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Sensitivity Constrained Optimal Control Synthesis

Abstract: A constraint on the sensitivity to plant parameter uncertainties is introduced in the synthesis of optimal linear controls by adding a sensitivity function to a quadratic form performance criterion. The synthesis is carried out in the frequency domain using both conventional and Bode sensitivity functions; open-loop and closed-loop controls are of necessity treated separately because the sensitivity functions are different for each case. The optimal control based on the closed-loop Bode sensitivity is shown to satisfy a scalar Wiener-Hopf equation; the same type of equation is also satisfied by the optimal control based on the open-loop conventional sensitivity, but the response characteristics are different. Using the conventional closed-loop sensitivity, the optimal control is shown to satisfy an equation analogous to the Wiener-Hopf equation, but which is more difficult to solve because the control enters quadratically rather than linearly. Examples illustrate the application of derived results.

Introduction

The mathematical model for a plant usually includes one or more (constant) parameters whose values are subject to some uncertainty. Therefore, when optimal control synthesis is performed using a set of assigned nominal values for the uncertain parameters, it becomes necessary to consider the sensitivity of the optimal system to perturbations of these parameters away from the assigned nominal values. Of course, after the optimal control is synthesized using the assumed nominal values for the uncertain parameters, a sensitivity analysis can be performed on the resulting system to determine whether the sensitivity is "sufficiently small" and what modifications, if any, are necessary. However, a more satisfactory approach consists of initially formulating the optimal control synthesis problem such that an appropriate sensitivity constraint relative to plant parameter uncertainties is inherent in the derived control.

Although sensitivity to uncertainties in plant parameters plays an essential role in classical control theory,[1] sensitivity considerations have largely been ignored in optimal control theory until relatively recently.[2–8] In optimal systems, the sensitivity of either the performance index,[2,3] the system response,[4–7] or both can be constrained; the choice depends on what is most meaningful relative to the problem at hand. For example, in linear systems with a quadratic performance index, the performance index sensitivity is often of little interest compared to the response sensitivity.

In this paper a linear, time-invariant plant with a single control variable, and a quadratic performance index is considered. A constraint on the sensitivity of the system response (output sensitivity) to plant parameter uncertainties is introduced in the optimal control synthesis by including sensitivity functions in the quadratic performance index. That is, the optimality of the control is defined both with respect to sensitivity as well as to the error and control action. Improvement of the system sensitivity is obtained by increasing the sensitivity function weighting factors. The use of sensitivity functions in a quadratic performance index was independently proposed by Siljak and Dorf,[4] and applied by Siljak and Burzio[5] to the problem of controller parameter optimization. The present paper may be considered as an extension of that work to the synthesis problem.

Two definitions of sensitivity are considered: the conventional one in which the sensitivity is just the first partial derivative of the system time response with respect to a plant parameter, and the classical frequency domain definition due to Bode. Because the sensitivity to a plant parameter differs for open-loop and feedback systems having the same nominal response, it is necessary to treat the synthesis of optimal open-loop and feedback controls separately. This is in contrast to the case without a sensitivity constraint, where it is often more convenient to first synthesize an open-loop control and then from it construct an equivalent feedback system. The synthesis is carried out entirely in the frequency domain, and the optimal controls are shown to satisfy scalar equations of the transformed Wiener-Hopf type.

When more general classes of systems with other types of sensitivity constraints and performance criteria are to be considered, it is necessary to formulate the sensitivity constrained optimal control synthesis problem in the time domain[7,8] rather than in the frequency domain. Although a study of the more general problem is outside the scope of the present paper, it is of interest to note that except for

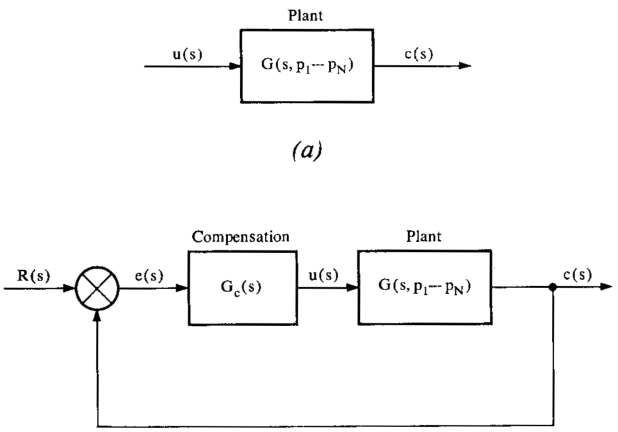

Figure 1 Linear control system; (a) open-loop control, (b) closed-loop control.

the intermediate step of introducing sensitivity equations to form an expanded state vector, optimization with sensitivity constraints is similar to optimization with state variable constraints.

Problem statement

Consider the linear control system shown in Fig. 1 where the plant transfer function is $G(s, p_1, \ldots p_N)$, and $p_1, \ldots, p_N$ are the values of the N constant plant parameters. These parameters are either the poles and zeros of $G(s)$, the coefficients of the numerator and denominator polynomials of $G(s)$, or the values of other quantities, e.g., stability derivatives if $G(s)$ is an airframe transfer function. If the values of these parameters are all known, the control $u^*(s)$ which optimizes the quadratic form performance index

$$J = \int_0^\infty [e^2(t) + k^2u^2(t)]\,dt; \quad k = \text{constant}, \tag{1}$$

is

$$u^*(s) = \frac{1}{(k^2 + G\bar{G})^+}\left[\frac{R\bar{G}}{(k^2 + G\bar{G})^-}\right]_+, \tag{2}$$

where

$e(t) = R(t) - c(t)$ is the error,

$G = G(s)$, $\bar{G} = G(-s)$, $R = R(s)$,

$(\cdot)^-$ = factors of $(\cdot)$ with poles and zeros in the right-half s-plane (RHP),

$(\cdot)^+$ = factors of $(\cdot)$ with poles and zeros in the left-half s-plane (LHP),

$[\cdot]_+$ = that part of the partial fraction expansion of $[\cdot]$ with poles in the left-half s-plane (LHP).

However, if the plant parameters are subject to some uncertainty, the assumed nominal values used to synthesize the optimal control, Eq. (2), may differ from the actual values. Therefore, as mentioned in the Introduction, it is desirable initially to reformulate the problem such that appropriate sensitivity constraints relative to plant parameter uncertainties are inherent in the derived optimal control. The specific means considered here for constraining the sensitivity is arrived at by the following reasoning. For simplicity, assume initially that only one parameter, denoted here by p, is subject to uncertainty. Then if p_0 is the assumed nominal parameter value and Δp the uncertainty,

$$p = p_0 \pm \Delta p. \tag{3}$$

Also assume that a sensitivity function, $S_p^c(t)$, can be defined which provides a measure of the sensitivity of the response $c(t)$ to a perturbation in the plant parameter, p. Then, if the control $u(t)$ is optimized so that among other things $S_p^c(t)$ is made "sufficiently small" (ideally, zero), the required type of sensitivity constraint will be incorporated. To avoid the mathematical difficulties involved in placing an explicit bound on $S_p^c(t)$ (i.e., $|S_p^c(t)| \leq S_0$; S_0 = constant), it appears more convenient to consider constraining $S_p^c(t)$ indirectly by minimizing the integral square

$$J = \int_0^\infty [S_p^c(t)]^2\,dt. \tag{4}$$

However, choosing the control $u(t)$ to optimize Eq. (4) means that other important aspects of system response such as error and control effort minimization, may be ignored. Therefore, a better approach is to choose $u(t)$ to optimize the more general quadratic form

$$J = \int_0^\infty \{e^2(t) + k^2u^2(t) + \sigma^2[S_p^c(t)]^2\}\,dt, \tag{5a}$$

in which the constant weighting factors k, σ are adjusted according to the relative importance of the quantities they multiply. By applying Parseval's theorem to (5a) the corresponding frequency domain expression is

$$J = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \{e(s)e(-s) + k^2u(s)u(-s) + \sigma^2 S_p^c(s)S_p^c(-s)\}\,ds. \tag{5b}$$

It can be seen that the performance functional (5) has the same form as (1) except for the addition of the sensitivity function. As noted earlier, reduced sensitivity of the reponse to plant parameter variations is achieved by increasing the value of the sensitivity weighting constant, σ.

If there are N uncertain plant parameters an equal number of sensitivity functions and weighting factors must be introduced, and the performance index becomes

$$J = \int_0^\infty \{e^2(t) + k^2u^2(t) + \sum_{j=1}^{N} \sigma_j^2[S_{p_j}^c(t)]^2\}\,dt. \tag{6}$$

Sensitivity functions

The control which optimizes (5) or (6) is derived using two different sensitivity functions: a "conventional" sensitivity function which is usually defined directly in the time domain, and a Bode sensitivity function which is defined in the frequency domain. These sensitivity functions must be written in a form suitable for subsequent use in performing the frequency domain synthesis. In addition, it is important to understand the significance of using the open-loop versus the closed-loop sensitivity function in the synthesis of the sensitivity constrained optimal control.

- *Conventional sensitivity*

A conventional sensitivity function is defined [9] directly in the time domain by

$$S_p^c(t) = \frac{\partial c(t)}{\partial p}, \tag{7}$$

which, in the frequency domain, becomes

$$S_p^c(s) = \mathcal{L}\left[\frac{\partial c(t)}{\partial p}\right] = \frac{\partial c(s)}{\partial p} = C_p(s). \tag{8}$$

The first partial derivative, (7), is recognized as the coefficient of the linear term of a Taylor series expansion of the perturbation in the response, $\Delta c(t)$, due to a perturbation in a plant parameter, p.

If the definition, (8), is applied to the open-loop system Fig. 1, the open-loop sensitivity function written in terms of the control, $u(s)$, becomes

$$S_p^c(s) = \frac{\partial}{\partial p}[G(s)u(s)] = u(s)\frac{\partial G(s)}{\partial p} = u(s)G_p(s). \tag{9}$$

Similarly, if (8) is applied to the closed-loop system of Fig. 1, the open-loop sensitivity function written in terms of the control $u(s)$ is

$$\begin{aligned} S_p^c(s) &= \frac{\partial}{\partial p}\left[\frac{R(s)G_c(s)G(s)}{1 + G_c(s)G(s)}\right] \\ &= \left[1 - \frac{G(s)u(s)}{R(s)}\right]u(s)G_p(s). \end{aligned} \tag{10}$$

Thus, the control enters linearly in the case of the open-loop sensitivity, but quadratically in the case of the closed-loop sensitivity.

Because the sensitivities of the open- and closed-loop systems are different, the approach usually taken in finding the optimal control must be slightly modified when a sensitivity constraint is imposed. The usual approach is to first find the optimal open-loop control and then from it, if possible, an equivalent (same response) closed-loop law. If this same approach is taken when a sensitivity constraint is imposed it might at first seem appropriate to use the open-loop sensitivity to obtain initially the open-loop control. However, although it would still be possible to find a feedback law with the same nominal response, the imposed sensitivity constraint would apply explicitly only to the sensitivity of the open-loop system, but not to the closed-loop one. Thus to have the desired sensitivity constraint apply explicitly to the closed-loop system, it is necessary to use the closed-loop sensitivity function in the synthesis of the optimal control. The point here is that because of the difference in open- and closed-loop sensitivity, it is, strictly speaking, essential to know at the outset of the synthesis whether an open- or closed-loop control is required so that the correct corresponding sensitivity function is employed. Of course, in many cases it is likely that if a sensitivity constraint is formally imposed on the open-loop response, a corresponding reduction in closed-loop sensitivity would also result. The solution found in this way would then represent a useful approximation to the exact problem which, as will be shown later, is difficult to solve when a conventional definition of sensitivity is employed. In fact, this type of approximate solution may be the only one available whenever the form of the feedback law is unknown since, in that event, an expression for the closed-loop sensitivity could not be written.

- *Bode sensitivity*

Since the Bode definition of feedback system sensitivity plays an important role in classical feedback theory, it is of interest to consider it here in connection with the optimization of the quadratic performance functional, Eq. (5). A Bode type sensitivity function is defined directly in the complex frequency domain by

$$S_p^T(s) = \frac{\partial \ln T(s)}{\partial \ln p}, \tag{11}$$

where p is a parameter of the plant transfer function, $G(s)$, and $T(s)$ is the closed-loop transfer function,

$$T(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)}. \tag{12}$$

(Actually, the Bode definition is the inverse of Eq. (11).) Using (11), (12), and the fact that $C(s) = T(s)R(s)$, it follows that the Bode (closed-loop) sensitivity function (11) can be written in terms of the control $u(s)$ as

$$S_p^T(s) = \left[\frac{1}{G(s)} - \frac{u(s)}{R(s)}\right]\frac{\partial G(s)}{\partial p} = pC_p(s)/C(s), \tag{13}$$

where $C(s)$, and $C_p(s)$ denote the closed-loop response, and closed-loop (response) sensitivity, respectively. Thus, the Bode sensitivity is just the conventional closed-loop sensitivity (8) normalized in the frequency domain with respect to the response. Note further that in contrast to the conventional closed-loop sensitivity, the control enters only linearly in the Bode (closed-loop) sensitivity.

Optimal control synthesis

The synthesis of the optimal control which minimizes the quadratic performance index (5) or (6) is performed here entirely in the frequency domain for three separate cases: (a) optimal control using Bode (closed-loop) sensitivity, (b) optimal control using conventional open-loop sensitivity, and (c) optimal control using conventional closed-loop sensitivity. Since the details of the method used are well known, only the main steps of the derivation are outlined. In the first two cases, the optimal control is found to satisfy a transformed Wiener-Hopf equation which is solved by the technique of spectral factorization. In the last case the optimal control is found to satisfy an equation which is analogous to the Wiener-Hopf equation, but more difficult to solve because it involves the control quadratically rather than linearly. In all three cases, several significant properties of the optimal system are investigated and compared with the solution (2) obtained without sensitivity considerations. To simplify the notation, the independent Laplace variable s is dropped in all transfer functions whenever it is convenient to do so.

- *Optimal control using Bode (closed-loop) sensitivity*

For simplicity assume initially that there is only one uncertain plant parameter, p. Using the sensitivity function (13), and referring to Fig. 1, the performance index (5) becomes

$$J = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Big[(R - Gu)(\bar{R} - \bar{G}\bar{u}) + k^2\, u\bar{u} + (p\sigma)^2\Big(\frac{G_p}{G} - \frac{G_p u}{R}\Big)\Big(\frac{\bar{G}_p}{\bar{G}} - \frac{\bar{G}_p\bar{u}}{\bar{R}}\Big)\Big]\, ds. \qquad (14)$$

To obtain the variation of J, let

$$u = u^* + \epsilon u_1 \qquad (15a)$$

$$\bar{u} = \bar{u}^* + \epsilon\bar{u}_1, \qquad (15b)$$

where u^* and u, are the stable optimum control and some arbitrary control, respectively, and ϵ is a constant parameter. Substituting (15) into (14) and then setting $(\partial J/\partial\epsilon)_{\epsilon=0} = 0$ gives the following necessary condition for J to be stationary:

$$0 = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Big[\Big(k^2 + G\bar{G} + p^2\sigma^2\frac{G_p\bar{G}_p}{R\bar{R}}\Big)u^* - R\bar{G} - p^2\sigma^2\frac{G_p\bar{G}_p}{G\bar{R}}\Big]\bar{u}_1\, ds. \qquad (16)$$

This equation is satisfied if the bracketed term of the integrand has all of its poles in the RPH, i.e., if

$$\Big(k^2 + G\bar{G} + p^2\sigma^2\frac{G_p\bar{G}_p}{R\bar{R}}\Big)u^* - \Big(R\bar{G} + p^2\sigma^2\frac{G_p\bar{G}_p}{G\bar{R}}\Big) = \bar{x}, \qquad (17)$$

and the integrand of (16) is of order $1/s^2$ in the limit as $s \to \infty$. Equation (17) is recognized as a transformed scalar Wiener-Hopf equation, and it can be shown that if $(m - n + 1) \geq (v - q)$, where

n, m = degree of numerator and denominator, respectively of G; $m > n$,

q, v = degree of numerator and denominator, respectively, of R; $v > q$,

it can be solved by spectral factorization to yield the optimal control

$$u_B^* = \frac{1}{\Big(k^2 + G\bar{G} + p^2\sigma^2\dfrac{G_p\bar{G}_p}{R\bar{R}}\Big)^+} \times \left[\frac{R\bar{G} + \sigma^2 p^2\dfrac{G_p\bar{G}_p}{G\bar{R}}}{\Big(k^2 + G\bar{G} + p^2\sigma^2\dfrac{G_p\bar{G}_p}{R\bar{R}}\Big)^-}\right]_+, \qquad (18)$$

where the notation is the same as that defined in relation to (2), and the subscript B denotes the fact that the Bode definition of sensitivity applies. Note that (18) has the same form as (2) and reduces to the latter when σ, the sensitivity weighting factor, is set equal to zero. If there are N uncertain plant parameters, the optimal control is found using the more general performance index (6). The result is

$$u_B^* = \frac{1}{\Big(k^2 + G\bar{G} + \sum\limits_{j=1}^{N}\sigma_j^2 p_j^2\dfrac{G_{pj}\bar{G}_{pj}}{R\bar{R}}\Big)^+} \times \left[\frac{R\bar{G} + \sum\limits_{j=1}^{N}\sigma_j^2 p_j^2\dfrac{G_{pj}\bar{G}_{pj}}{G\bar{R}}}{\Big(k^2 + G\bar{G} + \sum\limits_{j=1}^{N}\sigma_j^2 p_j^2\dfrac{G_{pj}\bar{G}_{pj}}{R\bar{R}}\Big)^-}\right]_+. \qquad (19)$$

To study the effect of σ, the sensitivity weighting factor, on the closed-loop transfer characteristics obtained using u_B^*, it is convenient to write each of the transfer functions in (18) as a ratio of polynomials. If it is assumed that the uncertain parameter is a plant pole s_i, or plant zero z_i (this pole or zero being denoted by p), the transfer functions of interest can be written as

$$G = N_G/D_G, \qquad (20a)$$

$$R = N_R/D_R, \qquad (20b)$$

$$G_p = -N_G/D_G(s + p); \qquad p = s_i, \quad \text{or} \qquad (20c)$$

$$G_p = +N_G/D_G(s + p); \qquad p = z_i. \qquad (20d)$$

Inspection of (18) shows that for stable, minimum-phase plants, the poles of the term $[\cdot]_+$ are the poles of the input R and the pole at $s = -p$ (recall that p is used to denote a plant pole or zero). Thus, for the case in which only one pole or zero is subject to uncertainty, the closed-loop transfer function $T_B^* = Gu_B^*/R$ can be written in the form

$$T_B^* = N_G F_B/((p + s)(p - s) N_R \bar{N}_R(k^2 D_G \bar{D}_G + N_G \bar{N}_G) + \sigma^2 p^2 D_R \bar{D}_R N_G \bar{N}_G)^+, \qquad (21)$$

where F_B is the numerator of the term $[\cdot]_+$ in (18). The (closed-loop) poles of T_B^* are the left half s-plane roots of $(\cdot) = 0$ which can be written as

$$1 + \frac{p^2\sigma^2 N_G \bar{N}_G D_R \bar{D}_R}{(k^2 D_G \bar{D}_G + N_G \bar{N}_G)(p + s)(p - s) N_R \bar{N}_R} = 0. \qquad (22)$$

Reference to (2) shows that the LHP roots of the term $(k^2 D_G \bar{D}_G + N_G \bar{N}_G)$ are poles of the optimal system obtained when sensitivity is ignored, i.e., when $\sigma = 0$. When $\sigma \neq 0$ and $p = z_i$ (a plant zero), the factors $(p \pm s)$ are cancelled by identical factors of $N_G \bar{N}_G$, and it follows that the number of poles of the optimal closed-loop system obtained with a sensitivity term in (5) is the same as that obtained ignoring sensitivity ($\sigma = 0$); however, the pole locations are different in each case. When $\sigma \neq 0$ and $p = s_i$ (a plant pole), the factors $(p \pm s)$ are not cancelled, and the optimal system found by including a sensitivity term in (5) has one additional pole compared to the optimal system found by ignoring sensitivity.

It is useful to consider the behavior of T_B^* obtained when the sensitivity weighting factor is increased to obtain reduced sensitivity to a plant pole or zero uncertainty. From (22) it can be seen that in the limiting case as $\sigma \to \infty$, $(m - n + q - v + 1)$ poles of T_B^* tend to infinity along straight line asymptotes; the remainder approach plant zeros, and zeros due to the roots of D_R (e.g., the poles of the input R). Thus, for sufficiently large σ, the poles approaching the plant zeros (which are also zeros of T_B^*) will have small residues, and it would at first appear that the poles of T_B^* that approach the roots of D_R become the dominant closed-loop poles. However, before the dominance of these poles can be established it is necessary to consider the location of the zeros of F_B which also depend on σ and, in certain cases, may be very close to the poles approaching D_R.

If the input is taken to be a unit step function, $N_R = 1$ and $D_R = s$. Thus T_B^* will have a pole on the negative real axis at $s = -\gamma$, and by the previous analysis $\gamma \to 0$ as $\sigma \to \infty$. The term $[\cdot]_+$ in (18) has LHP poles at $s = 0$, $s = -p$ and therefore can be written

$$[\cdot]_+ = \frac{r_1}{s + p} + \frac{r_2}{s} = \frac{(r_1 + r_2)\left[s + \dfrac{p}{1 + r_1/r_2}\right]}{s(s + p)} = \frac{F_B}{s(s + p)}, \qquad (23)$$

which indicates that for this case F_B has only one zero determined by the ratio of the residues r_1, r_2. If (20) is substituted into the term $[\cdot]_+$ in (18), an evaluation of the residues gives

$$r_1 = \sigma^2 p^3 \bar{N}_G(-p)/(\cdot)^-_{s=-p} \qquad (24a)$$

and

$$r_2 = p^2 \bar{N}_G(0)/(\cdot)^-_{s=0}. \qquad (24b)$$

The previous analysis also indicates that as $\sigma \to \infty$, the term $(\cdot)^-$ is given approximately by

$$(\cdot)^- \approx \bar{N}_G(s - \gamma)P_\infty, \qquad (25)$$

where P_∞ denotes the product of those factors of $(\cdot)^-$ whose roots approach infinity as $\sigma \to \infty$. By substituting (25) into (24) the approximation obtained for the ratio of the residues as $\sigma \to \infty$ is

$$\frac{r_1}{r_2} \approx P\gamma\sigma^2. \qquad (26)$$

Since γ satisfies the equation $(\cdot) = 0$, and it is known that $\gamma \to 0$ as σ increases, it follows that the approximate form for the variation of γ as a function of σ for $\sigma \to \infty$ can be obtained by approximating $(\cdot) = 0$ with the two lowest order terms in s (i.e., the s^2 and s^0 terms). Then since the s^2 term is multiplied by $p^2\sigma^2 +$ a constant (independent of σ), setting $s = \gamma$ and solving leads to the result that $\gamma \to 1/p\sigma$ as $\sigma \to \infty$. This implies, see (26), that $r_1/r_2 \to 0$ as $\sigma \to \infty$ and so the zero of F_B given in (23) approaches zero as $\sigma \to \infty$. That is, when the input is a step function $D_R = s$ and one pole of T_B^* approaches the origin as $\sigma \to \infty$; however, this pole is approximately cancelled by the zero of T_B^* due to F_B which also approaches the origin as $\sigma \to \infty$ and will not be the dominant pole as an analysis ignoring the zero of F_B might indicate. Thus, when σ is increased to a sufficiently large value (to obtain reduced sensitivity to a plant pole or zero uncertainty) the closed-loop bandwidth tends to increase; for "intermediate" values of σ the residues at the poles of T_B^* approaching the plant zeros and the origin will be larger and contribute more significantly to the response.

Example 1: To illustrate the application of (18), consider the case where the plant transfer function is

$$G(s) = \frac{a}{s(s + b)}. \qquad (27)$$

and the nominal values of the plant parameters are $a_0 = b_0 = 1$. Assuming the parameter a to be fixed, the plant sensitivity to a perturbation in b is

$$G_b(s) = \frac{-1}{s(1 + s)^2}. \qquad (28)$$

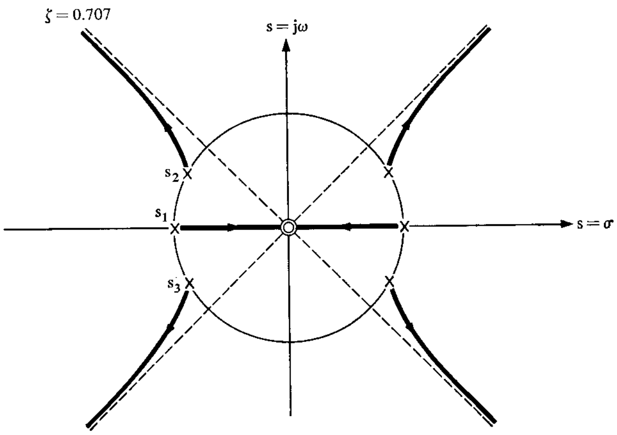

Figure 2 Root-square locus of optimal system poles; Example 1.

Setting $k^2 = 1$ and $R = 1/s$ (unit step input), reference to (18) gives:

$$\left(k^2 + G\bar{G} + \sigma^2 \frac{G_b \bar{G}_b}{R\bar{R}}\right)^+ = (-s^6 + 2s^4 - 2s^2 + 1 - \sigma^2 s^2)^+ / s(1+s)^2 = Y, \tag{29a}$$

$$\left(k^2 + G\bar{G} + \sigma^2 \frac{G_b G_b}{R\bar{R}}\right)^- = (-s^6 + 2s^4 - 2s^2 + 1 - \sigma^2 s^2)^- / (-s)(1-s)^2 = \bar{Y}, \tag{29b}$$

$$\left[\frac{R\bar{G}}{\bar{Y}} + \sigma^2 \frac{G_b \bar{G}_b}{G\bar{R}\bar{Y}}\right] = (1 - s^2 - \sigma^2 s^2)/(s+1)(s) \times (-s^6 + 2s^4 - 2s^2 + 1 - \sigma^2 s^2)^-, \tag{29c}$$

$$\left[\frac{R\bar{G}}{\bar{Y}} + \sigma^2 \frac{G_b \bar{G}_b}{G\bar{R}\bar{Y}}\right]_+ = \frac{r_1}{s+1} + \frac{r_2}{s}, \tag{29d}$$

where r_1 and r_2 denote the residues at the poles $s = -1$, $s = 0$, respectively.

Substituting (29) into (18) and evaluating the optimal closed-loop transfer function gives

$$T_B^* = \frac{G}{R} u_B^* = \frac{(r_1 + r_2)s + r_2}{(-s^6 + 2s^4 - 2s^2 + 1 - \sigma^2 s^2)^+}. \tag{31}$$

The optimal system poles, which are the left-half s-plane roots of

$$Y\bar{Y} = s^6 - 2s^4 + 2s^2 - 1 + \sigma^2 s^2 = 0, \tag{32}$$

are shown in the root-square locus diagram, Fig. 2, as a function of σ, the sensitivity weighting factor. When $\sigma \rightarrow 0$, the LHP-roots of (32) lie on the unit circle at the points

$$s_1 = -1, \tag{33a}$$

$$s_2 = -(\sqrt{3} + j)/2, \tag{33b}$$

$$s_3 = -(\sqrt{3} - j)/2. \tag{33c}$$

When σ is increased, the complex pair move away from the origin along the straight line asymptotes, but the real pole moves toward the origin. As $\sigma \rightarrow \infty$ the zero of T_B^* is near the real pole and so the residue at that pole tends to zero. Hence for sufficiently large σ the bandwidth of T_B^* will tend to increase due to the complex poles approaching infinity along the asymptotes.

- *Optimal control using conventional open-loop sensitivity*

When the conventional open-loop sensitivity definition (8) is used, the performance functional (5) becomes

$$J = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} [(R - Gu)(\bar{R} - \bar{G}\bar{u}) + k^2 u\bar{u} + \sigma^2 G_p \bar{G}_p u\bar{u}]\, ds. \tag{34}$$

A necessary condition for J in (34) to be stationary is the transformed Wiener-Hopf equation

$$(k^2 + G\bar{G} + \sigma^2 G_p \bar{G}_p) u^* - R\bar{G} = \bar{x}, \tag{35}$$

which can be solved by spectral factorization to yield the optimal control

$$u_{c0}^* = \frac{1}{(k^2 + G\bar{G} + \sigma^2 G_p \bar{G}_p)^+} \times \left[\frac{R\bar{G}}{(k^2 + G\bar{G} + \sigma^2 G_p \bar{G}_p)^-}\right]_+. \tag{36}$$

The notation used in (36) is again the same as that defined in relation to (2) and the subscript $c0$ denotes the fact that the conventional open-loop sensitivity function (8) applies. The form of (36) is also the same as (2) and reduces to the latter when $\sigma = 0$. For the case of N uncertain plant parameters, the corresponding optimal control found using (6) is

$$u_{c0}^* = \frac{1}{\left(k^2 + G\bar{G} + \sum_{i=1}^{N} \sigma_i^2 G_{p_i} \bar{G}_{p_i}\right)^+} \times \left[\frac{R\bar{G}}{\left(k^2 + G\bar{G} + \sum_{i=1}^{N} \sigma_i^2 G_{p_i} \bar{G}_{p_i}\right)^-}\right]_+. \tag{37}$$

Although u_{c0}^* was obtained using the *open-loop* sensitivity function, it is still possible to consider obtaining u_{c0}^* by means of a feedback structure as pointed out earlier in the paper. The closed-loop transfer function $T_{c0}^* = Gu_{c0}^*/R$ can be examined as a function of σ using the notation defined in (20).

Inspection of (36) shows that the only poles of the term $[\cdot]_+$ are the poles of R. Thus, for the case with only one pole, or zero perturbation, the closed-loop transfer function T_{c0}^* can be written in the form

$$T_{c0}^* = \frac{N_G(p+s)F_{c0}}{N_R((p+s)(p-s)(k^2 D_G \bar{D}_G + N_G \bar{N}_G) + \sigma^2 N_G \bar{N}_G)^+}, \tag{38}$$

where $p = s_i$ or z_i and F_{c0} denotes the numerator of $[\cdot]_+$ in (36). Note that the uncertain plant pole or zero is always a zero of T_{c0}^*. The closed-loop poles are determined by the zeros of the input R, and the left-half s-plane roots of $(\cdot) = 0$, which can be written in the form

$$1 + \frac{\sigma^2 N_G \bar{N}_G}{(k^2 D_G \bar{D}_G + N_G \bar{N}_G)(p + s)(p - s)} = 0. \tag{39}$$

Noting again the fact that the roots of the term $(k^2 D_G \bar{D}_G + N_G \bar{N}_G)$ are the poles of the optimal system obtained without a sensitivity term ($\sigma = 0$) in (5), it follows that when $\sigma \neq 0$ and $p = z_i$ the factors $(p \pm s)$ are cancelled by corresponding factors of $N_G \bar{N}_G$, and the number of closed-loop poles is the same with or without the sensitivity term in (5); when $\sigma \neq 0$, and $p = s_i$ the factors $(p \pm s)$ are not cancelled, and the optimal system found by including the sensitivity term in (5) has one more pole than the optimal system found by ignoring sensitivity.

When $\sigma \longrightarrow \infty$ to obtain reduced sensitivity, inspection of (39) indicates that $(m - n + 1)$ poles of T_{c0}^* go to infinity along straight line asymptotes; the remaining poles approach the plant zeros (which are also zeros of T_{c0}^*) and so the residues at these poles become small for sufficiently large σ. To determine which of the remaining poles are dominant for large σ it is necessary to consider the location of the other zeros of T_{c0}^* which include the zeros of F_{c0} (which depend on σ) and the fixed zero at $s = -p$. If the input is taken to be a unit step function as in the previous section the term $[\cdot]_+$ in (36) is simply r_0/s where r_0 is the residue of $[\cdot]$ at $s = 0$. Thus, for this case $F_{c0} = r_0$ and does not contribute any finite zeros to T_{c0}^*. This implies that by increasing σ to obtain reduced sensitivity to a plant pole or zero uncertainty the bandwidth of the closed-loop transfer function T_{c0}^* is increased for sufficiently large σ.

Example 2: For the same plant and other assumptions of Example 1, reference to (36) gives

$$(k^2 + G\bar{G} + \sigma^2 G_b \bar{G}_b)^+ = (s^6 - 2s^4 + 2s^2 - 1 - \sigma^2)^+/s(1 + s)^2 = Y, \tag{40a}$$

$$(k^2 + G\bar{G} + \sigma^2 G_b \bar{G}_b)^- = (s^6 - 2s^4 + 2s^2 - 1 - \sigma^2)^-/s(1 - s)^2 = \bar{Y}, \tag{40b}$$

$$[R\bar{G}/\bar{Y}] = -(1 - s)/s(s^6 - 2s^4 + 2s^2 - 1 - \sigma^2)^-, \tag{40c}$$

$$[R\bar{G}/\bar{Y}]_+ = \frac{r_0}{s}; \quad r_0 = \text{residue at } s = 0. \tag{40d}$$

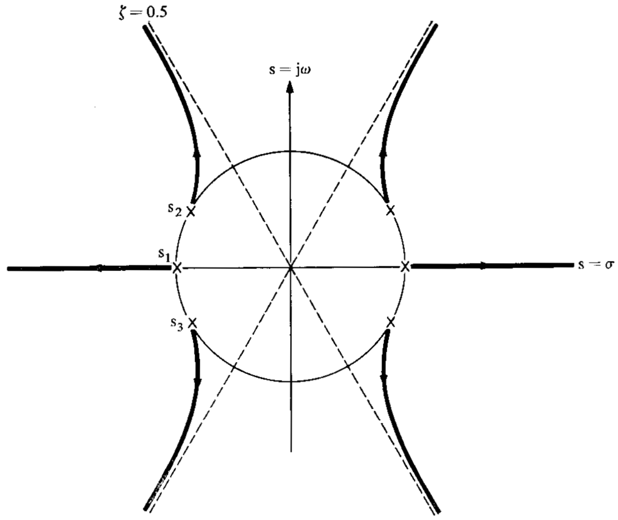

Figure 3 Root-square locus of optimal system poles; Example 2.

Substituting (40) into (36) and then evaluating Gu_{c0}^*/R gives the optimal closed-loop transfer function,

$$T_{c0}^* = \frac{G}{R} u_{c0}^* = \frac{r_0(s + 1)}{(s^6 - 2s^4 + 2s^2 - 1 - \sigma^2)^+}. \tag{41}$$

The optimal system poles, which are the left-half s-plane roots of

$$Y\bar{Y} = (s^6 - 2s^4 + 2s^2 - 1 - \sigma^2) = 0 \tag{42}$$

are shown in the root locus diagram in Fig. 3 as a function of the sensitivity weighting factor, σ. When $\sigma \longrightarrow 0$, the LHP-roots of (42) lie on the unit circle and have the same values given by (33). When σ is increased to obtain reduced sensitivity, the poles move away from the origin with the result that the closed-loop system bandwidth is increased.

- *Optimal control using conventional closed-loop sensitivity*

The performance functional (5) obtained using the conventional closed-loop sensitivity function (10) is

$$J = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left[(R - Gu)(\bar{R} - \bar{G}\bar{u}) + k^2 u\bar{u} + \sigma^2\left(G_p u - \frac{G}{R}G_p u^2\right)\left(\bar{G}_p \bar{u} - \frac{\bar{G}}{\bar{R}}\bar{G}_p \bar{u}^2\right)\right] ds \tag{43}$$

and the necessary condition for J to be stationary becomes:

$$-\bar{G}R + (k^2 + G\bar{G})u + \sigma^2 G_p \bar{G}_p u \times \left(1 - \frac{2\bar{u}\bar{G}}{\bar{R}}\right)\left(1 - \frac{uG}{R}\right) = \bar{x}. \tag{44}$$

The problem of finding u from (44) is analogous to the Wiener-Hopf problem, but more difficult because $\bar{x}$ involves u quadratically as well as linearly (and also involves

$\bar{u}$). Although a general solution is not attempted, the form of the solution and certain significant properties of it will be described. For this purpose, it is convenient to introduce the abbreviations:

$$H = (k^2 + G\bar{G}), \tag{45a}$$

$$W = \bar{G}R/H^-, \tag{45b}$$

$$A = G_p\bar{G}_p(1 - 2\bar{u}\bar{G}/\bar{R})/H^-, \tag{45c}$$

$$B = GA/R, \tag{45d}$$

where the notation used is the same as that used in (2). Substituting (45) into (44) and dividing by H^- gives

$$\frac{\bar{x}}{H^-} + [W]_- = -[W]_+ + H^+u + \sigma^2(Au - Bu^2). \tag{46}$$

Since the left side of this equation is regular only in the LHP, the first two terms on the right side, which are regular only in the RHP, cannot contribute to the left side. The only term on the right that can contribute to the left side is the last one which has singularities in both the LHP and the RHP. Since the last term is of order σ^2, the left side must be of order σ^2 also and so (46) can be written as

$$\sigma^2\bar{y} = -[W]_+ + H^+u + \sigma^2(Au - Bu^2), \tag{47}$$

where $\bar{y}$ is given by

$$\bar{y} = [Au - Bu^2]_- = \left[\frac{G_p\bar{G}_p(1 - 2\bar{u}\bar{G}/\bar{R})(1 - uG/R)u}{H^-}\right]_-. \tag{48}$$

Equation (47) can be solved for u to give

$$u = \frac{1}{2\sigma^2 B}\{(H^+ + \sigma^2 A) - \sqrt{(H^+ + \sigma^2 A)^2 - \sigma^2 B([W]^+ + \sigma^2\bar{y})}\} \tag{49a}$$

$$= \frac{2([W]_+ + \sigma^2\bar{y})}{(H^+ + \sigma^2 A) + \sqrt{(H^+ + \sigma^2 A) - \sigma^2 B([W]_+ + \sigma^2\bar{y})}}, \tag{49b}$$

where the sign in front of the square-root is chosen to satisfy the requirement that u reduce to the known solution $[W]_+/H^-$ given by (2) when $\sigma^2 = 0$. Although (49) holds everywhere it is expedient to make use of it only in the LHP where $\bar{y}$ and $\bar{u}$ are free of singularities. Thus u can be written

$$u = \left[\frac{1}{2\sigma^2 B}\{(H^+ + \sigma^2 A) - \sqrt{(H^+ + \sigma^2 A)^2 + \sigma^2 B([W]_+ + \sigma^2\bar{y})}\}\right]_+. \tag{50}$$

It is apparent from (50) that the singularities of u may include *branch points* as well as poles. The branch points occur when

$$(H^+ + \sigma^2 A)^2 = \sigma^2 B([W]_+ + \sigma^2\bar{y}), \tag{51}$$

which, for sufficiently small values of σ, reduces to

$$H \approx -\sigma^2 A \pm \sigma\sqrt{[W]_+ B}. \tag{52}$$

The approximate locations of the branch points can be determined from (52) if the zero-th order approximation for u obtained from (2) is substituted into (45) to get A_0 and B_0. Then if ζ is used to denote a LHP zero of H, expanding (52) about the point ζ and retaining only the lowest order terms leads to

$$(s - \zeta)H'(\zeta) + \sigma^2 A(\zeta) \pm \sigma\sqrt{[W(\zeta)]_+ B(\zeta)} = 0. \tag{53}$$

Thus, the branch points are approximately given by the equation

$$s_{\mathrm{B.P.}} \approx \zeta + \frac{1}{H'(\zeta)}[-\sigma^2 A_0(\zeta) \pm \sigma\sqrt{[W]_+ B_0(\zeta)}], \tag{54}$$

which shows that the pole of $u(s)$ at ζ goes over into a short *branch cut* whose length is of order σ, and whose direction is determined by

$$\sqrt{[W(\zeta)]_+ B_0(\zeta)}/H'(\zeta) \tag{55}$$

which may be real or complex. An exceptional case occurs if $B_0(\zeta)$ is zero; then a pole of u_0 may go over into a pole of u even for $\sigma \neq 0$. Other possible LHP branch points may occur near poles of A, B or $[W]_+$, but the working out of these branch point locations will not be considered further here.

The possibility of the optimal control u which satisfies (44) having branch cuts arises because u enters the conventional closed loop sensitivity function (10) quadratically rather than linearly. Since the control enters the Bode closed-loop sensitivity function (13) only linearly, the corresponding optimal control u_B^* cannot have branch cuts and its singularities are restricted to poles. The fact that u may have branch cuts has important physical significance because it implies that u could not be implemented by means of a lumped parameter network. Thus, even if an exact analytical solution of (44) for u could be found, a rational polynomial approximation to the result would be required for practical implementation. Note that whenever the weighting factor σ is sufficiently small for the iteration to converge, the optimal control can be found by solving (48) and (50) iteratively starting with the zero-th order approximation to u given by (2).

It is of interest to consider whether the problem of branch cuts would be eliminated by carrying out the synthesis of u (i.e. with a conventional closed-loop sensitivity function) in the time domain using a state-variable approach rather than in the frequency domain considered here. A corresponding difficulty can be expected in the time domain since the possibility of u having branch cuts implies that u cannot be obtained as a solution to a finite set of ordinary differential equations.

Conclusion

In this paper the synthesis of optimal controls with reduced sensitivity to plant parameter uncertainties is approached by adding sensitivity functions to a quadratic form performance index. It is shown that the synthesis results in optimal controls of different forms depending on the definition used for the sensitivity function.

When the synthesis is based on a Bode (closed-loop) sensitivity or a conventional open-loop sensitivity, the resulting optimal control singularities are restricted to poles. The application to specific problems of the general solutions obtained in the paper for these two cases is straightforward. One useful procedure suggested by the form of these solutions consists of first solving for the optimal control without regard to sensitivity considerations, and then applying graphical root-square locus techniques to determine the optimal control as a function of the sensitivity weighting factors. Under certain conditions it is shown that the bandwidth of the optimal closed-loop transfer function increases as the sensitivity weighting factor for a plant pole or zero sensitivity function is increased, but further study is required to determine whether this result applies in general.

When the synthesis is based on a conventional closed-loop sensitivity it is shown that the optimal control singularities may in general include branch cuts as well as poles. Since the possible existence of branch cuts implies that the optimal control could not be implemented (except approximately) with a lumped parameter network, it is apparent that future research should be directed at finding a general method for approximating the solution with a rational polynomial function.

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