this is the classical Whiddington's law. But for kilovolt-range electrons, n is between 1.5 and 1.7.

The range R of the electron may be defined as the distance z when the energy E of the electrons diminishes to zero. Therefore, $R = E_0{}^n/b$.

When $n = 2$, this is the Schonland's formula.[12]

Thus we may write $E^n = b(R - z)$.

Since $E = (m_0/2)v^2 = (m_0/2)(dz/dt)^2$, the above equation may be transformed into

$$dt = \sqrt{m_0/2}\, b^{-\frac{1}{2}n}(R - z)^{-\frac{1}{2}n}dz.$$

Integrating from $z = 0$ to $z = R$, we obtain the time the electron spends inside the target before it is stopped completely as:

$$t = \frac{2n}{2n - 1}\,\frac{R}{v_0}\,,$$

where v_0 is the initial electron velocity.

For a 100 keV electron hitting an iron or aluminum target, the dwelling times are of the order of 2 to 6 $\times 10^{-13}$ seconds. During this time, the kinetic energy is transferred from the beam electrons to the target electrons.

The total duration for the conversion of the kinetic energy of the electron into the thermal energy of the lattice is much longer than the above dwelling time, because the process of transferring the energy of target electrons to the whole lattice is much slower. However, the total duration is still probably much shorter compared to the time scale of heat conduction.

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