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A Computer Model for Global Study of the General Circulation of the Atmosphere

Abstract: A mathematical model is developed for global prediction of large-scale movements and mean properties of the atmosphere from simulated initial weather conditions. A novel feature is the development of the formulation—one of considerably greater complexity than those for previous weather models adapted to machine calculation—in a concise tensor form. The organization of the computational task illustrates how a very complex problem with vast data requirements may be solved on a computer with limited high-speed storage.

Contents

Introduction

This paper describes a joint project of IBM and the U. S. Weather Bureau in the area of large-scale global weather forecasting on the IBM STRETCH (7030) Computer. The basic formulation chosen for the project was proposed by the General Circulation Research Section of the U. S. Weather Bureau, under the guidance of Drs. J. Smagorinsky and S. Manabe, as well as J. L. Holloway, Jr. and R. F. Strickler. The STRETCH program, designated "Global Weather Simulator," resulting from this endeavor was used in modified forms, during several years, for experimental studies at the General Circulation Research Laboratory in Washington, D. C.[1,2] The most extensive of these modifications, all of which were made by Weather Bureau personnel, reduced the region of treatment to a hemisphere. Alternative methods which enable treatment of global models have been developed and/or used by Y. Kurihara[3] and K. Bryan[4] and also by H. L. Kuo, Y. Mintz, and J. Smagorinsky in unpublished work.

The formulation of the basic problem is developed in terms of tensor algebra which, we believe, is a substantial simplification and should have interest even for the meteorologist. In the discretization for numerical calculation, the atmosphere is stratified into nine layers in the vertical, with up to more than 10,000 points taken in each layer. Different mappings are adopted for the northern and southern hemispheres, and special interpolation techniques are developed to relate data at points near the transitional equatorial boundary. The discussion of the data processing problem at the core of this project seems to be, as the authors have found out over the past years, of recurrent interest to the computing field. Special attention is given to the organization, and plan for efficient computer manipulation, of the vast amounts of data required. The computer program is also designed to facilitate experimentation with "submodels" of varying complexity for several alternative choices of the number of points in the spatial grid.

Even within the framework of this rather lengthy paper, some important areas, in particular that of "radiation," had to be omitted.

Formulation of the global weather model

In the following we wish to describe as concisely as possible the formulation on which the model is based. We can be quite brief as far as physical aspects and meteorological significance of the model are concerned, referring the interested reader for details to the pioneering work of Eliassen,[5-7] Phillips,[8] Charney,[9] and, for this model in particular, of Smagorinsky.[1,2,10-12] However, we shall expend some effort on the formal mathematical derivation of the basic equations used by the computer. We found that the various transformations of the physical equations, necessitated by a mapping of the earth on two Cartesian coordinate systems, lead to many difficulties, e.g., in the area of interpolation around the equator. These can be handled only if the transformation properties of the various variables are clearly understood. As a consequence we were naturally led towards employing the well-known procedures of tensor calculus. For a different, more meteorologically oriented approach to a part of this material we refer to Refs. 7 and 13.

• *1. The basic physical equations*

The prime objective of the global weather model is to determine a set of seven physical variables, taken to be characteristic for the state of the atmosphere, from the integration of a corresponding set of partial differential equations with physically appropriate boundary conditions. We may take as unknowns the variables

$\mathbf{w} = \{w^1, w^2, w^3\}$ = wind velocity
p = pressure
ρ = density
T = absolute temperature
h = relative humidity.

Correspondingly, the determining equations are:

a) The equations of motion (Navier-Stokes) (compare, e.g., Ref. 5)

$$\mathbf{T} \equiv \frac{\partial \mathbf{w}}{\partial t} + (\mathbf{w}\cdot\nabla)\mathbf{w} + 2\boldsymbol{\Omega} \times \mathbf{w} + \frac{1}{\rho}\,\mathrm{grad}\, p - \mathbf{g} - \frac{1}{\rho}\mathbf{f} = 0, \quad \text{where} \qquad (1)$$

$\boldsymbol{\Omega}$ = angular velocity of earth
$\mathbf{g}$ = gravitational acceleration (assumed to have constant magnitude)
$\mathbf{f}$ = frictional force vector.

b) The equation of continuity:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \mathbf{w}) = 0. \qquad (2)$$

c) The equation of state (of air):

$$p = \rho R T, \qquad (3)$$

where

R = universal gas constant (if necessary, modified to account for deviations of the atmosphere from the ideal gas state)

d) Thermodynamic energy equation [compare, e.g., (1), (2)]

$$\dot{\theta} = \frac{1}{c_p T}\,\theta q = \frac{q}{c_p}\left(\frac{p_{st}}{p}\right)^{\kappa}, \qquad \theta \equiv T\left(\frac{p_{st}}{p}\right)^{\kappa} \qquad (4)$$

θ = potential temperature
c_p, c_v = specific heat of air at constant pressure (volume)

$$\kappa = \frac{c_p - c_v}{c_p}$$

p_{st} = standard pressure (1000 mb)
q = rate of heating per unit mass of air.

The dot above θ denotes total differentiation with respect to time. The specific entropy of a perfect gas is equal to $c_p \cdot \ln \theta$

e) Humidity tendency equation: $\dot{r} = W$ (5)

r = mixing ratio ($\mathrm{gm}_{\text{water vapor}}/\mathrm{gm}_{\text{dry air}}$)
W = rate of water vapor increase (per unit mass of air) due to external sources or internal changes (e.g., evaporation) of the system.

• *2. The equations of motion in stereographic map systems*

It was considered desirable to project the surface of the spherical earth stereographically onto two planes tangent at the poles. The equations are to be solved in the Cartesian system which obtains if a point P in the atmosphere vertically above a surface point SP is represented under the transformation by a point the same distance vertically above the image point of SP. We consider four coordinate systems, the local Cartesian system with origin at SP (oriented as shown in Fig. 1), the spherical coordinate system E (Figs. 1, 2, 3) and the two Cartesian map systems N and S. To indicate the system used at any instant, we employ indices (Σ, E, N, S) as subscripts or superscripts (whichever is more convenient). For the coordinates and

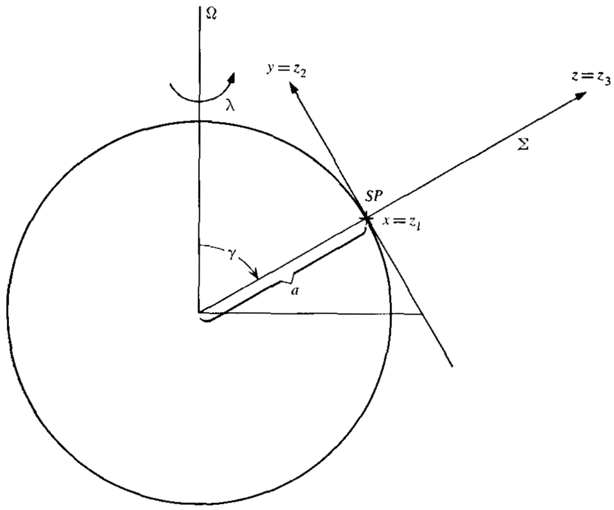

Figure 1 The local Cartesian system Σ with origin at typical earth surface point SP.

velocities we use the following symbols ($i = 1, 2, 3$):

coordinates	*velocity*
Σ: $(z_1, z_2, z_3) = (z^1, z^2, z^3)$,	$\mathbf{w} = w_i = \{w_1, w_2, w_3\} = \{w^1, w^2, w^3\} = w^i = \{\dot{z}^1, \dot{z}^2, \dot{z}^3\}$
E: $(y^1, y^2, y^3) = (r, \gamma, \lambda)$,	$v^i = \{\dot{r}, \dot{\gamma}, \dot{\lambda}\}$
N: $(x^1, x^2, x^3) = (x, y, z)$,	$u^i = \{\dot{x}, \dot{y}, \dot{z}\}$
S: $(\xi^1, \xi^2, \xi^3) = (\xi, \eta, \zeta)$.	$v^i = \{\dot{\xi}, \dot{\eta}, \dot{\zeta}\}$.

Triplets enclosed by { } and symbols such as v^i represent vectors in the sense of tensor algebra. Superscripts (subscripts) are used to denote contravariant (covariant) tensor character.

We use the convention of summing over repeated indices and we assume some familiarity with the basic tensors g_{ik} and g^{ik}, e_{abc} and e^{abc}, the operations of raising and lowering indices, the Christoffel symbols, covariant differentiation, etc. (see, e.g., Ref. 14).

In the Σ system the vectors $\mathbf{\Omega}$ and $\mathbf{g}$ have the form:

$$\begin{aligned}\mathbf{\Omega}_{(\Sigma)} &= \{\Omega_1, \Omega_2, \Omega_3\} = \{0, \Omega\sin\gamma, \Omega\cos\gamma\},\\ \mathbf{g}_{(\Sigma)} &= \{0, 0, -g\}.\end{aligned} \tag{6}$$

The fact that the equations have to be solved on a grid with a finite number of points suggests that the variables be considered as mean values of the corresponding physical entities over a neighborhood surrounding a grid point. It is shown easily (in any text on meteorological turbulence) that (1) is indeed satisfied by such mean values, if $\mathbf{f}$ is interpreted to represent not only viscous forces (which we neglect) but also the "turbulence" or "diffusion" forces caused by the transport of air by small scale motions (perturbations). In Σ, $\mathbf{f}$ can be written in terms of a "stress" tensor τ_{ij} (or in general coordinates R_{ij}, R^{ij})

$$f_i = \frac{\partial}{\partial z_1}\tau_{i1} + \frac{\partial}{\partial z_2}\tau_{i2} + \frac{\partial}{\partial z_3}\tau_{i3}, \qquad i = 1, 2, 3. \tag{7}$$

Equation (1) can be viewed as derived from a general tensor equation, valid in all coordinate systems, by suitable interpretation of the variables (e.g., $u_i \rightarrow w_i$). The covariant form of the tensor equation is:

$$\begin{aligned}T_i = \frac{\partial u_i}{\partial t} &+ u^\alpha u_{i,\alpha} + 2e_{ism}\Omega^s u^m\\ &+ \frac{1}{\rho}\frac{\partial p}{\partial x^i} - G_i - \frac{1}{\rho}g_{ij}R^{j\alpha}_{,\alpha} = 0.\end{aligned} \tag{8}$$

The contravariant form $T^i = 0$ would be somewhat less convenient. The operation denoted by a comma preceding a subscript is that of "covariant differentiation," which corresponds to the usual partial differentiation in Cartesian coordinates and ensures, in more general coordinates, that the resulting entity has tensor character. For general tensors A^i, A_i and A^{ij} the covariant derivatives are:

$$\begin{aligned}A^i_{,k} &= \frac{\partial A^i}{\partial x^k} + \begin{Bmatrix} i \\ m\ k \end{Bmatrix} A^m,\\ A_{i,k} &= \frac{\partial A_i}{\partial x^k} - \begin{Bmatrix} m \\ i\ k \end{Bmatrix} A_m,\\ A^{ij}_{,k} &= \frac{\partial A^{ij}}{\partial x^k} + \begin{Bmatrix} i \\ m\ k \end{Bmatrix} A^{mj} + \begin{Bmatrix} j \\ m\ k \end{Bmatrix} A^{im},\end{aligned} \tag{9}$$

where the expressions in brackets are "Christoffel symbols," defined in terms of the fundamental tensors. In orthogonal coordinates the Christoffel symbols can be obtained as follows (in this scheme summation over doubly occurring symbols is suspended and the comma denotes the usual partial differentiation):

$$\begin{aligned}&\text{a)}\quad k \neq s \neq m \neq k: \begin{Bmatrix} m \\ k\ s \end{Bmatrix} = 0;\\ &\text{b)}\quad s = k,\ m \neq k: \begin{Bmatrix} m \\ k\ k \end{Bmatrix} = -\tfrac{1}{2}g^{mm}g_{kk,m}\\ &\text{c)}\quad m = k,\ s \neq k: \begin{Bmatrix} k \\ k\ s \end{Bmatrix} = \begin{Bmatrix} k \\ s\ k \end{Bmatrix} = \tfrac{1}{2}g^{kk}g_{kk,s};\\ &\text{d)}\quad m = s = k: \begin{Bmatrix} k \\ k\ k \end{Bmatrix} = \tfrac{1}{2}g^{kk}g_{kk,k}.\end{aligned} \tag{10}$$

To interpret Eq. (8) in the Cartesian system Σ we require the fundamental tensors for Σ:

$$(ds)^2 = g_{ik}\, dx^i\, dx^k \rightarrow g_{ik}^{(\Sigma)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = g_{(\Sigma)}^{ik},$$

$$g_{(\Sigma)} \equiv |g_{ik}^{(\Sigma)}| = 1 \tag{11}$$

$e_{ism} = \sqrt{g} \rightarrow e_{ism}^{(\Sigma)} = 1$ when (i, s, m) is an even permutation of (1, 2, 3),

$e_{ism} = -\sqrt{g} \rightarrow e_{ism}^{(\Sigma)} = -1$ when (i, s, m) is an odd permutation of (1, 2, 3)

$$e_{ism} = 0 \rightarrow e_{ism}^{(\Sigma)} = 0 \text{ otherwise.} \tag{12}$$

Equation (8) can then be seen to reduce to Eq. (1) if the obvious changes in notation are made (e.g., $u^\alpha \rightarrow w_\alpha$, $R^{i\alpha} \rightarrow \tau_{i\alpha}$). From the validity of $T_i^{(\Sigma)} = 0$, moreover, we can infer the validity of (8) for all coordinate systems.

As an intermediate step we consider the transformation from Σ to spherical coordinates E.

$$dz^1 = r \sin\gamma\, d\lambda, \quad dz^2 = -r\, d\gamma, \quad dz^3 = dr \tag{13}$$

or

$$\frac{\partial z^i}{\partial y^s} = \begin{bmatrix} 0 & 0 & r\sin\gamma \\ 0 & -r & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\frac{\partial y^i}{\partial z^s} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\dfrac{1}{r} & 0 \\ \dfrac{1}{r\sin\gamma} & 0 & 0 \end{bmatrix}.$$

For the time being we retain the distinction between r (radial coordinate of the spherical system) and a (radius of the earth, assumed constant).

Before writing similar relations for the transformations $E \rightarrow N$, $E \rightarrow S$, we define the mapping factors m and μ:

$$m = \frac{t}{a\sin\gamma}, \qquad \mu = \frac{\tau}{a\sin\gamma}. \tag{14}$$

(Refer to Figs. 2 and 3. Note that both quantities are defined so as to be independent of r and z.) It is now easy to derive the following relations and transformation equations:

$$\begin{aligned} t &= 2a\tan\frac{\gamma}{2} = 2a\frac{\sin\gamma}{1+\cos\gamma}, \\ \tau &= 2a\tan\left(\frac{\pi-\gamma}{2}\right) = 2a\frac{\sin\gamma}{1-\cos\gamma} \end{aligned} \tag{15}$$

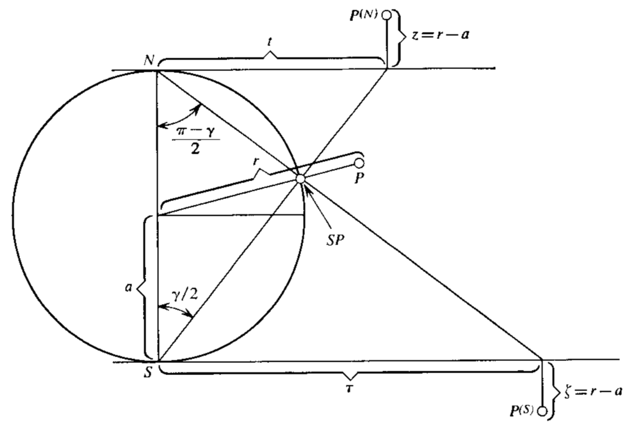

Figure 2 Relation of radial map distances (t and τ) and coordinates (z and ζ) of Cartesian map systems N and S to coordinates r and γ of spherical system E for typical atmosphere point P.

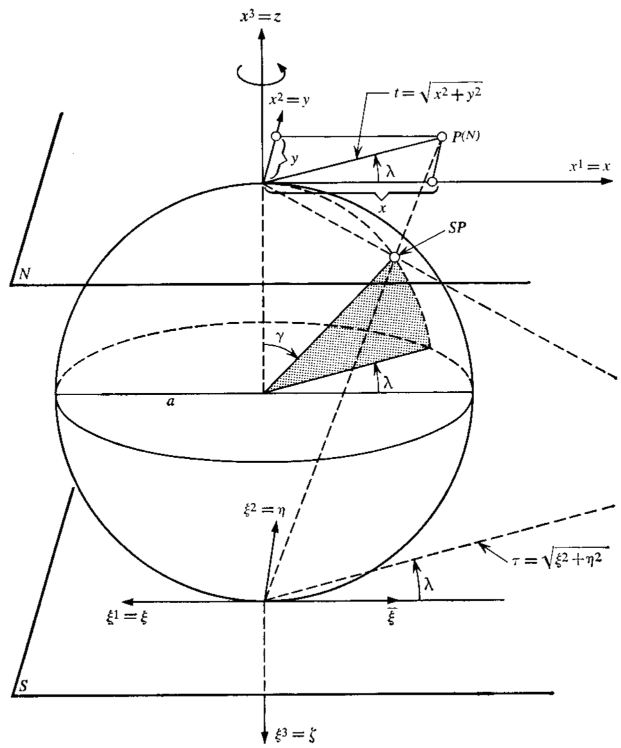

Figure 3 Relation between coordinates x, y, η and ξ of Cartesian map systems N and S and of coordinates γ and λ of spherical system E for typical earth surface point SP.

$$\begin{aligned} m &= \frac{2}{1+\cos\gamma} = 1 + \left(\frac{t}{2a}\right)^2, \\ \mu &= \frac{2}{1-\cos\gamma} = 1 + \left(\frac{\tau}{2a}\right)^2 \end{aligned} \tag{16}$$

$$\begin{aligned} dt &= ma\, d\gamma, \\ d\tau &= -\mu a\, d\gamma, \end{aligned} \tag{17}$$

$$
\begin{aligned}
x &= x^1 = t \cos\lambda = ma \sin\gamma \cos\lambda, \\
\xi &= \xi^1 = -\bar{\xi} = -\tau \cos\lambda = -\mu a \sin\gamma \cos\lambda \\
y &= x^2 = t \sin\lambda = ma \sin\gamma \sin\lambda, \\
\eta &= \xi^2 = \tau \sin\lambda = \mu a \sin\gamma \sin\lambda \\
z &= x^3 = r - a, \\
\zeta &= \xi^3 = r - a.
\end{aligned} \tag{18}
$$

Taking advantage of (17) one easily obtains:

$$
\frac{\partial x^k}{\partial y^i} = ma \begin{bmatrix} 0 & \cos\lambda & -\sin\gamma \sin\lambda \\ 0 & \sin\lambda & \sin\gamma\cos\lambda \\ 1/ma & 0 & 0 \end{bmatrix},
\qquad
\frac{\partial \xi^k}{\partial y^i} = \mu a \begin{bmatrix} 0 & \cos\lambda & \sin\gamma \sin\lambda \\ 0 & -\sin\lambda & \sin\gamma\cos\lambda \\ 1/\mu a & 0 & 0 \end{bmatrix} \tag{19}
$$

We now combine (13) and (19) to give the composite transformations $\Sigma \to N, S$.

$$
\left.\begin{aligned}
dy^i &= \frac{\partial y^i}{\partial z^s}\, dz^s \\
dx^k &= \frac{\partial x^k}{\partial y^i}\, dy^i \\
d\xi^k &= \frac{\partial \xi^k}{\partial y^i}\, dy^i
\end{aligned}\right\}
\quad
\begin{aligned}
dx^k &= \frac{\partial x^k}{\partial z^s}\, dz^s = \frac{\partial x^k}{\partial y^i}\frac{\partial y^i}{\partial z^s}\, dz^s \\
d\xi^k &= \frac{\partial \xi^k}{\partial z^s}\, dz^s = \frac{\partial \xi^k}{\partial y^i}\frac{\partial y^i}{\partial z^s}\, dz^s
\end{aligned} \tag{20}
$$

$$
\to \frac{\partial x^k}{\partial z^s} = m\cdot\frac{a}{r} \begin{bmatrix} -\sin\lambda & -\cos\lambda & 0 \\ \cos\lambda & -\sin\lambda & 0 \\ 0 & 0 & r/ma \end{bmatrix},
\qquad
\frac{\partial \xi^k}{\partial z^s} = \mu\,\frac{a}{r} \begin{bmatrix} \sin\lambda & -\cos\lambda & 0 \\ \cos\lambda & \sin\lambda & 0 \\ 0 & 0 & r/\mu a \end{bmatrix} \tag{21}
$$

The fundamental tensors and the line element in N and S are easily computed

$$
\left(\text{e.g.,}\quad g^{ik}_{(N)} = \frac{\partial x^i}{\partial z^s}\frac{\partial x^k}{\partial z^m}\, g^{sm}_{(\Sigma)}\right):
$$

$$
g^{ik}_{(N)} = a^2/r^2 \begin{bmatrix} m^2 & 0 & 0 \\ 0 & m^2 & 0 \\ 0 & 0 & r^2/a^2 \end{bmatrix},
\qquad
g^{ik}_{(S)} = a^2/r^2 \begin{bmatrix} \mu^2 & 0 & 0 \\ 0 & \mu^2 & 0 \\ 0 & 0 & r^2/a^2 \end{bmatrix}, \tag{22}
$$

$$
g^{(N)}_{ik} = r^2/a^2 \begin{bmatrix} m^{-2} & 0 & 0 \\ 0 & m^{-2} & 0 \\ 0 & 0 & a^2/r^2 \end{bmatrix},
\qquad
g^{(S)}_{ik} = r^2/a^2 \begin{bmatrix} \mu^{-2} & 0 & 0 \\ 0 & \mu^{-2} & 0 \\ 0 & 0 & a^2/r^2 \end{bmatrix}, \tag{23}
$$

$$
\begin{aligned}
g_{(N)} &= m^{-4} r^4 a^{-4}, \\
g_{(S)} &= \mu^{-4} r^4 a^{-4},
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
(ds)^2_{(N)} &= \frac{r^2}{a^2 m^2}\,[(dx)^2 + (dy)^2] + (dz)^2, \\
(ds)^2_{(S)} &= \frac{r^2}{a^2 \mu^2}\,[(d\xi)^2 + (d\eta)^2] + (d\zeta)^2
\end{aligned} \tag{25}
$$

$$
e^{(N)}_{ism} = m^{-2} r^2 a^{-2}, \qquad e^{(S)}_{ism} = \mu^{-2} r^2 a^{-2}
$$

for (i, s, m) an even permutation of (1, 2, 3)

$$
= -m^{-2} r^2 a^{-2}, \qquad = -\mu^{-2} r^2 a^{-2}
$$

for (i, s, m) an odd permutation of (1, 2, 3)

$$
= 0, \qquad = 0 \quad \text{otherwise.}
$$

Utilizing the scheme (10) we arrive at the following non-vanishing Christoffel symbols (concentrating for the time being on N only):

$$
\begin{aligned}
&\begin{Bmatrix} 1 \\ 2\ 2 \end{Bmatrix} = -\begin{Bmatrix} 1 \\ 1\ 1 \end{Bmatrix} = -\begin{Bmatrix} 2 \\ 2\ 1 \end{Bmatrix} = -\begin{Bmatrix} 2 \\ 1\ 2 \end{Bmatrix} \\
&\qquad = \frac{1}{m}\frac{\partial m}{\partial x} = \frac{x}{2ma^2} \\
&\begin{Bmatrix} 2 \\ 1\ 1 \end{Bmatrix} = -\begin{Bmatrix} 2 \\ 2\ 2 \end{Bmatrix} = -\begin{Bmatrix} 1 \\ 1\ 2 \end{Bmatrix} = -\begin{Bmatrix} 1 \\ 2\ 1 \end{Bmatrix} \\
&\qquad = \frac{1}{m}\frac{\partial m}{\partial y} = \frac{y}{2ma^2} \\
&\begin{Bmatrix} 1 \\ 3\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1\ 3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 3\ 2 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 2\ 3 \end{Bmatrix} = \frac{1}{r}, \\
&\begin{Bmatrix} 3 \\ 1\ 1 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 2\ 2 \end{Bmatrix} = -\frac{1}{m^2}\frac{r}{a^2}.
\end{aligned} \tag{26}
$$

We are now in a position to assign explicit expressions to the various members of $T^{(N)}_i$. We consider first the term

$$
S_i = \frac{\partial u_i}{\partial t} + u^\alpha \cdot u_{i,\alpha},
$$

$$
u^\alpha = g^{\alpha i} u_i = \left\{\left(\frac{ma}{r}\right)^2 u_1,\ \left(\frac{ma}{r}\right)^2 u_2,\ u_3\right\}.
$$

The covariant derivatives of u_i are obtained from (9) and (26):

$$u_{i,j} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x} + \dfrac{u_1}{m}\dfrac{\partial m}{\partial x} - \dfrac{u_2}{m}\dfrac{\partial m}{\partial y} + \dfrac{u_3}{m^2}\dfrac{r}{a^2} & \dfrac{\partial u_1}{\partial y} + \dfrac{u_1}{m}\dfrac{\partial m}{\partial y} + \dfrac{u_2}{m}\dfrac{\partial m}{\partial x} & \dfrac{\partial u_1}{\partial z} - \dfrac{u_1}{r} \\ \dfrac{\partial u_2}{\partial x} + \dfrac{u_1}{m}\dfrac{\partial m}{\partial y} + \dfrac{u_2}{m}\dfrac{\partial m}{\partial x} & \dfrac{\partial u_2}{\partial y} - \dfrac{u_1}{m}\dfrac{\partial m}{\partial x} + \dfrac{u_2}{m}\dfrac{\partial m}{\partial y} + \dfrac{u_3}{m^2}\dfrac{r}{a^2} & \dfrac{\partial u_2}{\partial z} - \dfrac{u_2}{r} \\ \dfrac{\partial u_3}{\partial x} - \dfrac{u_1}{r} & \dfrac{\partial u_3}{\partial y} - \dfrac{u_2}{r} & \dfrac{\partial u_3}{\partial z} \end{bmatrix}. \tag{27}$$

Denoting by a dot the "chain rule" operator $\partial/\partial t + u^\alpha \cdot \partial/\partial x^\alpha$, we have for S_i:

$$S_i = \dot{u}_i + \frac{m}{2r^2}(u_1^2 + u_2^2)\, x^i \qquad i = 1, 2 \tag{28}$$

$$S_3 = \dot{u}_3 - \frac{m^2 a^2}{r^3}(u_1^2 + u_2^2).$$

The angular velocity and Coriolis term ($C_i = 2\, e_{ikm} \cdot \Omega^k u^m$) components become:

$$\Omega^k_{(N)} = \Omega^l_{(\Sigma)} \frac{\partial x^k}{\partial z^l}$$

$$= m\Omega \frac{a}{r}\left\{-\sin\gamma\cos\lambda,\ -\sin\gamma\sin\lambda,\ \frac{r}{ma}\cos\gamma\right\}, \tag{29}$$

$$C_i^{(N)} = 2\Omega \frac{r}{a}\left\{\frac{-1}{m}\sin\gamma\ \sin\lambda\ u_3 - \frac{a}{r}\cos\gamma\ u_2,\right.$$

$$\frac{a}{r}\cos\gamma\ u_1 + \frac{1}{m}\sin\gamma\ \cos\lambda u_3,$$

$$\left.\left(\frac{a}{r}\right)^2 m\sin\gamma[-\cos\lambda\ u_2 + \sin\lambda\ u_1]\right\}. \tag{30}$$

It is evident that the pressure and gravitational terms in (8) remain of the same form in N as in Σ. Postponing a detailed discussion of the "friction" term f_i, we can now write the equations of motion in the form:

$$T_1^{(N)} = \dot{u}_1 + \frac{m}{2r^2}(u_1^2 + u_2^2)\, x - \frac{2\Omega r}{a}\left(\underline{\frac{u_3}{m}\sin\gamma\sin\lambda}\right.$$

$$\left. + \frac{u_2}{r}\, a\cos\gamma\right) + \frac{1}{\rho}\left(\frac{\partial p}{\partial x} + f_1^{(N)}\right) = 0$$

$$T_2^{(N)} = \dot{u}_2 + \frac{m}{2r^2}(u_1^2 + u_2^2) y + \frac{2\Omega r}{a}\left(\frac{u_1}{r}\, a\cos\gamma\right.$$

$$\left. + \underline{\frac{u_3}{m}\sin\gamma\cos\lambda}\right) + \frac{1}{\rho}\left(\frac{\partial p}{\partial y} + f_2^{(N)}\right) = 0$$

$$T_3^{(N)} = \underline{\dot{u}_3}$$

$$\underline{-\frac{m^2 a^2}{r^3}(u_1^2 + u_2^2) + \frac{2\Omega a}{r}\, m\sin\gamma(u_1\sin\lambda - u_2\cos\lambda)}$$

$$+ g + \frac{1}{\rho}\left(\frac{\partial p}{\partial z}\right) + \underline{\frac{1}{\rho} f_3^{(N)}} = 0. \tag{31}$$

In dynamic meteorology it is customary to make (in the coordinate systems Σ and E) the "quasistatic" approximations (see Ref. 7, A. Eliassen and E. Kleinschmidt, p. 20), which amount to dropping the underlined terms above. In the sequel we shall adhere to this convention and shall also set $r = a$, treating the atmosphere as an essentially infinitesimally thin layer. We also treat terms of order a^{-1} as negligibly small in expressions containing terms of order 1.

Retracing the above arguments for the S-system, we find the only difference to be one in the sign of $\sin\lambda$ in the Coriolis terms.

3. *Pressure and relative pressure as independent variables*

It was established by Eliassen[5] that, within the framework of the quasistatic approximations, the equations of motion take an especially simple form (in the Σ coordinate system) if p is introduced as independent variable in place of z_3. In particular the equation of continuity becomes ($\Sigma(p)$ = Σ system with p as third coordinate):

$$\left(\frac{\partial \omega^1}{\partial z_1}\right)_p + \left(\frac{\partial \omega^2}{\partial z_2}\right)_p + \frac{\partial \omega}{\partial p} = 0, \quad \omega \equiv \dot{p} \quad [\text{in } \Sigma(p)]. \tag{32}$$

The equation in the map system can be transformed partially in the same manner:

$$\frac{\partial \rho}{\partial t} + (\rho u^i)_{,i} = \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u^i)}{\partial x^i} + \rho u^s\left\{\begin{matrix} k \\ s\ k \end{matrix}\right\}$$

$$= 0 \qquad [\text{in } N(z)], \tag{33}$$

$$\rightarrow \left[\left(\frac{\partial u^1}{\partial x}\right)_p + \left(\frac{\partial u^2}{\partial y}\right)_p + \frac{\partial \omega}{\partial p}\right]$$

$$- \frac{2}{m}\left[u^1\left(\frac{\partial m}{\partial x}\right)_z + u^2\left(\frac{\partial m}{\partial y}\right)_z\right] = 0. \tag{34}$$

Since m does not depend on z nor on p, $(\partial m/\partial x)_z$ can be replaced by $(\partial m/\partial x)_p$. One then obtains immediately:

$$m^2\left[\left(\frac{\partial u_1}{\partial x}\right)_p + \left(\frac{\partial u_2}{\partial y}\right)_p\right] + \frac{\partial \omega}{\partial p} = 0. \tag{35}$$

As a consequence one can transform the chain rule operator d/dt as follows (denoting by α any function):

$$\frac{d\alpha}{dt} = \left(\frac{\partial\alpha}{\partial t}\right)_p + u^1\left(\frac{\partial\alpha}{\partial x}\right)_p + u^2\left(\frac{\partial\alpha}{\partial y}\right)_p + \omega\frac{\partial\alpha}{\partial p}$$
$$= \left(\frac{\partial\alpha}{\partial t}\right)_p + m^2\left[\frac{\partial(u_1\ \alpha)}{\partial x}\bigg|_p + \frac{\partial(u_2\ \alpha)}{\partial y}\bigg|_p\right] + \frac{\partial(\omega\alpha)}{\partial p}. \tag{36}$$

From the hydrostatic relation ($\partial p/\partial z = -\rho g$) one may infer that $(\partial p/\partial x)_z$ can be replaced by $\rho g(\partial z/\partial x)_p$, so that the equations of motion (excluding f_i) for the $N(p)$ system become:

$$\frac{\partial u_1}{\partial t} = -m^2\left[\frac{\partial(u_1)^2}{\partial x} + \frac{\partial(u_1u_2)}{\partial y}\right] - \frac{\partial(\omega u_1)}{\partial p} - \frac{m}{2a^2}(u_1^2 + u_2^2)\,x + 2\Omega\cos\gamma\, u_2 - g\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial u_2}{\partial t} = -m^2\left[\frac{\partial(u_1u_2)}{\partial x} + \frac{\partial(u_2)^2}{\partial y}\right] - \frac{\partial(\omega u_2)}{\partial p} - \frac{m}{2a^2}(u_1^2 + u_2^2)\,y - 2\Omega\cos\gamma\, u_1 - g\frac{\partial z}{\partial y} = 0. \tag{37}$$

$$\frac{\partial z}{\partial p} = -\frac{1}{g\rho} = -\frac{RT}{gp} \qquad \text{(using the ideal gas law).}$$

Finally we introduce (see Ref. 15, N. A. Phillips) the ratio of p to p_* (surface pressure at point $(x, y, z = 0)$) as a new variable:

$$Q \equiv p/p_*, \qquad \omega \equiv \dot{p} = p_*\dot{Q} + Q\dot{p}_* \equiv p_*\bar{\omega} + Q\dot{p}_*. \tag{38}$$

Considering the total differential of Q with $dp = 0$, one has:

$$\left(\frac{\partial Q}{\partial x^i}\right)_p = -\frac{Q}{p_*}\frac{\partial p_*}{\partial x^i} \qquad (i = 1, 2) \tag{39}$$

$$\frac{\partial\alpha}{\partial p} = \frac{1}{p_*}\frac{\partial\alpha}{\partial Q}, \qquad \left(\frac{\partial\alpha}{\partial t}\right)_p = \left(\frac{\partial\alpha}{\partial t}\right)_Q - \frac{Q}{p_*}\left(\frac{\partial p_*}{\partial t}\right)_Q\frac{\partial\alpha}{\partial Q}$$

$$\left(\frac{\partial\alpha}{\partial x^i}\right)_p = \left(\frac{\partial\alpha}{\partial x^i}\right)_Q - \frac{Q}{p_*}\left(\frac{\partial p_*}{\partial x^i}\right)_Q\frac{\partial\alpha}{\partial Q} \qquad (i = 1, 2).$$

After some algebra, using the chain rule

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t}\right)_Q + m^2\left[u_1\left(\frac{\partial}{\partial x}\right)_Q + u_2\left(\frac{\partial}{\partial y}\right)_Q\right] + \bar{\omega}\frac{\partial}{\partial Q} \tag{40}$$

one obtains in analogy to (33), (34), (35) the relations:

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u^i)}{\partial x^i} = \rho\left[\left(\frac{\partial u^1}{\partial x}\right)_Q + \left(\frac{\partial u^2}{\partial y}\right)_Q + \frac{\partial\bar{\omega}}{\partial p} + \frac{\dot{p}_*}{p_*}\right]$$
$$= \rho\left\{\left(\frac{\partial u^1}{\partial x}\right)_Q + \left(\frac{\partial u^2}{\partial y}\right)_Q + \frac{\partial\bar{\omega}}{\partial Q} + \frac{1}{p_*}\left[\left(\frac{\partial p_*}{\partial t}\right)_Q + u^1\left(\frac{\partial p_*}{\partial x}\right)_Q + u^2\left(\frac{\partial p_*}{\partial y}\right)_Q\right]\right\} \tag{41}$$

$$p_*\frac{\partial\bar{\omega}}{\partial Q} + \left(\frac{\partial p_*}{\partial t}\right)_Q + m^2\left[\frac{\partial(p_*u_1)}{\partial x}\bigg|_Q + \frac{\partial(p_*u_2)}{\partial y}\bigg|_Q\right] = 0. \tag{42}$$

In the computer program the equation of continuity (42) is used in a two-fold manner, to step p_* in time and to compute $\bar{\omega} = \dot{Q}$ (from which both ω and u_3 can be obtained) at the various Q-levels. To this end one integrates (42) with respect to Q, using the boundary conditions $\bar{\omega} = 0$ at $Q = 0$, $Q = 1$:

$$\bar{\omega}(Q) = -\frac{Q}{p_*}\left(\frac{\partial p_*}{\partial t}\right) - \frac{m^2}{p_*}\int_0^Q\left[\frac{\partial}{\partial x}(p_*u_1) + \frac{\partial}{\partial y}(p_*u_2)\right]dQ \tag{43}$$

$$\frac{\partial p_*}{\partial t} = -m^2\int_0^1\left[\frac{\partial}{\partial x}(p_*u_1) + \frac{\partial}{\partial y}(p_*u_2)\right]dQ \tag{44}$$

(partial derivatives here and in the sequel in the $N(Q)$ system).

Combining (42) with (40) one has (in analogy to (36)):

$$\frac{d\alpha}{dt} = \frac{1}{p_*}\frac{\partial(\alpha p_*)}{\partial t} + \frac{m^2}{p_*}\left[\frac{\partial(p_*u_1\alpha)}{\partial x} + \frac{\partial(p_*u_2\alpha)}{\partial y}\right] + \frac{\partial(\bar{\omega}\alpha)}{\partial Q}. \tag{45}$$

Using the "geopotential" $\Phi = gz$ instead of z, we may replace the horizontal derivatives of z as follows [see (39)]:

$$g\left(\frac{\partial z}{\partial x^i}\right)_p = \left(\frac{\partial\Phi}{\partial x^i}\right)_p = \left(\frac{\partial\Phi}{\partial x^i}\right)_Q - \frac{Q}{p_*}\left(\frac{\partial p_*}{\partial x^i}\right)_Q\frac{\partial\Phi}{\partial Q}$$
$$= \left(\frac{\partial\Phi}{\partial x^i}\right)_Q + \frac{RT}{p_*}\left(\frac{\partial p_*}{\partial x^i}\right)_Q. \tag{46}$$

The equations of motion can now be taken from (31), (45) and (46):

$$\frac{\partial(p_*u_1)}{\partial t} = -m^2\left[\frac{\partial(p_*u_1^2)}{\partial x} + \frac{\partial(p_*u_1u_2)}{\partial y}\right] - p_*\frac{\partial\bar{\omega}u_1}{\partial Q} + 2\Omega\,p_*u_2\cos\gamma - \frac{m^2p_*}{2a^2}(u_1^2 + u_2^2)\,x - p_*\frac{\partial\Phi}{\partial x} - RT\frac{\partial p_*}{\partial x} = 0, \tag{47.a}$$

$$\frac{\partial(p_* u_2)}{\partial t} = -m^2\left[\frac{\partial(p_* u_1 u_2)}{\partial x} + \frac{\partial(p_* u_2^2)}{\partial y}\right]$$

$$- p_* \frac{\partial \bar{\omega} u_2}{\partial Q} - 2\Omega\ p_* u_1 \ \cos\gamma - \frac{m^2 p_*}{2a^2}(u_1^2 + u_2^2)\ y$$

$$- p_* \frac{\partial \Phi}{\partial y} - RT\frac{\partial p_*}{\partial y} = 0 \tag{47.b}$$

$$\frac{\partial \Phi}{\partial Q} = -\frac{RT}{Q}. \tag{47.c}$$

It is quite clear that the same equations hold for S, (u_1, u_2, x, y, m^2) being replaced by $(v_1, v_2, \xi, \eta, \mu^2)$.

The covariant velocity components (u_1, u_2, u_3) are not to be confused with actual physical velocity components. To obtain the physical velocity components on earth, we have to use the appropriate tensor transformations:

$$w^i = w_i = \frac{\partial x^k}{\partial z^i} u_k = m\left\{-\sin\lambda\ u_1 + \cos\lambda\ u_2,\right.$$

$$\left. -\cos\lambda\ u_1 - \sin\lambda\ u_2, \frac{u_3}{m}\right\}. \tag{48}$$

In the computer program it is also necessary, for interpolation purposes at the equator, to relate the coordinates and the velocity components of N and S. Utilizing relations (15)–(17) we established:

$$\begin{aligned}\left(\frac{x}{\xi}, \frac{y}{\eta}\right) &= \left(-\frac{t\tau}{\tau^2}, \frac{t\tau}{\tau^2}\right) = \frac{4a^2}{\xi^2 + \eta^2}(-1, 1),\\ \left(\frac{\xi}{x}, \frac{\eta}{y}\right) &= \left(-\frac{\tau t}{t^2}, \frac{\tau t}{t^2}\right) = \frac{4a^2}{x^2 + y^2}(-1, 1)\end{aligned} \tag{49}$$

$$\frac{\partial x^k}{\partial \xi^i} = \frac{4a^2}{(\xi^2 + \eta^2)^2}\begin{bmatrix} \xi^2 - \eta^2 & 2\xi\eta & 0 \\ -2\xi\eta & \xi^2 - \eta^2 & 0 \\ 0 & 0 & \dfrac{(\xi^2 + \eta^2)^2}{4a^2}\end{bmatrix} \tag{50}$$

$$= \frac{1}{4a^2}\begin{bmatrix} x^2 - y^2 & 2xy & 0 \\ -2xy & x^2 - y^2 & 0 \\ 0 & 0 & 4a^2\end{bmatrix}$$

$$\frac{\partial \xi^l}{\partial x^m} = \frac{4a^2}{(x^2 + y^2)^2}\begin{bmatrix} x^2 - y^2 & 2xy & 0 \\ -2xy & x^2 - y^2 & 0 \\ 0 & 0 & \dfrac{(x^2 + y^2)^2}{4a^2}\end{bmatrix} \tag{51}$$

$$= \frac{1}{4a^2}\begin{bmatrix} \xi^2 - \eta^2 & 2\xi\eta & 0 \\ -2\xi\eta & \xi^2 - \eta^2 & 0 \\ 0 & 0 & 4a^2\end{bmatrix}.$$

The velocity transformations

$$u_m = \frac{\partial \xi^l}{\partial x^m} v_l, \qquad v_j = \frac{\partial x^k}{\partial \xi^j} u_k$$

can now be read off without difficulty.

- *4. The energy and humidity equations in the map systems*

In (4) and (5) the terms $\dot{\theta}$ and $\dot{r}$ represent qualities of a particular parcel of air. In terms of the fields $\theta(x, y, z, t)$ and $r(x, y, z, t)$ they are total time derivatives. On the other hand the quantities q and W should be regarded as functions of (x, y, z, t) to be specified directly by some auxiliary formulations. In the computer formulation we have to link the local variations $\partial\theta/\partial t$, $\partial r/\partial t$, $\partial u/\partial t$, etc. to these "outside" influences (e.g., radiation, precipitation, changes of state, diffusion).

Using results from Refs. 10 and 11 we can write the energy equation in the form

$$\frac{1}{\theta}\dot{\theta} = \delta\frac{\Gamma\omega}{p} + \frac{1}{c_p T}(q_R + q_D). \tag{52}$$

The three right-hand terms represent, in succession, the effects of "pseudoadiabatic" thermodynamic processes, radiation processes and diffusion processes. It suffices to regard the terms as being computed at a point (x, y, z, t) from the variables stored in the machine at that instant. To obtain the local variation of (θp_*) with time we apply the operation implicit in (45):

$$\frac{1}{p_*}\frac{\partial(\theta p_*)}{\partial t} + \frac{m^2}{p_*}\left[\frac{\partial(p_* u_1\theta)}{\partial x} + \frac{\partial(p_* u_2\theta)}{\partial y}\right] + \frac{\partial(\bar{\omega}\theta)}{\partial Q}$$

$$= \theta\left[\delta\frac{\Gamma\omega}{p_* Q} + \frac{1}{c_p T}(q_R + q_D)\right]. \tag{53}$$

Introducing absolute temperature T by $\theta = T(p_{st}/p)^\kappa$, substituting from (38) and the chain rule, and cancelling a factor $p_{st}^\kappa(Qp_*)^{-\kappa}$, we arrive at

$$\frac{\partial p_* T}{\partial t} = -m^2\left[\frac{\partial(p_* T u_1)}{\partial x} + \frac{\partial(p_* T u_2)}{\partial y}\right]$$

$$- p_*\frac{\partial(\bar{\omega}T)}{\partial Q} + (\kappa + \delta\Gamma)\frac{T\omega}{Q} + \frac{p_*}{c_p}(q_R + q_D) \tag{54}$$

Turning to the humidity equation (5), we break W into two parts, one (W_E) standing for the change of water vapor contents of air at a point due to external agents, the other (W_I) representing the change due to internal processes. We also introduce as new variable the relative humidity h (the ratio of mixing ratio to saturation mixing ratio r_s).

$$\dot{r} = W_E + W_I, \qquad r = hr_s \tag{55}$$

$$\dot{h} = \frac{1}{r_s}(W_I - h\cdot\dot{r}_s) + \frac{1}{r_s}W_E \tag{56}$$

According to Ref. 11 the equation can be rewritten in the form:

$$\dot{h} = \frac{\omega}{p}\,[(1 - h)\ \delta\cdot\gamma_m - (1 - \delta)h\cdot\gamma_d] + \frac{1}{r_s}\,W_E, \quad (57)$$

where the first term accounts for pseudo-adiabatic processes (in a fraction δ of the air around a grid point), the second for adiabatic processes. The parameters δ, γ_m, γ_d are suitable functions of T, h and the coefficients characteristic for a realistic air-vapor mixture.

Proceeding in the same fashion as above, we finally obtain for the local variation of $p_* h$:

$$\frac{\partial p_* h}{\partial t} = -m^2\left[\frac{\partial(p_* h u_1)}{\partial x} + \frac{\partial(p_* h u_2)}{\partial y}\right] - p_*\,\frac{\partial(\bar{\omega} h)}{\partial Q}$$

$$+ \frac{\omega}{Q}\,[(1 - h)\ \delta\cdot\gamma_m - (1 - \delta)h\cdot\gamma_d] + \frac{p_* W_E}{r_s}. \quad (58)$$

In this paper we shall omit discussions of the terms q_R and W_E. The computation of the first term is executed in a highly complicated "radiation program" (for a partial formulation see Ref. 16). In the actual model Eq. (58) was replaced by a simpler formulation, keeping the mixing ratio constant, in the upper layers of the atmosphere.

A summary of the computational procedure

In the initial formulation the independent variables are x, y, z, t and the dependent variables are $w_1, w_2, w_3, \rho, p, T, r$. In the final formulation p and z have interchanged places and new coordinates as well as new auxiliary variables have made their appearance. The basic prognostic equations are (47a), (47b), (44), (54) and (58). For most meteorological purposes the quantities $p_* u_1$, $p_* u_2$, $p_* T$ and $p_* h$, apparently with little difficulty (i.e., after appropriate scaling) can be interpreted directly as horizontal velocity components, absolute temperature and relative humidity. The total time derivatives of the vertical coordinates p and Q, namely $\omega = \dot{p}$ and $\bar{\omega} = \dot{Q}$, take the place of vertical velocity. It may be helpful to give a brief schematic description of a possible sequence of computational steps leading to the evaluation of the basic quantities: velocity components (w_1, w_2, w_3), geopotential (Φ), vertical elevation (z), density (ρ), absolute temperature (T), relative humidity (h), mixing ratio (r), ground pressure (p_*) and pressure (p). In the following, "$\rightarrow$" means "yields":

$$\left.\begin{array}{l}(47) \rightarrow p_* u_1,\ p_* u_2,\ \Phi \\ (43),\ (44) \rightarrow \bar{\omega},\ p_*\end{array}\right\} \rightarrow u_1,\ u_2$$

$$(38) \rightarrow \omega, p; \quad (54) \rightarrow T; \quad (58) \rightarrow h; \quad (55) \rightarrow r; \quad \Phi \rightarrow z$$

$$(3) \rightarrow \rho; \quad w^i = \frac{\partial x^k}{\partial z^i}\,u_k \rightarrow w^1 = w_1, \quad w^2 = w_2;$$

$$(2) \rightarrow w_3 = u_3.$$

• 5. *Formulation of momentum diffusion*

The nature of the diffusion terms f_i is only incompletely understood at the present time. In the following we borrow, with Ref. 12, a formalism from elasticity theory. In the process of developing the formalism into a physical description we are guided by results from one-dimensional flows (Prandtl hypothesis). Yet we must still regard the final formulation as tentative, subject to modifications, hopefully only of parameters, in accordance with the results obtained by the model.

We consider the contravariant vector $F^i \equiv R^{i\alpha}_{\ ,\alpha}$ of Eq. (8). It is well known (see, e.g., Ref. 17, p. 67), that the stress tensor $R^{i\alpha}$ is symmetric:

$$R^{i\alpha} = R^{\alpha i}. \quad (59)$$

Our objective is to express the "force components" F^i in terms of known properties of the flow, e.g., velocities and velocity gradients. To this end we investigate what special form the general stress-strain relation

$$R^{ik} = c^{ik}_{rs} e^{rs} \quad (60)$$

takes when we assume certain reasonable symmetries in our model. The strains e^{rs} are, in general, defined by:

$$\begin{aligned} e^{rs} &= g^{rm} g^{sn} e_{mn} = e^{sr} \\ e_{mn} &= \tfrac{1}{2}(u_{m,n} + u_{n,m}). \end{aligned} \quad (61)$$

The tensor c^{ik}_{rs} has 81 components. However, by virtue of the symmetry properties (see Ref. 17, p. 156):

$$c^{ik}_{rs} = c^{ik}_{sr} = c^{ki}_{sr} = c^{ki}_{rs} = c^{rs}_{ik}, \quad (62)$$

which hold in a Cartesian coordinate system (say Σ), the number of independent "elastic coefficients" reduces to 21 and the stress-strain relations can be represented by the matrix equation (entries represented by points are to be filled in by symmetry):

$$R_{(\Sigma)} = \begin{bmatrix} c^{11}_{11} & c^{11}_{22} & c^{11}_{33} & c^{11}_{23} & c^{11}_{31} & c^{11}_{12} \\ \cdot & c^{22}_{22} & c^{22}_{33} & c^{22}_{23} & c^{22}_{31} & c^{22}_{12} \\ \cdot & \cdot & c^{33}_{33} & c^{33}_{23} & c^{33}_{31} & c^{33}_{12} \\ \cdot & \cdot & \cdot & c^{23}_{23} & c^{23}_{31} & c^{23}_{12} \\ \cdot & \cdot & \cdot & \cdot & c^{31}_{31} & c^{31}_{12} \\ \cdot & \cdot & \cdot & \cdot & \cdot & c^{12}_{12} \end{bmatrix} E_{(\Sigma)} \quad (63)$$

$$R = \begin{bmatrix} R^{11} \\ R^{22} \\ R^{33} \\ R^{23} \\ R^{31} \\ R^{12} \end{bmatrix}, \qquad E = \begin{bmatrix} e^{11} \\ e^{22} \\ e^{33} \\ e^{23} \\ e^{31} \\ e^{12} \end{bmatrix}. \quad (64)$$

We now assume, in the Σ system, "elastic" symmetry with respect to two perpendicular planes, say $z_3 = 0$ and $z_1 = 0$. It is shown in Ref. 17, p. 159, that this implies symmetry with respect to $z_2 = 0$ as well, and causes a reduction of the number of independent nonzero elastic coefficients to 9 (orthotropic symmetry). Finally we impose the additional requirement of symmetry with respect to rotations about the $z_3 = z$ axis (hexagonal symmetry), which leaves us with only 5 independent nonvanishing elastic coefficients (say a, b, c, d, e):

$$R_{(\Sigma)} = \begin{bmatrix} a & b & c & 0 & 0 & 0 \\ b & a & c & 0 & 0 & 0 \\ c & c & d & 0 & 0 & 0 \\ 0 & 0 & 0 & e & 0 & 0 \\ 0 & 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 & 0 & \tfrac{1}{2}(a-b) \end{bmatrix} E_{(\Sigma)}. \tag{65}$$

To obtain a further simplification, we now refer to the physical meaning of $R^{ii} = \tau_{ii}$ (see Ref. 14) and demand that there exists a partial "equipartition" of kinetic energy of the turbulent disturbances u_i'. That is, we assume that the energy associated with perturbations of the vertical wind velocity is on the average equal to the arithmetic mean of the energies associated with the two horizontal components:

$$\overline{(w_3')^2} = \tfrac{1}{2}[\overline{(w_1')^2} + \overline{(w_2')^2}] \rightarrow R^{33}_{(\Sigma)} = \tfrac{1}{2}(R^{11}_{(\Sigma)} + R^{22}_{(\Sigma)}) \tag{66}$$

We can consider the strains as arbitrary, so that (65) implies:

$$a + b = 2c, \qquad c = d. \tag{67}$$

In addition we choose, as is frequently done, to rearrange the sum of the two terms $\partial p/\partial x^i$ and $- g_{ij}R^{j\alpha}_{,\alpha}$ of (8) so as to modify both p (to $\tilde{p}$) and $R^{i\alpha}$ (to $\tilde{R}^{i\alpha}$), keeping the combined effect unchanged.

$$g_{ij}R^{j\alpha}_{,\alpha} - \partial p/\partial x^i \equiv g_{ij}\tilde{R}^{j\alpha}_{,\alpha} - \partial\tilde{p}/\partial x^i. \tag{68}$$

The modified stress tensor is chosen so that its relation with a correspondingly modified strain tensor is especially simple:

$$\begin{aligned} \tilde{R}^{ik} &\equiv R^{ik} - S\cdot g^{ik} \equiv \tilde{c}^{ik}_{rs}\cdot e^{rs} \\ S &= \tfrac{1}{3}R^l_l = \tfrac{1}{3}R^{lm}g_{ml} = \tfrac{1}{3}c^{lm}_{rs}g_{ml}e^{rs}, \\ \tilde{p} &= p - S. \end{aligned} \tag{69}$$

We verify that (68) is indeed satisfied [by virtue of $(g_{ij}\cdot S\cdot g^{j\alpha})_{,\alpha} = \delta^\alpha_i \cdot \partial S/\partial x^\alpha = \partial S/\partial x^i$] and that the modified elastic coefficients are given by:

$$\begin{aligned} &\tilde{c}^{ik}_{rs} = c^{ik}_{rs} - \tfrac{1}{3}c^{lm}_{rs}g_{ml}g^{ik} \\ &\tilde{c}^{11}_{11} = a - c, \qquad \tilde{c}^{11}_{22} = b - c, \qquad \tilde{c}^{11}_{33} = 0 \\ &\tilde{c}^{23}_{23} = \tilde{c}^{31}_{31} = e, \qquad \tilde{c}^{12}_{12} = c^{12}_{12} = \tfrac{1}{2}(a-b). \end{aligned} \tag{70}$$

Utilizing (67) we can finally introduce the new coefficient χ:

$$a - c = c - b = \chi, \quad a - b = 2(c - b) = 2\chi, \tag{71}$$

in terms of which the modified coefficient matrix becomes:

$$\tilde{c}^{ik}_{rs(\Sigma)} = \begin{bmatrix} \chi & -\chi & 0 & 0 & 0 & 0 \\ -\chi & \chi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e & 0 & 0 \\ 0 & 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 & 0 & \chi \end{bmatrix}. \tag{72}$$

In the sequel we shall drop the tildes, but must remember to interpret p, R^{ik} and c^{ik}_{rs} as being the modified quantities.

We now resume our endeavor to represent $f_i^{(N)}$ in terms of the basic quantities in the N-system.

$$f_i^{(N)} = g_{ij}^{(N)}(c^{j\alpha}_{rs(N)}\cdot e^{rs}_{(N)})_{,\alpha} = g_{ij}^{(N)}\cdot f^j. \tag{73}$$

Fortunately, the structure of $c^{ik}_{rs(N)}$ is exactly the same as that of $c^{ik}_{rs(\Sigma)}$ (as in Eq. (65) or in Eq. (72)).

This can be seen by evaluating each term in

$$c^{ik}_{rs(N)} = \frac{\partial z^p}{\partial x^r}\frac{\partial z^l}{\partial x^s}\frac{\partial x^i}{\partial z^m}\frac{\partial x^k}{\partial z^n}\cdot c^{mn}_{pl(\Sigma)} \tag{74}$$

with the partial derivatives [(21) and its inverse]:

$$\begin{aligned} \frac{\partial x^i}{\partial z^m} &= m\begin{bmatrix} -\sin\lambda & -\cos\lambda & 0 \\ \cos\lambda & -\sin\lambda & 0 \\ 0 & 0 & \dfrac{r}{am} \end{bmatrix}, \\ \frac{\partial z^p}{\partial x^r} &= \frac{1}{m}\begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\cos\lambda & -\sin\lambda & 0 \\ 0 & 0 & \dfrac{am}{r} \end{bmatrix}. \end{aligned} \tag{75}$$

For instance:

$$\begin{aligned} c^{11}_{11(N)} = &\left(\frac{\partial z^1}{\partial x^1}\frac{\partial x^1}{\partial z^1}\right)^2 c^{11}_{11(\Sigma)} \\ &+ \left(\frac{\partial z^2}{\partial x^1}\frac{\partial x^1}{\partial z^1}\right)^2 c^{11}_{22(\Sigma)} + \left(\frac{\partial z^1}{\partial x^1}\frac{\partial x^1}{\partial z^2}\right)^2 c^{22}_{11} \\ &+ \left(\frac{\partial z^2}{\partial x^1}\frac{\partial x^1}{\partial z^2}\right)^2 c^{22}_{22(\Sigma)} + 4\left(\frac{\partial z^1}{\partial x^1}\frac{\partial z^2}{\partial x^1}\frac{\partial x^1}{\partial z^1}\frac{\partial x^1}{\partial z^2}\right)c^{12}_{12} \\ = &\, a(\sin^4\lambda + \cos^4\lambda) \\ &+ [2b + \tfrac{4}{2}(a-b)]\sin^2\lambda\cos^2\lambda = a. \end{aligned}$$

Hence we can take $c^{ik}_{rs(N)}$ to have the form of (72). We still require the strain tensors defined in (61). They are obtained from (27) and (22) in the following form:

$$e^{(N)}_{mn} = \begin{bmatrix} \frac{\partial u_1}{\partial x} + \frac{u_1}{m}\frac{\partial m}{\partial x} - \frac{u_2}{m}\frac{\partial m}{\partial y} & \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) + \frac{u_1}{m}\frac{\partial m}{\partial y} + \frac{u_2}{m}\frac{\partial m}{\partial x} & \frac{1}{2}\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right) \\ \cdot & \frac{\partial u_2}{\partial y} - \frac{u_1}{m}\frac{\partial m}{\partial x} + \frac{u_2}{m}\frac{\partial m}{\partial y} & \frac{1}{2}\left(\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y}\right) \\ \cdot & \cdot & \frac{\partial u_3}{\partial z} \end{bmatrix} \tag{76}$$

$$e^{rs}_{(N)} = \begin{bmatrix} m^4\frac{\partial u_1}{\partial x} + m^3 u_1\frac{\partial m}{\partial x} - m^3 u_2\frac{\partial m}{\partial y} & \frac{m^2}{2}\left(\frac{\partial u_1 m^2}{\partial y} + \frac{\partial u_2 m^2}{\partial x}\right) & \frac{m^2}{2}\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right) \\ \cdot & m^4\frac{\partial u_2}{\partial y} - m^3 u_1\frac{\partial m}{\partial x} + m^3 u_2\frac{\partial m}{\partial y} & \frac{m^2}{2}\left(\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y}\right) \\ \cdot & \cdot & \frac{\partial u_3}{\partial z} \end{bmatrix} \tag{77}$$

We now implement (60) (e.g., $R^{11} = c^{11}_{11}\ e^{11}_{11} + c^{11}_{22}\,e^{22} = -\,R^{22}$):

$$\begin{aligned} R^{11}_{(N)} &= -R^{22}_{(N)} = \chi m^2\left\{\frac{\partial m^2 u_1}{\partial x} - \frac{\partial m^2 u_2}{\partial y}\right\}, \\ R^{12}_{(N)} &= R^{21}_{(N)} = \chi m^2\left\{\frac{\partial m^2 u_1}{\partial y} + \frac{\partial m^2 u_2}{\partial x}\right\}, \\ R^{13}_{(N)} &= R^{31}_{(N)} = e\cdot m^2\left\{\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right\}, \\ R^{23}_{(N)} &= R^{32}_{(N)} = e m^2\left\{\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y}\right\}. \end{aligned} \tag{78}$$

To evaluate $f_i^{(N)}$ we apply the last equation of (9) in its contracted form or use standard expressions for the contracted Christoffel symbols:

$$\begin{aligned} R^{i\alpha}_{,\alpha} &= \frac{\partial R^{i\alpha}}{\partial x^\alpha} + \begin{Bmatrix} j \\ m\ \alpha \end{Bmatrix} R^{m\alpha} + \begin{Bmatrix} \alpha \\ m\ \alpha \end{Bmatrix} R^{im}, \\ \begin{Bmatrix} \alpha \\ l\ \alpha \end{Bmatrix} &= \frac{1}{2}\frac{\partial}{\partial x^l}\ln g = -\frac{1}{m^2}\frac{\partial m^2}{\partial x^l} \end{aligned} \tag{79}$$

After some simple algebra we obtain:

$$\begin{aligned} \frac{1}{\rho} f_1^{(N)} &= \frac{m^2}{\rho}\left\{\frac{\partial}{\partial x}\left(\frac{\chi}{m^2} D_T\right) + \frac{\partial}{\partial y}\left(\frac{\chi}{m^2} D_S\right)\right\} \\ &\quad + \frac{1}{\rho m^2}\frac{\partial}{\partial z}\left\{e m^2\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right)\right\}, \\ \frac{1}{\rho} f_2^{(N)} &= \frac{m^2}{\rho}\left\{\frac{\partial}{\partial x}\left(\frac{\chi}{m^2} D_S\right) - \frac{\partial}{\partial y}\left(\frac{\chi}{m^2} D_T\right)\right\} \\ &\quad + \frac{1}{\rho m^2}\frac{\partial}{\partial z}\left\{e m^2\left(\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y}\right)\right\}, \end{aligned} \tag{80}$$

where we have set:

$$\begin{aligned} D_T &\equiv \left(\frac{\partial m^2 u_1}{\partial x} - \frac{\partial m^2 u_2}{\partial y}\right), \\ D_S &\equiv \left(\frac{\partial m^2 u_1}{\partial y} + \frac{\partial m^2 u_2}{\partial x}\right). \end{aligned} \tag{81}$$

In order to use these relations in the computer program, we still have to relate the "elastic coefficients" χ and e to the local properties of the flow.

According to Eq. (31), the $1/\rho\, f_i^{(N)}$ are the contributions (except for sign) of "diffusion" to du_i/dt. We may then compare (80) to the general diffusion equation in a turbulent wind:

$$\frac{d\eta}{dt} = \frac{\partial}{\partial x}\left(M_x\frac{\partial\eta}{\partial x}\right) + \frac{\partial}{\partial y}\left(M_y\frac{\partial\eta}{\partial y}\right) + \frac{\partial}{\partial z}\left(M_z\frac{\partial\eta}{\partial z}\right), \tag{82}$$

M_x, M_y, M_z being exchange coefficients.

While equations (80) are more complicated than (82), they have a strong structural resemblance to a simpler system of form:

$$\begin{aligned} \frac{dU_i}{dt} &= \frac{m^2}{\rho}\left\{\frac{\partial}{\partial x}\left(\chi\frac{\partial U_1}{\partial x}\right)\right. \\ &\quad \left. + \frac{\partial}{\partial y}\left(\chi\frac{\partial U_1}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{e}{m^2}\frac{\partial U_1}{\partial z}\right)\right\}. \end{aligned} \tag{83}$$

For this system the substitution

$$d\eta_i = \frac{\rho}{m^2}\,dU_i, \qquad \frac{\partial\eta_i}{\partial x} = \frac{\rho}{m^2}\frac{\partial U_i}{\partial x}, \qquad \text{etc.}, \tag{84}$$

would lead to an equation of type (82) with

$$M_x = M_y = m^2\chi/\rho, \qquad M_z = e/\rho. \tag{85}$$

We conclude that M_x, M_y, and M_z should be introduced in (80) to represent the "exchange" or "eddy diffusion" coefficients in the sense of Prandtl's mixing length theory

$$\frac{1}{\rho} f_1^{(N)} = \frac{m^2}{\rho} \left\{ \frac{\partial}{\partial x} \left(\frac{\rho}{m^4} M_x D_T \right) + \frac{\partial}{\partial y} \left(\frac{\rho}{m^4} M_x D_S \right) + \frac{\partial}{\partial z} \left[\rho \frac{M_z}{m^2} \left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} \right) \right] \right\},$$
$$\frac{1}{\rho} f_2^{(N)} = \frac{m^2}{\rho} \left\{ \frac{\partial}{\partial x} \left(\frac{\rho}{m^4} M_x D_S \right) - \frac{\partial}{\partial y} \left(\frac{\rho}{m^4} M_x D_T \right) + \frac{\partial}{\partial z} \left[\rho \frac{M_z}{m^2} \left(\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} \right) \right] \right\}. \tag{86}$$

The explicit construction of M_x and M_z is left until after the discussion of temperature and humidity diffusion. At this point we may mention that in the computer program the first two terms of each equation are computed separately (in a "horizontal diffusion" subroutine) from the last ("vertical diffusion"). One reason for this is that the "vertical diffusion" term is modified in accordance with temperature distribution considerations (Richardson number). Since the details of that formulation are still subject to change, we shall not dwell on them. We may also point out that it was considered unnecessary, at this stage, to change the partial derivatives [e.g., $(\partial/\partial x)_z \to (\partial/\partial x)_p \to (\partial/\partial x)_Q$] in accordance with the developments of Section 3.

- *6. Temperature and humidity diffusion*

In this section we are concerned with the variation of a scalar function $\Psi(x, y, z, t)$, in particular potential temperature θ and mixing ratio r, due to turbulent diffusion. We assume that these processes can be modeled after the process of molecular heat diffusion, which is represented formally by

$$\dot{T} = \frac{1}{c_p \cdot \rho} \operatorname{div} (K \cdot \operatorname{grad} T) = \frac{1}{c_p \cdot \rho} \operatorname{div} F, \tag{87}$$

where

$F =$ heat flux, $K =$ coefficient (or coefficient matrix) of heat conduction.

We set similarly:

$$\dot{\Psi} = \frac{1}{c_1} \operatorname{div} (c_1 \cdot c \operatorname{grad} \Psi) \tag{88}$$

or in tensor form:

$$\dot{\Psi} = \frac{1}{c_1} [c_1 \cdot F^i]_{,i}, \qquad F^i = c^{ii} \Psi_i. \tag{89}$$

In both cases (θ, r) the coefficients c^{ii} of the "eddy diffusion" tensor c^{ii} have dimension (length)^2/time. To simplify the structure of c^{ii} we may demand, e.g., in the Σ system, that the relation $F^i = c^{ii}\Psi_i$ remain valid (on account of natural symmetries) after the transformation:

$$z' = Az \tag{90}$$

a) $A = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$ rotation about z_3-axis

b) $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ reflection with respect to the z_2 z_3-plane.

It is seen easily that one must have $A\,c^{ii} = c^{ii}\,A$ and that the final form of c^{ii} must be:

$$c^{ii}_{(\Sigma)} = \begin{bmatrix} c^{11} & 0 & 0 \\ 0 & c^{11} & 0 \\ 0 & 0 & c^{33} \end{bmatrix} \equiv \begin{bmatrix} \Gamma & 0 & 0 \\ 0 & \Gamma & 0 \\ 0 & 0 & \tilde{\Gamma} \end{bmatrix}. \tag{91}$$

From the tensor character one obtains for the N map:

$$c^{ii}_{(N)} = \begin{bmatrix} \Gamma m^2 & 0 & 0 \\ 0 & \Gamma m^2 & 0 \\ 0 & 0 & \tilde{\Gamma} \end{bmatrix} \tag{92}$$

and

$$\begin{aligned} c_1 \cdot \dot{\Psi}_{(N)} &= [c_1 \cdot c^{ii}_{(N)} \cdot \Psi_i^{(N)}]_{,i} \\ &= \frac{\partial}{\partial x}\left[c_1 \cdot c^{(11)}_{(N)} \frac{\partial \Psi}{\partial x}\right] + \left\{ \begin{matrix} i \\ 1\ i \end{matrix} \right\} \left[c_1 c^{(11)} \frac{\partial \Psi}{\partial x} \right] + \cdots \\ &= \frac{\partial}{\partial x}\left[c_1 \cdot \Gamma m^2 \frac{\partial \Psi}{\partial x}\right] - c_1 \Gamma \frac{\partial \Psi}{\partial x} \frac{\partial m^2}{\partial x} + \cdots \end{aligned} \tag{93}$$

$$\dot{\Psi}_N = \frac{m^2}{c_1}\left[\frac{\partial}{\partial x}\left(c_1 \cdot \Gamma \frac{\partial \Psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(c_1 \cdot \Gamma \frac{\partial \Psi}{\partial y}\right)\right] + \frac{1}{c_1}\frac{\partial}{\partial z}\left[c_1 \tilde{\Gamma} \frac{\partial \Psi}{\partial z}\right].$$

By an argument analogous to that used around equations (82) to (85) we see that the "exchange coefficients" are:

$$(K_x, K_y, K_z) = (m^2/c_1)(\Gamma, \Gamma, \tilde{\Gamma}). \tag{94}$$

Table 1 Approximate corresponding values of "vertical" structure variables for ICAO standard atmosphere adjusted for height $z = 0$ at $p = p_* = 1000$ mb.

K	σ_K	$p_* = 1000\, Q_K$ (*mb.*)	Δp_* (*mb.*)	z_K (*km.*)
$\frac{1}{2}$	0	0		—
1	.0556	9	34	31.6
$1\frac{1}{2}$	.1111	34		23.0
2	.1667	74	92	18.0
$2\frac{1}{2}$	.2222	126		14.7
3	.2778	189	133	12.0
$3\frac{1}{2}$	.3333	259		10.1
4	.3889	336	158	8.3
$4\frac{1}{2}$	.4444	417		6.8
5	.5	500	166	5.5
$5\frac{1}{2}$	.5556	583		4.3
6	.6111	664	158	3.3
$6\frac{1}{2}$	.6667	741		2.5
7	.7222	811	133	1.7
$7\frac{1}{2}$	.7778	874		1.1
8	.8333	926	92	.64
$8\frac{1}{2}$	.8889	966		.3
9	.9444	991	34	.07
$9\frac{1}{2}$	1	1000		0

In the case of temperature diffusion we have $c_1 = c_p \cdot \rho$, in the case of humidity diffusion c_1 was assumed to be essentially independent of x, y, z.

$$\left.\frac{d\theta}{dt}\right|_{\text{diff.}} = \frac{m^2}{\rho}\left[\frac{\partial}{\partial x}\left(\Gamma\rho\,\frac{\partial\theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(\Gamma\rho\,\frac{\partial\theta}{\partial y}\right)\right] + \frac{1}{\rho}\left[\frac{\partial}{\partial z}\left(\tilde{\Gamma}\rho\,\frac{\partial\theta}{\partial z}\right)\right]$$
$$\left.\frac{dr}{dt}\right|_{\text{diff.}} = m^2\left[\frac{\partial}{\partial x}\left(\Gamma\,\frac{\partial r}{\partial x}\right) + \frac{\partial}{\partial y}\left(\Gamma\,\frac{\partial r}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left(\tilde{\Gamma}\,\frac{\partial r}{\partial z}\right). \tag{95}$$

• *7. Eddy diffusion coefficients*

According to the "mixing length hypothesis" of Prandtl, an "eddy diffusion coefficient" should be proportional to the product of two factors. One is the square of a characteristic length or scale of the mean motion, the other is usually (for one-dimensional motions) taken as the gradient of the mean velocity. In the three-dimensional case confronting us, the local horizontal variations of velocity seem to be best described by the terms D_S and D_T of (81). We shall therefore relate the first two eddy diffusion coefficients K_x, K_y to the quantity:

$$D \equiv (D_T^2 + D_S^2)^{1/2}, \tag{96}$$

which has the convenient property of being invariant (a scalar) under the transformation $N \Leftrightarrow S$ (Ref. 18). To characterize the velocity variation in the vertical we take the scalar:

$$D_V = \left\{\left[\frac{\partial}{\partial z}\,(mu_1)\right]^2 + \left[\frac{\partial}{\partial z}\,(mu_2)\right]^2\right\}^{1/2} = \left|\frac{\partial\, m\mathbf{V}_H}{\partial z}\right|, \qquad \mathbf{V}_H = (u_1, u_2). \tag{97}$$

Since we are constructing the diagonal elements of a contravariant tensor of rank 2, we choose as our characteristic length terms the scalars $(1/m^2)\,[(dx)^2 + (dy)^2]$ and $(dz)^2$, multiplied by the appropriate terms of the fundamental tensor g^{ik}.

$$K^{ii}_{(N)} = \begin{bmatrix} K_x & 0 & 0 \\ 0 & K_y & 0 \\ 0 & 0 & K_z \end{bmatrix} = \text{const.}\begin{bmatrix} \dfrac{\Gamma m^2}{c_1} & 0 & 0 \\ 0 & \dfrac{\Gamma m^2}{c_1} & 0 \\ 0 & 0 & \dfrac{\tilde{\Gamma} m^2}{c_1} \end{bmatrix}$$
$$= \text{const.}\begin{bmatrix} D\cdot g^{11}\,\dfrac{(dx)^2 + (dy)^2}{m^2} & 0 & 0 \\ 0 & D\cdot g^{22}\,\dfrac{(dx)^2 + (dy)^2}{m^2} & 0 \\ 0 & 0 & D_v\cdot g^{33}(dz)^2 \end{bmatrix} \tag{98}$$

$$\Gamma = \text{const.}\ D\,\frac{(dx)^2 + (dy)^2}{m^2}, \qquad \tilde{\Gamma} = \text{const.}\ D_v(dz)^2.$$

In the computer program the difference equations corresponding to (95) are solved on a grid with uniform horizontal spacing $\delta s = \delta x = \delta y$ and vertical spacing δz. The scalar $(ds)_h^2 = m^{-2}[(dx)^2 + (dy)^2]$ of (95) is then replaced by const. $(\delta s)^2$. This is tantamount to having the characteristic length of Prandtl correspond to a definite length on earth, e.g., the largest size of a particular type eddy, at every point of the grid.

Taking out factors which are not affected by differentiations, we can replace (95) by:

$$\left.\frac{d\theta}{dt}\right|_{\text{diff.}} = C_1\,(\delta s)^2\,\frac{m^2}{\rho}\left[\frac{\delta}{\delta x}\left(\rho D\,\frac{\delta\theta}{\delta x}\right) + \frac{\delta}{\delta y}\left(\rho\ D\,\frac{\delta\theta}{\delta y}\right)\right] + C_2(\delta z)^2\,\frac{1}{\rho}\left[\frac{\delta}{\delta z}\left(\rho D_V\,\frac{\delta\theta}{\delta z}\right)\right]$$
$$\left.\frac{dr}{dt}\right|_{\text{diff.}} = C_3(\delta s)^2 m^2\left[\frac{\delta}{\delta x}\left(D\,\frac{\delta r}{\delta x}\right) + \frac{\delta}{\delta y}\left(D\,\frac{\delta r}{\delta y}\right)\right] + C_4(\delta z)^2\left[\frac{\delta}{\delta z}\left(D_V\,\frac{\delta\theta}{\delta z}\right)\right]. \tag{99}$$

The constants C_1, C_2, C_3, C_4 have been left unspecified. They are best chosen from empirical data or, as in the case of the program described here, in accordance with previous computational experience.

We are now also in a position to reformulate the equations of momentum diffusion. Similarly to (98), we have:

$$M^{ij}_{(N)} = \begin{bmatrix} M_x & 0 & 0 \\ 0 & M_y & 0 \\ 0 & 0 & M_z \end{bmatrix} = \frac{1}{\rho}\begin{bmatrix} m^2\chi & 0 & 0 \\ 0 & m^2\chi & 0 \\ 0 & 0 & e \end{bmatrix}$$

$$= \text{const.}\begin{bmatrix} D[(dx)^2+(dy)^2] & 0 & 0 \\ 0 & D[(dx)^2+(dy)^2] & 0 \\ 0 & 0 & D_V\cdot(dz)^2 \end{bmatrix} \tag{100}$$

where

$\chi = \text{const. } D\cdot\rho\cdot[(dx)^2+(dy)^2]$, $e = \text{const. } D_V\cdot\rho\cdot(dz)^2$.

After transition to the difference scheme, we obtain:

$$\chi = \text{const. } D\cdot\rho\cdot m^2(\delta s)^2, \quad e = \text{const. } D_V\cdot\rho\cdot(\delta z)^2. \tag{101}$$

$$\frac{1}{\rho}f_1 = \frac{m^2}{\rho}\left\{C_5\,(\delta s)^2\left[\frac{\delta}{\delta x}\left(\rho\frac{DD_T}{m^2}\right)+\frac{\delta}{\delta y}\left(\rho\frac{DD_S}{m^2}\right)\right]\right.$$
$$\left.+\,C_6\,(\delta z)^2\left[\rho\frac{D_V}{m^2}\left(\frac{\delta u_1}{\delta z}+\frac{\delta u_3}{\delta x}\right)\right]\right.$$
$$\frac{1}{\rho}f_2 = \frac{m^2}{\rho}\left\{C_5(\delta s)^2\left[\frac{\delta}{\delta x}\left(\rho\frac{DD_S}{m^2}\right)-\frac{\delta}{\delta y}\left(\rho\frac{DD_T}{m^2}\right)\right]\right.$$
$$\left.+\,C_6(\delta z)^2\left[\rho\frac{D_V}{m^2}\left(\frac{\delta u_2}{\delta z}+\frac{\delta u_3}{\delta y}\right)\right]\right. . \tag{102}$$

The authors know of no other detailed attempt at derivation of formulas (80)–(102). Some of the assumptions made may, therefore, be open to doubt.

• 8. Discrete formulation

"*Vertical*" *structure*

The proposal of Smagorinsky to take $Q = p/p_* = \sigma^2(3-2\sigma)$ with equal σ-intervals, based upon considerations of resolution in the "vertical" structure of the atmosphere, was adopted (see Ref. 16, S. Manabe and F. Möller). For the selected maximum number of nine σ-intervals "$\Delta\sigma$", it follows that $\Delta\sigma = 1/9$ from the fact that σ ranges from 0 to 1 as p varies from its value of 0 at the top of the atmosphere to p_* at the surface of the earth. Because of the variability of p and p_* in space and time, the Q-intervals "ΔQ" obtained from this σ-subdivision correspond to nonuniform and variable distances. Thus, for $\Delta\sigma = 1/9$, Q_K may be determined from $\sigma_K = (K-1/2)\,\Delta\sigma = (K-1/2)1/9$ where $K = 1/2, 1, 1\frac{1}{2}, 2, \ldots, 9\frac{1}{2}$. Q_K will be called a "level" for K an integer, otherwise a "half-level". Table 1 illustrates this "vertical" structuring under standard atmospheric conditions.

Table 2 Table of approximate map grid interval Δs vs number n of intervals from pole to equator.

n	5	10	20	40
$\Delta s(km.)$	2548	1274	637.1	318.6

"*Horizontal*" *structure*

A uniform rectangular grid is superimposed on the stereographic map associated with each Q_K for each of the two hemispheres. Each grid arises from the introduction of lines parallel to the rectangular coordinate axes. Both sets of parallels have the same interval of spacing, equal to the radial map distance divided by an integer n. The radial map distance from pole to equator is the constant $2a$, where a is the mean earth radius, map magnification for each hemisphere is equivalent to the map "factor" previously introduced in (14), and the spacing interval is $2a/n$ for all grids. Table 2 illustrates the relationship between the spacing interval $2a/n$, denoted by "Δs", and the parameter n for several values of n within the range handled by the computer program. Since the map magnification varies from 1 at the pole to 2 at the equator, the earth distance corresponding to the map distance Δs tends with increasing n to Δs at the pole and to $\frac{1}{2}\,\Delta s$ at the equator.

Each grid is regarded as comprising not only "interior" points inside or on the boundary of the circular stereographic representation of the equatorial boundary of the hemisphere, but also "exterior" points obtained from continuation of the grid—with the same uniform spacing—outside the equatorial circle. The extent of the grid is as required to evaluate for the next time position five "basic" variables at interior grid points of each hemisphere, using the finite-difference equations chosen to approximate discretely the continuous formulation over the grid. The basic variables integrated over the grid with respect to time from these difference equations are p_*, v_1p_*, v_2p_*, Tp_*, and hp_* in N, and the same except for v_1p_* and v_2p_* instead of u_1p_* and u_2p_* in S. (Because of the form of these equations, a minor economy in computing time is effected by computing and storing values of T rather than of the "time-differenced" variable Tp_*.) This evaluation assumes, for each Q_K, that each interior grid point is the center of a

Table 3 Table of numbers of interior and exterior grid points for each Q_K of N or S versus the number n of equal subdivisions from pole to equator.

	Number of grid points for each Q_K of N or S	
n	*Interior*	*Exterior*
5	81	68
10	317	124
20	1,257	236
40	5,025	460

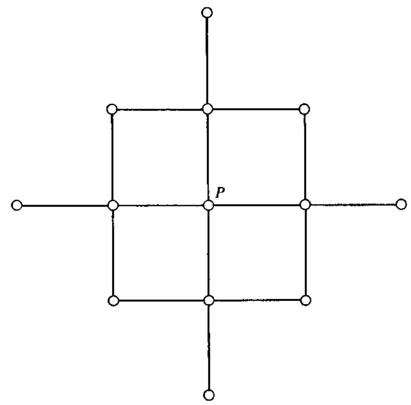

Figure 4 The 13-point star of neighbor grid points for interior point P.

"13-point star" of neighbor grid points. In fact, the grid is limited to contain only each interior point and each exterior point contained in the 13-point star of neighbor grid points for some interior point. (See Fig. 4.)

The numbers of interior and exterior points in the grid for each Q_K of N or S are illustrated in Table 3 using the previously selected values of the parameter n. The configuration of grid points for $n = 5$ is indicated in Fig. 5.

The method of evaluating the basic variables at exterior grid points from their values computed from the difference equations at interior grid points will be briefly indicated now, and will be described in detail in a later section. Corresponding positions in N and S are related by the geometry of inversion with respect to the equatorial circle. Consequently, a grid point exterior to one hemisphere represents the same point in the atmosphere as represented by an interior point—not in general an interior grid point—of the other hemisphere. Bivariate linear interpolation was chosen as the means of expressing the values of each of the basic variables at such an interior point not in the grid as a linear combination of its values at the surrounding four neighbor—not necessarily entirely interior—grid points. However, in relating values at the corresponding interior and exterior points, the basic variables which are vector components (u_1p_*, u_2p_*, v_1p_*, v_2p_*) require coordinate transformation from N to S and/or S to N.

Approximation of derivatives and integrals using finite differences

The methods of approximation described below were selected for initial experimentation and basically express time and space derivatives (in "compact" or "contracted" form, instead of expanding derivatives of products) as central difference quotients. Later, in an attempt to avoid possibilities of nonlinear instability such as experienced by N. Phillips[19] and to conserve certain integral properties, a modified difference method was adopted by the Weather Bureau for hemispherical experiments. The modification was proposed by D. Lilly[20] and is similar to a method of A. Arakawa.[21]

For any variable Ψ, let $\Psi^{\tau}_{i,j,K;N}$ denote the value of Ψ at time $t = \tau\Delta t$ at space point ($x = i\Delta s$, $y = j\Delta s$, Q_K) in N. The time interval Δt is regarded as constant for one experiment (computer "run"), in which $\Delta s = 2a/n$ and the number of Q_K positions are also fixed. The symbols τ, i and j represent integers. (When not pertinent, subscripts and superscripts will be omitted.)

The derivative $\partial\Psi^{\tau}/\partial x|_{i,j,K;N}$ is approximated by the central difference quotient

$$\frac{1}{2\Delta s}\left(\Psi^{\tau}_{i+1,j,K;N} - \Psi^{\tau}_{i-1,j,K;N}\right)$$

and similarly $\partial\Psi^{\tau}/\partial y|_{i,j,K;N}$ by

$$\frac{1}{2\Delta s}\left(\Psi^{\tau}_{i,j+1,K;N} - \Psi^{\tau}_{i,j-1,K;N}\right).$$

The calculation of derivatives in S is completely analogous to that in N, requiring only the replacement, in the preceding and following developments, of x, y and N by ξ, η and S, respectively.

The derivative $\partial\Psi^{\tau}/\partial t|_{i,j,K;N}$ is approximated in one of two ways. For $t = \tau = 0$, and from the assumption that initial data for the basic variables are specified only at this one time position, $\partial\Psi^{0}/\partial t|_{i,j,K;N}$ is approximated by the noncentral difference quotient

$$\frac{1}{\Delta t}\left(\Psi^{1}_{i,j,K;N} - \Psi^{0}_{i,j,K;N}\right).$$

For $\tau > 0$, central differencing is used to approximate $\partial\Psi^{\tau}/\partial t|_{i,j,K;N}$ by

$$\frac{1}{2\Delta t}\left(\Psi^{\tau+1}_{i,j,K;N} - \Psi^{\tau-1}_{i,j,K;N}\right).$$

An approximation to $\partial\Psi^{\tau}/\partial Q|_{i,j,K;N}$ for each integer value of K is required. For this purpose, the central difference quotient

$$\frac{1}{\Delta Q_K}\,(\Psi^{\tau}_{i,j,K+1/2;N} - \Psi^{\tau}_{i,j,K-1/2;N}),$$

where $\Delta Q_K = Q_{K+1/2} - Q_{K-1/2}$, is used. For these derivatives, Ψ may be expressed as a product of two factors, the first factor being $\bar{\omega}(=\dot{Q})$ and the second being one of the basic variables other than p_*. E.g., the term

$$p_*\,\frac{\partial(\bar{\omega}h)}{\partial Q} = \frac{\partial(\bar{\omega}hp_*)}{\partial Q}$$

[cf. (58)] yields $\Psi = \bar{\omega}\cdot(hp_*)$, hp_* being one of the basic variables other than p_*. The expression

$$\frac{1}{\Delta Q_K}\,(\Psi^{\tau}_{i,j,K+1/2} - \Psi^{\tau}_{i,j,K-1/2})$$

is evaluated as

$$\frac{1}{\Delta Q_K}\,[\bar{\omega}^{\tau}_{i,j,K+1/2;N}\cdot(hp_*)^{\tau}_{i,j,K+1/2;N} - \bar{\omega}^{\tau}_{i,j,K-1/2;N}\cdot(hp_*)^{\tau}_{i,j,K-1/2;N}].$$

As each basic variable in the second factor of Ψ is computed from the difference equations only for integer values of K, the arithmetic mean approximations

$$(hp_*)^{\tau}_{i,j,K+1/2;N} = \tfrac{1}{2}[(hp_*)^{\tau}_{i,j,K;N} + (hp_*)^{\tau}_{i,j,K+1;N}]$$

and

$$(hp_*)^{\tau}_{i,j,K-1/2;N} = \tfrac{1}{2}[(hp_*)^{\tau}_{i,j,K-1;N} + (hp_*)^{\tau}_{i,j,K;N}]$$

are used.

Furthermore,

$$\bar{\omega} = -\frac{Q}{p_*}\frac{\partial p_*}{\partial t} - \frac{m^2}{p_*}\int_0^Q \left[\frac{\partial(u_1'p_*)}{\partial x} + \frac{\partial(u_2'p_*)}{\partial y}\right] dQ'$$

in N [cf. (43)], $Q' = p'/p_*$ denoting the dummy variable of integration,

$$u_1' = u_1(x, y, Q; t)]^{Q=Q'}, \qquad u_2' = u_2(x, y, Q; t)]^{Q=Q'}$$

and

$$\frac{\partial p_*}{\partial t} = -m^2\int_0^1 \left[\frac{\partial(u_1p_*)}{\partial x} + \frac{\partial(u_2p_*)}{\partial y}\right] dQ$$

in N [cf. (44)].

These two integrals are approximated by the customary Riemann sums, requiring only values of the integrands at the Q_K-levels to obtain $\bar{\omega}$ at the half-levels. Introduction of the boundary conditions $\bar{\omega} = 0$ at the top and bottom half-levels (where $Q = 0$ and $Q = 1$, respectively), it is noted, renders unnecessary any special evaluation of hp_* in approximating $\partial\Psi^{\tau}/\partial Q|_{i,j,K;N}$ at the top and bottom Q_K-levels. The treatment of the other $\partial\Psi/\partial Q$ terms is completely analogous.

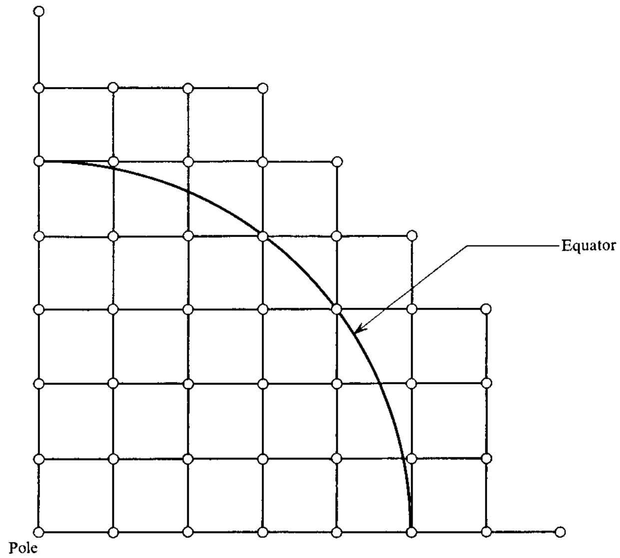

Figure 5 Interior and exterior grid points in one quadrant for each Q_K of N or S when $n = 5$.

From (38) and (40), $\bar{\omega}(=\dot{p})$ may be expressed by

$$\omega = p_*\bar{\omega} + Q\left[\frac{\partial p_*}{\partial t} + m^2\left(u_1\frac{\partial p_*}{\partial x} + u_2\frac{\partial p_*}{\partial y}\right)\right]$$

in N. The value $\omega^{\tau}_{i,j,K;N}$ is approximated using the previously described approximations for the terms in this relationship.

Consider next the treatment of the geopotential terms. From Eq. (47): $\partial\Phi/\partial Q = -RT/Q$. Integration yields

$$\Phi\Big]^{Q'=Q} = \Phi\Big]^{Q'=1} + R\int_Q^1 \frac{T}{Q'}\,dQ'$$

$$= gz_* + R\int_Q^1 T d(\ln Q'),$$

where z_* is the height of the earth's surface, and Q' a dummy variable. This relation is differentiated to obtain the required derivatives

$$\frac{\partial\Phi}{\partial x},\quad \frac{\partial\Phi}{\partial y},\quad \frac{\partial\Phi}{\partial\xi},\quad \text{and}\quad \frac{\partial\Phi}{\partial\eta}.$$

E.g.,

$$\frac{\partial\Phi}{\partial x} = g\frac{\partial z_*}{\partial x} + R\int_Q^1 \frac{\partial T}{\partial x}\,d(\ln Q').$$

The derivatives

$$\frac{\partial T}{\partial x},\quad \frac{\partial T}{\partial y},\quad \frac{\partial T}{\partial\xi},\quad \text{and}\quad \frac{\partial T}{\partial\eta}$$

are assumed in approximation to vary linearly with $\ln Q$

in each interval between adjacent Q_K-levels. For nine Q_K-levels, the resulting approximation:

$$\frac{\partial \Phi^\tau}{\partial x}\bigg|_{i,j,K;N} \cong g\frac{\partial z_*}{\partial x}\bigg|_{i,j;N} + R\left\{\sum_{K'=K}^{9} \frac{1}{2}\left[\frac{\partial T^\tau}{\partial x}\bigg|_{i,j,K';N} + \frac{\partial T^\tau}{\partial x}\bigg|_{i,j,K'+1;N}\right]\ln\left(\frac{Q_{K'+1}}{Q_{K'}}\right) + \mathfrak{L}\left(\frac{\partial T^\tau}{\partial x}\bigg|_{i,j,8;N}, \frac{\partial T^\tau}{\partial x}\bigg|_{i,j,9;N}\right)\right\}$$

is obtained from the customary Riemann sum approximation for $\int_{Q_K}^{Q_*}$ and a linear extrapolation approximation for $\int_{Q_*}^{1}$. The expression

$$\mathfrak{L}\left(\frac{\partial T^\tau}{\partial x}\bigg|_{i,j,8;N}, \frac{\partial T^\tau}{\partial x}\bigg|_{i,j,9;N}\right)$$

denotes a linear combination of the values $\partial T^\tau/\partial x_{\ i,j,K';N}$ at the lowest two $Q_{K'}$-levels, the coefficients of which are constants. (These same constants apply to the other Φ-derivative approximations.) The derivatives in this expression for $\partial\Phi^\tau/\partial x_{i,j,K;N}$ are approximated as described previously, and the treatment of the other Φ-derivatives is completely analogous to that for $\partial\Phi/\partial x$.

The only derivatives which appear explicitly in the final equations of the continuous formulation are of first order, and the method of approximation for each such derivative is indicated in the above discussion. Higher order derivatives would appear in the expansion of derivatives of products occuring in diffusion terms in the final equations. However, derivatives of products (not only in diffusion terms, but also in all other occurrences in the final formulation) are approximated in contracted form, and the computations for such terms at most require a few successive applications of the above differencing techniques for first order derivatives.

Let Ψ denote one of the basic variables. Then, $\partial\Psi^\tau/\partial t_{i,j,K;N(\text{or }S)}$ and consequently $\Psi^{\tau+1}_{i,j,K;N(\text{or }S)}$ are approximated from these difference methods. E.g., let

$$\frac{1}{2\Delta t}\left[(u_1 p_*)^{\tau+1} - (u_1 p_*)^{\tau-1}\right]_{i,j,K;N}$$

represent the approximation of $\partial(u_1p_*)^\tau/\partial t_{i,j,K;N}$ for a typical value of τ, and $\mathfrak{F}_{i,j,K;N}$ the approximate value of this derivative obtained from applying a composition of the above differencing techniques to the expression for this derivative in terms of the other variables obtained from the continuous formulation. Basically, $\mathfrak{F}_{i,j,K;N}$ is developed from values of the variables which were computed for time $t = \tau\Delta t$ at interior grid point ($x = i\Delta s$, $y = j\Delta s$, Q_K) in N—the derivatives at this position requiring the use of values of variables at neighboring grid points also—by addition, multiplication, division, square root, etc. Approximations for

$$\frac{\partial(u_2p_*)}{\partial t}, \frac{\partial p_*}{\partial t}, \frac{\partial(Tp_*)}{\partial t}, \frac{\partial(hp_*)}{\partial t}$$

in N (or for

$$\frac{\partial(v_1p_*)}{\partial t}, \frac{\partial(v_2p_*)}{\partial t},$$

etc., in S) are developed similarly. During these computations, use is made of certain coefficients which, being independent of time, are computed only once (prior to other calculations for $\tau = 0$). However, provision was made—for economy of computing time—to permit the calculation of radiation, "horizontal" diffusion, and "vertical" diffusion terms periodically in time. Another exception is that the "horizontal" diffusion terms are "lagged in time" in accordance with a suggestion of Smagorinsky based on stability considerations (J. Smagorinsky,[10] p. 464). E.g., $\mathfrak{F}_{i,j,K;N}$ may be separated into two parts $\mathcal{G}^\tau_{i,j,K;N}$ and $\mathcal{H}^{\tau-1}_{i,j,K;N}$.
Then,

$$(u_1p_*)^{\tau+1}_{i,j,K;N} = (u_1p_*)^{\tau-1}_{i,j,K;N} + 2\Delta t(\mathfrak{F}_{i,j,K;N})$$
$$= [(u_1p_*)^{\tau-1} + 2\Delta t(\mathcal{G}^\tau + \mathcal{H}^{\tau-1})]_{i,j,K;N}$$

for $\tau > 0$, where the "horizontal" diffusion effect $\mathcal{H}^{\tau-1}_{i,j,K;N}$ and part (in this case only for "vertical" diffusion) of $\mathcal{G}^\tau_{i,j,K;N}$ may have been calculated for $\tau - 1$ and τ respectively, or may have been obtained by merely referencing in storage previously computed values. The basic variables for each hemisphere are computed at $\tau + 1$ for each interior point of each Q-level, except for p_* which is computed only for each interior point of half-level $Q = 1$.

Approximation of exterior data using interpolation

As indicated previously, the four neighbor grid points which surround the interior "image" of an exterior grid point are not necessarily interior. In fact, as many as three of these four neighbor grid points may be exterior. For some of the image points having exterior neighbors, a special "simultaneous equations" relationship arises from the application of bivariate linear interpolation. From (49), the abscissas of each exterior point and its interior image have opposite signs, while the ordinates have the same sign. Consider the exterior grid points $P_1 = (x = i\Delta s,\ \ y = j\Delta s, Q_K)$ in N and $P_2 = (\xi = -x, \eta = y, Q_K)$ in S. Let P_1' and P_2' denote the interior images of P_1 and P_2, respectively. The simultaneous equations case arises when both P_1' has exterior point P_2 as one of its four surrounding neighbor grid points and P_1' is properly interior to its surrounding grid square, i.e., P_1' does not lie on a grid line. (The same statement holds with P_1' and P_2 replaced by P_2' and P_1, respectively.) For, in the simultaneous equations case, the approximation of the basic variables at exterior point P_1 using bivariate linear interpolation depends upon their values at P_2, while at P_2 it depends upon their values at P_1.

This bivariate linear interpolation may be described briefly as the formation of linear combinations of data with constant coefficients. Thus, the "interpolation coefficients" depend solely upon the coordinates of the exterior grid points (alternatively their interior image points) and the number n of equal subdivisions from pole to equator. From symmetry considerations, it would have been sufficient to have computed the interpolation coefficients for one octant of the circular grid of one Q_K-level in N or S, i.e. for, say, $x \geq y \geq 0$. (This may be seen from the interpolation formulas which will be described.)

Scalar exterior data

The basic variables p_*, Tp_*, and hp_*—whose values are invariant under coordinate change from N to S or vice versa—are evaluated at exterior grid points solely by interpolation from their values at some interior grid points. Denote by $P = (i\Delta s, j\Delta s, Q_K)$ an exterior grid point in either N or S and by $P' = (i'\Delta s, j'\Delta s, Q_K)$ its interior image in S or N, respectively. Denote

$$\mathcal{I}_{P'} = \langle |i'| \rangle - |i'|$$

and

$$\mathcal{J}_{P'} = \langle |j'| \rangle - |j'|,$$

where $\langle \alpha \rangle$ is the least integer $\geq \alpha$. The interpolation coefficients $\mathcal{A}_{P'}$, $\mathcal{B}_{P'}$, $\mathcal{C}_{P'}$, $\mathcal{D}_{P'}$, which may be regarded as "weights" for values of a variable at the four neighbor grid points surrounding the image point P', are given in Table 4.

It is easily seen that $\mathcal{A}_{P'}$, $\mathcal{B}_{P'}$, $\mathcal{C}_{P'}$, and $\mathcal{D}_{P'}$ are nonnegative and have sum 1. Special cases involving only two grid point neighbors of P', and consequently only two coefficients, arise when P' lies on a grid line.

In the nonsimultaneous equations case, the value of a scalar basic variable at exterior point P, denoted by $\mathcal{S}_P$ (i.e., $\mathcal{S} = p_*$, Tp_*, or hp_*), is obtained as a linear combination of its values at the grid point neighbors (not necessarily entirely interior) of its image point P':

$$\mathcal{S}_P = \mathcal{A}_{P'}\mathcal{S}_{\mathcal{A}_{P'}} + \mathcal{B}_{P'}\mathcal{S}_{\mathcal{B}_{P'}} + \mathcal{C}_{P'}\mathcal{S}_{\mathcal{C}_{P'}} + \mathcal{D}_{P'}\mathcal{S}_{\mathcal{D}_{P'}}, \tag{103}$$

where the subscripts on $\mathcal{S}$ in the right member indicate the neighbor grid points of P' according to their associated coefficients.

In the simultaneous equations case, the interpolated value, again denoted by $\mathcal{S}_P$, is obtained by a modification of the above procedure. As before, consider the exterior grid points $P_1 = (x = i\Delta s, y = j\Delta s, Q_K)$ in N and $P_2 = (\xi = -x, \eta = y, Q_K)$ in S, their interior images being P_1' in S and P_2' in N, respectively. In the simultaneous equations case, P_1 is one of the four surrounding neighbor grid points for P_2', and P_2 serves similarly for P_1'. Denote by $S_{P_2'}$ the linear combination of values of $\mathcal{S}$ at the neighbor grid points of P_2' in N other than P_1, and similarly $S_{P_1'}$, the linear combination of values of $\mathcal{S}$ at the neighbor grid points of P_1' in S other than P_2. By analogy with the nonsimultaneous equations case:

$$S_{P_1'} = \mathcal{A}_{P_1'}\mathcal{S}_{\mathcal{A}_{P_1'}} + \mathcal{B}_{P_1'}\mathcal{S}_{\mathcal{B}_{P_1'}} + \mathcal{C}_{P_1'}\mathcal{S}_{\mathcal{C}_{P_1'}}$$

$$S_{P_2'} = \mathcal{A}_{P_1'}\mathcal{S}_{\mathcal{A}_{P_2'}} + \mathcal{B}_{P_1'}\mathcal{S}_{\mathcal{B}_{P_2'}} + \mathcal{C}_{P_1'}\mathcal{S}_{\mathcal{C}_{P_2'}}$$

as

$$\mathcal{A}_{P_1'} = \mathcal{A}_{P_2'}, \quad \mathcal{B}_{P_1'} = \mathcal{B}_{P_2'}, \quad \mathcal{C}_{P_1'} = \mathcal{C}_{P_2'}$$

(and $\mathcal{D}_{P_1'} = \mathcal{D}_{P_2'}$) by symmetry.

Then

$$\mathcal{S}_{P_1} = S_{P_1'} + \mathcal{D}_{P_1'}\mathcal{S}_{P_2}$$

and

$$\mathcal{S}_{P_2} = S_{P_2'} + \mathcal{D}_{P_1'}\mathcal{S}_{P_1}.$$

Hence

$$\mathcal{S}_{P_1} = \frac{S_{P_1'} + \mathcal{D}_{P_1'}S_{P_2'}}{1 - (\mathcal{D}_{P_1'})^2}$$

and (104)

$$\mathcal{S}_{P_2} = \frac{S_{P_2'} + \mathcal{D}_{P_1'}S_{P_1'}}{1 - (\mathcal{D}_{P_1'})^2}.$$

Vector exterior data

The basic variables u_1p_* and u_2p_* in N transform covariantly into v_1p_* and v_2p_* in S and vice versa. (The covariant velocity components u_1 and u_2 in N are related to the contravariant "map velocity" components $u^1 = \dot{x}$ and $u^2 = \dot{y}$ in N by $m^2(u_1, u_2) = (u^1, u^2)$, and similarly for the components in S. The corresponding physical velocity components, expressed in stereographic coordinates, are obtained by multiplying the map velocity components by the reciprocal of the magnification.) The evaluation of u_1p_* and u_2p_* at each exterior grid point of N, or the evaluation of v_1p_* and v_2p_* at each exterior grid point of S, may be regarded as composed of two operations: (a) interpolation for components expressed in the coordinate system of the interior image point using the same interpolation coefficients as for scalar data, and (b) transformation of the interpolated components to obtain the corresponding components expressed in the coordinate system of the associated exterior grid point.

In the nonsimultaneous equations case, let $(u_1p_*)_P$ and $(u_2p_*)_P$ denote the values of u_1p_* and u_2p_* to be derived for exterior grid point $P = (i\Delta s, j\Delta s, Q_K)$ in N, and let $(v_1p_*)_{P'}$

and $(v_2p_*)_{P'}$ denote the corresponding components at the interior image point $P' = (i'\Delta s, j'\Delta s, Q_K)$ in S of p. Then, in accordance with (a) above, $(v_1p_*)_{P'}$ and $(v_2p_*)_{P'}$ are obtained as linear combinations of values of v_1p_* and v_2p_* at the grid point neighbors (not necessarily entirely interior) of P' in S using the interpolation coefficients in Table 4. Denoting, as usual, the number of equal subdivisions of the grid from pole to equator by n, in accordance with (b) above, (50) and (51) yield:

$$(u_1p_*)_P = \frac{1}{n^2}\{[(i')^2 - (j')^2](v_1p_*)_{P'} + 2i'j'(v_2p_*)_{P'}\}$$

$$(u_2p_*)_P = \frac{1}{n^2}\{-2i'j'(v_1p_*)_{P'} + [(i')^2 - (j')^2](v_2p_*)_{P'}\}. \tag{105}$$

The formulas for evaluating v_1p_* and v_2p_* at exterior grid point P in S are obtained by interchanging in (105) u_1p_* with v_1p_* and u_2p_* with v_2p_*, P' then denoting the interior image in N of P.

In the simultaneous equations case, the development of the formulas is completely analogous to the scalar simultaneous equations case, except for the additional operations (b) above using (50) and (51). The resulting formulas are:

$$(u_1p_*)_{P_1} = \frac{\mathfrak{D}_{P_1'}\left[\frac{(i')^2 + (j')^2}{n}\right]^2 (S_{u_1p_*})_{P_2'} + [(i')^2 - (j')^2](S_{v_1p_*})_{P_1'} + 2i'j'(S_{v_2p_*})_{P_1'}}{n^2 - (\mathfrak{D}_{P_1'})^2\left[\frac{(i')^2 + (j')^2}{n}\right]},$$

$$(u_2p_*)_{P_1} = \frac{\mathfrak{D}_{P_1'}\left[\frac{(i')^2 + (j')^2}{n}\right]^2 (S_{u_2p_*})_{P_2'} - 2i'j'(S_{v_1p_*})_{P_1'} + [(i')^2 - (j')^2](S_{v_2p_*})_{P_1'}}{n^2 - (\mathfrak{D}_{P_1'})^2\left[\frac{(i')^2 + (j')^2}{n}\right]^2} \tag{106}$$

where

$$P_1 = (i\Delta s, j\Delta s, Q_K) \quad \text{in} \quad N,$$

$$P_2 = (-i\Delta s, j\Delta s, Q_K) \quad \text{in} \quad S,$$

$$(S_{u_1p_*})_{P_2'} = \mathfrak{A}_{P_1'}(u_1p_*)_{\mathfrak{A}P_2'} + \mathfrak{B}_{P_1'}(u_1p_*)_{\mathfrak{B}P_2'} + \mathfrak{C}_{P_1'}(u_1p_*)_{\mathfrak{C}P_2'},$$

and similarly for the other subscripted S terms. The formulas for evaluation of v_1p_* and v_2p_* at exterior grid points P_2 are obtained by interchanging u_1p_* with v_1p_*, u_2p_* with v_2p_*, P_1 with P_2, and P_1' with P_2' in (106).

9. Data "flow"

Input

The first stage of data processing performed by the computer program develops initial data and initializes the program from input data stored on magnetic tape. In addition to the initial data which are required to be specified over the grid at time position $t = \tau = 0$ for the basic variables, other data are processed which describe the desired distributions of characteristics of the earth's surface (height, albedo, roughness, soil moisture, sea temperature, snow depth), of the atmosphere (cloud properties, long-wave absorption, solar absorption, carbon dioxide, ozone), and of solar radiation (zenith angle of the sun, duration of daylight). Initially, the atmosphere is assumed to be isothermal and not in motion. Provision was made to enable performance of experiments of different complexity. Thus, input control parameters not only specify the size of the grid, time interval, and frequencies of calculation of radiation, "horizontal" diffusion, and "vertical" diffusion effects, but determine whether moisture and cloud distributions are to be held constant in time or to be predicted.

Table 4 Bivariate linear interpolation coefficients for interior image point P′.

Coefficients for P′	*Relative positions of associated grid point neighbors of P′*
$\mathfrak{A}_{P'} = \mathfrak{s}_{P'}\mathfrak{J}_{P'}$	Nearest origin
$\mathfrak{B}_{P'} = (1 - \mathfrak{s}_{P'})\mathfrak{J}_{P'}$	Has same ordinate as that nearest origin
$\mathfrak{C}_{P'} = \mathfrak{s}_{P'}(1 - \mathfrak{J}_{P'})$	Has same abscissa as that nearest origin
$\mathfrak{D}_{P'} = (1 - \mathfrak{s}_{P'})(1 - \mathfrak{J}_{P'})$	Most distant from origin

Table 5 Data in high-speed storage used in computing $M_{i,N}^{\tau+1}$

$(M, I)_{i+2,N}^{\tau-1}$		
$(M, I)_{i+1,N}^{\tau-1}$	$(M, I, L)_{i+1,N}^{\tau}$	
$(M, I)_{i,N}^{\tau-1}$	$(M, I, L)_{i,N}^{\tau}$	$M_{i,N}^{\tau+1}$
$(M, I)_{i-1,N}^{\tau-1}$	$(M, I, L)_{i-1,N}^{\tau}$	
$(M, I)_{i-2,N}^{\tau-1}$		

Storage and data manipulation during numerical integration

The magnitude of the data requirements is indicated by the fact that, for the maximum size grid treated ($n = 40$, nine Q-levels), approximately 11,000 values of p_* and 395,000 values of the other basic variables collectively are computed from the difference equations and interpolation for each time point (cf. Table 3). Because of limited space in the high-speed magnetic core storage, only a small portion of these data for each of the three successive time points involved in the difference equations is contained in this storage at any instant of time, while all of these data for three time points are simultaneously present in the intermediate-speed magnetic disk storage. The storage requirements are further enlarged by non-time-dependent pointwise data (e.g., coefficients in the difference equations, height of the earth's surface), by pointwise variables computed periodically in time (e.g., land surface temperature, radiation and diffusion terms), by non-time-dependent non-pointwise data (e.g., tables), and by the instructions and temporary data storage of the computer program.

A complex plan is used for the allocation of storage and the manipulation of data in the computer program. This plan was chosen with the purpose of minimizing the time required for the numerical integration of the basic variables over the maximum size grid.

Denote by $M^{\tau}_{j,N}$ the set of all interior grid point values for the half-level $Q = 1$ of p_*, and for all Q-levels of u_1p_*, u_2p_*, T, and hp_*, at ordinate $y = j\Delta s$ and time point $t = \tau\Delta t$ in N. Denote by $I^{\tau}_{j,N}$ the same set of values except for exterior grid points instead of interior grid points. Let $L^{\tau}_{j,N}$ contain pointwise non-time-dependent data for both interior and exterior grid points and pointwise periodically computed data at interior grid points for ordinate $y = j\Delta s$ at time $t = \tau\Delta t$ (except the "lagged" diffusion terms are at time $t - \Delta t = [\tau - 1]\Delta t$) in N. Similar sets of data for S are denoted by $M^{\tau}_{j,S}$, $I^{\tau}_{j,S}$, and $L^{\tau}_{j,S}$ where $\eta = j\Delta s$.

The calculations required for one time "step" by the difference equations and bivariate linear interpolation may be described as the evaluation of $M^{\tau+1}_{j,N}$, $I^{\tau+1}_{j,N}$, $M^{\tau+1}_{j,S}$, $I^{\tau+1}_{j,S}$, and any needed modification of $L^{\tau-1}_{j,N}$ and $L^{\tau-1}_{j,S}$ to obtain $L^{\tau}_{j,N}$ and $L^{\tau}_{j,S}$, for all pertinent ordinates $y = j\Delta s$ and $\eta = j\Delta s$ of N and S. In fact, a "data flow plan" for these calculations, i.e., a plan of transference of data between the high-speed smaller capacity core storage and the intermediate-speed larger capacity disk storage during the computations, was selected which uses such M, I, and L "planes of data" as "read-write" transference units. The choice of these units seemed natural in view of the grid geometry, the requirements of the discrete formulation, and of the limited capacity of core storage. (The computer program was designed to be operable on a STRETCH installation including 65,536 words of core storage, approximately 2,000,000 words of disk storage, at least five magnetic tapes, card reader, printer, and console.) This choice enables these calculations for one time step to be performed in successive stages as required by high-speed storage space limitations while permitting data transferences between the high-speed storage and the intermediate-speed storage often to be performed simultaneously with the calculations.

The "grid sweep" selected for the calculations produces the output $M^{\tau+1}$ and L^{τ} data in the following order: L^{τ} (if no modification of $L^{\tau-1}$ data is required to obtain the L^{τ} data, no re-computation and no re-"writing" from core to disk of L data are performed), $M^{\tau+1}$ for $y = 0, \Delta s, 2\Delta s, \ldots, 2a, -\Delta s, -2\Delta s, \ldots, -2a$; $\eta = 0, \Delta s, \ldots, 2a, -\Delta s, \ldots, -2a$, where $2a$ is the radius of the circular stereographic representation of the equator (a = mean radius of earth). To simplify exposition of the data flow plan, a basic plan excluding the $I^{\tau+1}$ and modification of $L^{\tau-1}$ will be discussed first.

Table 5 indicates the data in high-speed storage used in the calculation of $M^{\tau+1}_{j,N}$ for a typical ordinate $y = j\Delta s$ in N, but is also valid for a typical ordinate $\eta = j\Delta s$ in S by replacing N by S.

The data $M^{\tau+1}_{j,N}$ (at time $t + \Delta t$) in the right column of the diagram are calculated from both data in middle column (at time t) and data in left column (at time $t - \Delta t$). The L data and the union of data $M \cup I$ are stored in core storage from left to right (in order of increasing abscissa in N and of decreasing abscissa in S) for each ordinate and each time, and the calculation of $M^{\tau+1}$ is performed in the same order. This arrangement of $M \cup I$ data permits the program's referencing of data in the same manner for a neighbor grid point which is exterior as for one which is interior. (For convenience in re-arranging M and I data "read" from the disk to obtain the $M \cup I$ arrangement in core storage, the core storage associated with each of the $M \cup I$ is that for $(M \cup I,{}_LI)$, i.e., $M \cup I$ and a repetition of the portion of I—denoted by ${}_LI$—contained in the left half of the N-Q coordinate space.)

In accordance with the stated order of grid sweep, the calculation proceeds from $M^{\tau+1}_{j,N}$ to $M^{\tau+1}_{j+1,N}$ or to $M^{\tau+1}_{j-1,N}$ for a typical ordinate $y = j\Delta s$ in N depending on whether $y \geq 0$ or $y < 0$, respectively. To avoid the delay which would otherwise occur in commencement of the calculation of $M^{\tau+1}$ for the next ordinate, additional core storage corresponding to one additional entry in each of the three columns of the diagram in Table 5 is used as "buffer" storage for writing on disk from core storage of $\tau + 1$ data or for reading from disk to core storage of τ and $\tau - 1$ data. However, a saving of core storage is made by having $M^{\tau+1}_{j,N}$ share space with $(M \cup I)^{\tau}$ for the previous ordinate ($y - \Delta s = [j - 1]\Delta s$ for $y \geq 0$, $y + \Delta s = [j + 1]\Delta s$ for

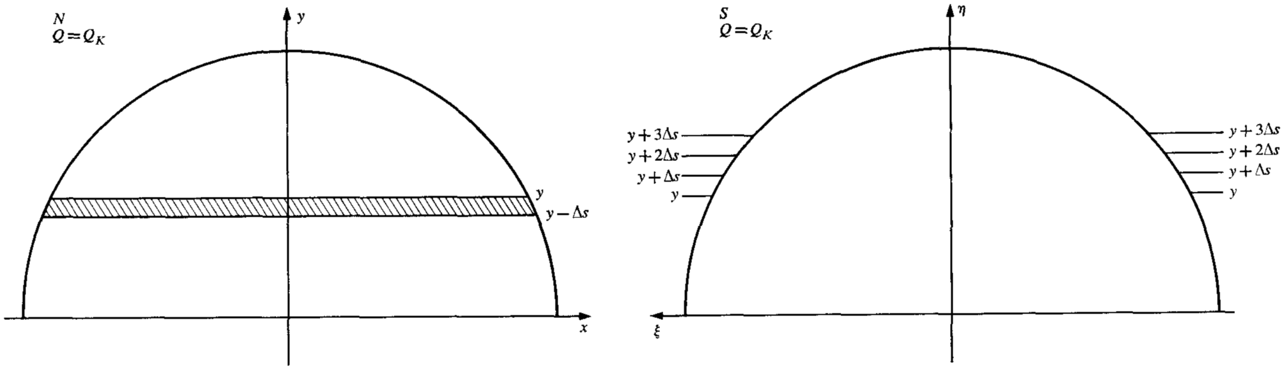

Figure 6 Interior image points in shaded region of N represent same physical points as selected exterior grid points having labelled ordinates in S.

$y < 0$), the data for the leftmost (interior) abscissa of $M_{j,N}^{\tau+1}$ replacing in core storage the data for the leftmost (exterior) abscissa of $(M \cup I)^{\tau}$ at the previous ordinate.

A minor revision of the basic data flow plan described above is necessary if $L^{\tau-1}$ data are modified to obtain L^{τ}. In the case where the calculation proceeds from $M_{j-1,N}^{\tau+1}$ to $M_{j,N}^{\tau+1}$, for example, after the calculation of $M_{j-1,N}^{\tau+1}$ but prior to calculation of $M_{j,N}^{\tau+1}$, the $L_{j,N}^{\tau}$ data are obtained as a result of modifications using data at τ and/or $\tau - 1$ and replace the $L_{j,N}^{\tau-1}$ data in both disk and core storage. The change in the basic plan required in the case where the calculation proceeds from $M_{j+1,N}^{\tau+1}$ to $M_{j,N}^{\tau+1}$, or in the case of S instead of N, is completely analogous. The writing of the L^{τ} data plane on disk from core storage is initiated as soon as feasible after its computation. This change in the basic data flow plan does not necessitate any increase in the core or disk storage requirements. (No additional core storage is required as a buffer for disk writing. Thus, there are in total four L data planes in core storage at any instant of time. There is only one L data plane per ordinate y in N and one per ordinate η in S stored on the disk, in contrast to the three M and I data planes for each typical ordinate in disk storage arising from the three successive time points.)

A more complex change in the basic data flow plan is made to include the development of the $I^{\tau+1}$ data for N and S. One of the reasons for this complexity is that some of the neighbor grid points which surround interior "images" of the exterior grid points are exterior. As the values of the variables at the exterior grid points of N which comprise $I_{j,N}^{\tau+1}$ can not be obtained until after computation of $M^{\tau+1}$ data in S, the $I^{\tau+1}$ data in S which depend upon $I_{j,N}^{\tau+1}$ can not be evaluated completely during the calculation of the $M^{\tau+1}$ data planes in N. The linear combinations used for interpolation, requiring values both at interior and exterior grid points to obtain the $I_{j,S}^{\tau+1}$ data planes, are restricted to values at interior grid points during the calculation of $M_{j,N}^{\tau+1}$. During the course of calculating the $M_{j,S}^{\tau+1}$ data planes, any of the linear combinations of data in the $M^{\tau+1}$ data planes for N which were "incomplete" are "completed" using $I^{\tau+1}$ data for N to obtain the $I_{j,S}^{\tau+1}$ data planes. Also during the course of calculating the $M_{j,S}^{\tau+1}$ data planes, "complete" linear combinations of data in $M^{\tau+1}$ and $I^{\tau+1}$ planes in S are formed to obtain the $I_{j,N}^{\tau+1}$ data planes.

Another reason the interpolation procedure introduces complexity is that the order of generating the $I^{\tau+1}$ data is quite different from their order of usage in subsequent calculations. The order of the generation of $I^{\tau+1}$ data is governed by the arrangement of the interior image points and by the order of calculations of the $M^{\tau+1}$ data planes, and results in the allotment of core storage space for several $I^{\tau+1}$ data regions.

In discussing in further detail the effects of interpolation upon the data flow plan, two properties of the arrangement of interior image points are required. Consider the linear combinations obtained using the data $M_{j,N}^{\tau+1}$ and $M_{j-1,N}^{\tau+1}$ which are simultaneously available in core storage prior to calculation of $M_{j+1,N}^{\tau+1}$ when $y > 0$. Each of these linear combinations involves values at the surrounding interior grid point neighbors of an (interior) image point whose ordinate, denoted by η', satisfies $y - \Delta s < \eta' \leq y$. Then, it may be shown that the ordinate η of the same physical point in S (an exterior grid point) satisfies $y \leq \eta \leq y + 3\Delta s$. (See Fig. 6.)

The second of the two properties of the arrangement of interior image points required is contained in the result that $y \leq \eta \leq y + \Delta s$ if the further restriction is imposed that the image point, having ordinate η' where $y - \Delta s < \eta' \leq y$, lie in a grid square not all of whose vertices are interior. This second property is schematically illustrated in Fig. 7. (Simple examples within the range of

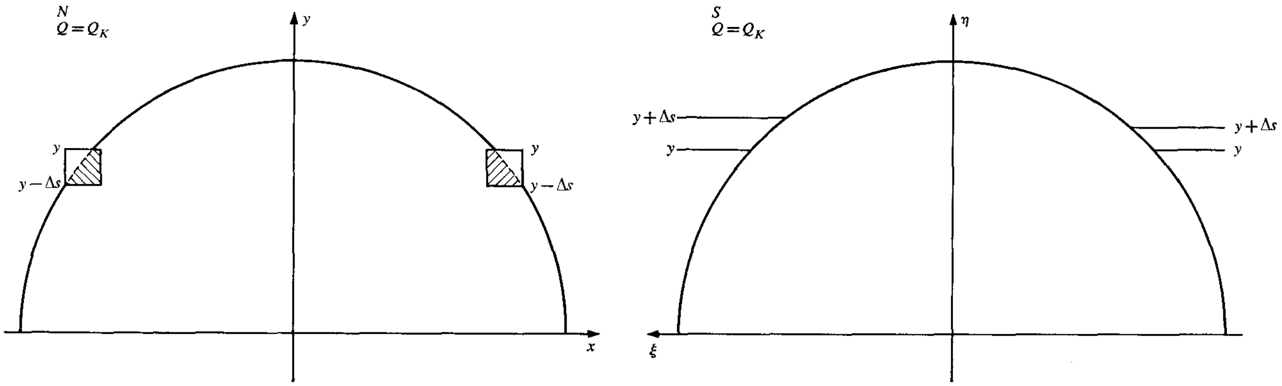

Figure 7 Interior image points in shaded portions of partially exterior grid squares in N represent same physical points as selected exterior grid points having labelled ordinates in S.

interest for n establish that these inequalities can not be sharpened by replacement of $3\Delta s$ with $2\Delta s$ in the first result or by elimination of Δs in the second.)

Entirely analogous results to those illustrated in Figs. 6 and 7 for $y > 0$ hold for $y < 0$ due to the symmetrical arrangement of the image points in N with respect to $y = 0$, and also by symmetry for y interchanged with η and N with S. In the special case $y = 0$ (or $\eta = 0$), the only image points for which the variables in $M^{\tau+1}$ are processed are those four for each Q_K having ordinate $\eta' = 0$ (or $y' = 0$), arising from the four exterior grid points on $\eta = 0$ (or $y = 0$) of coordinate plane $Q = Q_K$. In this special case, the linear combinations do not require "completion" using exterior data.

Only the first of these two properties of the arrangement of interior image points affects the data flow plan during the course of the $M^{\tau+1}_{j,N}$ calculations. The effect may be illustrated by considering a typical ordinate $y > 0$. Then, after the calculation of all linear combinations required from $M^{\tau+1}_{j-1,N}$ and $M^{\tau+1}_{j,N}$ where $y = j\Delta s$ by image points having ordinates η' satisfying $y - \Delta s = (j-1)\Delta s < \eta' \leq y = j\Delta s$, additional core storage is occupied by the resulting generally incomplete $I^{\tau+1}$ data planes denoted by ${}^4I^{\tau+1}_{j,S}$, ${}^3I^{\tau+1}_{j+1,S}$, ${}^2I^{\tau+1}_{j+2,S}$ and ${}^1I_{j+3,S}$. The pre-superscript notation is used to indicate that each of these sets of data may be updated, corresponding to incrementing the pre-superscript, during the calculation of the linear combinations for interpolation following the computation of the $M^{\tau+1}$ data planes succeeding that just computed. Although the ${}^4I^{\tau+1}_{j,S}$ data comprise linear combinations which are not in general entirely "complete", it follows from $0 < \eta' < \eta$ that no further $M^{\tau+1}$ data in N will cause modification of ${}^4I^{\tau+1}_{j,S}$. Consequently, as soon as feasible after calculation, the ${}^4I^{\tau+1}_{j,S}$ data are written on the disk from core storage. One additional $I^{\tau+1}$ core storage region is assigned in order to avoid possible delay in commencement of the next $I^{\tau+1}$ calculations following the computation of $M^{\tau+1}_{j+1,N}$. (Such delay would prevent re-usage of the core storage occupied by ${}^4I^{\tau+1}_{j,S}$ until the writing of ${}^4I^{\tau+1}_{j,S}$ on disk is finished.) An entirely analogous change in the data flow plan is introduced to handle interpolation during the course of calculating $M^{\tau+1}_{j,N}$ for a typical ordinate $y = j\Delta s < 0$.

Consider next the development of $I^{\tau+1}$ data during the computation of the $M^{\tau+1}_{j,S}$ data planes. Without use of the $I^{\tau+1}$ data obtained during the calculation of the $M^{\tau+1}_{j,N}$ data planes, the linear combinations for interpolation by analogy would produce in core storage ${}^4I^{\tau+1}_{j,N}$, ${}^3I^{\tau+1}_{j+1,N}$, ${}^2I^{\tau+1}_{j+2,N}$, and ${}^1I^{\tau+1}_{j+3,N}$ following computation of $M^{\tau+1}_{j,S}$ for a typical ordinate $\eta > 0$. However, from $0 < y' < y$, $0 < \eta' < \eta$, $y = \eta = j\Delta s$, it follows that, with appropriate use of the $I^{\tau+1}$ data obtained during the $M^{\tau+1}$ calculations for N, the data comprising entirely "complete" linear combinations $I^{\tau+1}_{j,N}$ and $I^{\tau+1}_{j,S}$ can be computed at this stage, rendering unnecessary the notation "${}^4I^{\tau+1}_{j,N}$". Thus, all linear combinations contained in ${}^3I^{\tau+1}_{j+1,N}$, ${}_2I^{\tau+1}_{j+2,N}$ and ${}^1I^{\tau+1}_{j+3,N}$ can also be "completed", but the pre-superscript notation is retained to indicate that in general these sets, in containing only data for image points having positive ordinates less than or equal $\eta = y$, do not contain data for all required exterior grid points. By the second of the two arrangement properties of image points, the ${}^4I^{\tau+1}_{j,S}$ data first become subject to updating after computation of $I^{\tau+1}_{j-1,N}$. Consequently, the ${}^4I^{\tau+1}_{j,S}$ data are read from the disk to core storage sufficiently early to be available for modification prior to computing $M^{\tau+1}_{j,S}$. The data resulting from this modification are denoted by ${}^5I^{\tau+1}_{j,S}$. After calculation of $M^{\tau+1}_{j,S}$ where $\eta = j\Delta s$, data ${}^5I^{\tau+1}_{j,S}$ are further updated to obtain $I^{\tau+1}_{j,S}$. As soon as feasible after the calculation of $I^{\tau+1}_{j,S}$ and $I^{\tau+1}_{j,N}$, these data are written from core storage on the disk. Sufficient core storage is allotted for the $I^{\tau+1}$ data to enable its use as buffer storage to avoid delays in disk writing

Table 6 Data in high-speed storage immediately prior to computing $M^{\tau+1}_{j,N}$ for a typical ordinate $y = j\Delta s > 0$ in N.

$(M \cup I)^{\tau-1}_{j+2,N}$		${}^{1}I^{\tau+1}_{j+2,S}$
$(M \cup I, L)^{\tau-1}_{j+1,N}$	$(M \cup I)^{\tau}_{j+1,N}$	${}^{2}I^{\tau+1}_{j+1,S}$
$(M \cup I, L)^{\tau-1}_{j,N}$	$(M \cup I)^{\tau}_{j,N}$	${}^{3}I^{\tau+1}_{j,S}$
$(M \cup I)^{\tau-1}_{j-1,N}$	$(M \cup I, L)^{\tau}_{j-1,N}$	$(M_{,N}, {}^{4}I_{,S})^{\tau+1}_{j-1}$
$(M \cup I)^{\tau-1}_{j-2,N}$		

Table 7 Data in high-speed storage immediately prior to computing $M^{\tau+1}_{j,S}$ for a typical ordinate $\eta = j\Delta s > 0$ in S.

$(M \cup I)^{\tau-1}_{j+2,S}$		${}^{1}I^{\tau+1}_{j+2,N}$
$(M \cup I, L)^{\tau-1}_{j+1,S}$	$(M \cup I)^{\tau}_{j+1,S}$	$({}^{2}I_{,N}, {}^{4}I_{,S})^{\tau+1}_{j+1}$
$(M \cup I, L)^{\tau-1}_{j,S}$	$(M \cup I)^{\tau}_{j,S}$	$({}^{3}I_{,N}, I_{,S})^{\tau+1}_{j}$
$(M \cup I)^{\tau-1}_{j-1,S}$	$(M \cup I, L)^{\tau}_{j-1,S}$	$(M_{,S}, I_{,N}, I_{,S})^{\tau+1}_{j-1}$
$(M \cup I)^{\tau-1}_{j-2,S}$		

and reading of $I^{\tau+1}$ data. The change in the data flow plan due to interpolation during the course of the $M^{\tau+1}_{j,S}$ calculations for a typical ordinate $\eta = j\Delta s < 0$ is completely analogous to that described for $\eta = j\Delta s > 0$.

Tables 6 and 7 indicate the contents of core storage immediately prior to computing $M^{\tau+1}_{j,N}$ and $M^{\tau+1}_{j,S}$ for typical ordinates $y = j\Delta s > 0$ and $\eta = j\Delta s > 0$, including the effects upon core storage of $I^{\tau+1}$ calculations and of updating $L^{\tau-1}$ data which occurs only during some time steps, but excluding additional buffer storage.

The amount of core storage allotted for the M, I, and L data planes, including buffer storage, in order to perform the calculations over the entire grid is that for eleven regions of $(M \cup I, {}_{L}I, I^{\tau+1})$ and four regions of L data. The size of each of these two types of regions is determined from consideration of the zero ordinate position where the space required is a maximum. The $M \cup I$, L, and $I^{\tau+1}$ data planes—although each is typically used in the same or updated form but in a different role during the calculations associated with each of several successively computed $M^{\tau+1}$ data planes—are not moved within core storage. Rather, in the progression from the calculations associated with one $M^{\tau+1}$ data plane to the next, the changing roles are obtained by cyclically permuting the reference addresses—within each of several sets of addresses—used during calculation, reading from the disk to core storage, and writing on the disk from core storage.

Output

In addition to identifying information (e.g., experiment number, values of input parameters, time and space positions at selected stages of the calculation), essentially two different types of data are provided (periodically or subject to manual control) as output at the printer and/or on magnetic tape. One type of output consists of the values of pointwise basic or intermediate variables. The other type consists of properties "in-the-large" obtained from approximate integrations (summations) of functions over various spatial regions for a fixed time point. Among the properties "in-the-large" computed are relative and absolute angular momentum, kinetic energy, potential energy, and air mass. Both of these types of output, but especially that consisting of properties "in-the-large," are helpful in the monitoring and interpretation of experiments performed with the computer program.

• *10. Refinement: A procedure for increasing "horizontal" resolution*

Subject to the maximum permissible number n of intervals from pole to equator being 40, a special procedure contained in the computer program called "refinement" may be used to compute initial data for calculations with $n = 2n_1$ from results obtained with $n = n_1$. E.g., a numerical integration of the basic variables over the earth from time $t = 0$ to $t = t_f$ might be accomplished in four stages of calculations: (i) from $t = 0$ to $t = t_1$ with $n = 5$, (ii) from $t = t_1$ to $t = t_2$ with $n = 10$, (iii) from $t = t_2$ to $t = t_3$ with $n = 20$, and (iv) from $t = t_3$ to $t = t_f$ with $n = 40$. At the commencement of each of the last three stages, some of the input parameters, including time interval Δt, could be changed, a bivariate linear interpolation of the basic variables would enable assignment of the values of these variables for the commencement time at the newly introduced spatial points, and the non-time-dependent pointwise data would be recomputed or supplemented from additional input data on magnetic tape for the new grid. The height z_* at each newly introduced point of the earth's surface is gradually adjusted from an initial value obtained from bivariate linear interpolation to the proper final value during the calculations performed for subsequent time positions.

Concluding remarks

The Global Weather Simulator contains more than 15,000 STRETCH instructions, and is comprised of five sections which share approximately 10,000 64-bit words of the high-speed magnetic core storage. The remainder of the 65,536 words of core storage is available for data.

The program sections include: GWS0 ("Loader," automatic restart from data on disk, processor of interrupts and input-output operations), GWS1 (Processor of input data), GWS2 (Calculations and data manipulations for numerical integration), GWS3 (Development of intermediate output on magnetic tape, and restart from such tape output), and GWS4 (Refinement). The intermediate tape output also serves as input for "out-of-stream" analyses with independent programs for development of final output in the required wide variety of graphically displayed forms.[1,2]

Preliminary studies were performed on the IBM 7090 computer with programs for developing approximate solutions of Poisson's equation and of the "wave equation" on the surface of a sphere. The same forms of stereographic mapping and "equatorial" interpolation were used in these studies as in the Global Weather Simulator. The comparative effectiveness of several types of differencing was evaluated with tentative conclusions drawn concerning the convergence, stability, and accuracy of the several differencing methods considered. The choice of a time interval Δt for early experiments with the Global Weather Simulator was based partly upon the "wave equation" experience, and partly upon the experience of others with simpler weather models (e.g., see Smagorinsky,[10] p. 461). The dependence of the usual choice of time step upon the "horizontal" spacing of grid points, in both the preliminary experiments for the global and hemispherical models and the later hemispherical experiments,[1,2] is shown in Table 8. The initial data for the time-dependent variables, the geophysical data describing the earth and its atmosphere, and the frequencies of calculating radiation and diffusion terms, were chosen by the Weather Bureau. The "running time" of this Simulator program is limited by the speed of STRETCH, on which the approximate time required for a floating-point multiplication is 2.7 μsec. The running time also depends in a complicated way upon the size of the grid, the choice of certain formulation options (e.g., the frequency selected for the highly complex radiation calculation), and the amount and frequency of the generation of output information. It was estimated that a basic global experiment for n in Table 8 equal to 40 would require 18 hours of calculation on STRETCH per atmosphere day. Table 9 provides timing information based upon actual hemispherical experiments for a "dry" model (having "moist convective adjustment") with $n = 40$. For a discussion of hemispherical experiments with the program, see J. Smagorinsky, S. Manabe, J. L. Holloway, Jr., and R. F. Strickler (Refs. 1 and 2).

With reference to Table 9, the "initializations" section required execution only once per "run"; the remaining sections were typically executed at every time point, except only once per 72 time points for the "radiation" section. On this basis, the entire program comprising these sections requires approximately 10.6 hours of STRETCH calculation per atmosphere day for the values of n and Δt specified in Table 9.

Table 8 Time step Δt for Global Weather Simulator experiments for nine Q-Levels (n = number of equal subdivisions from pole to equator in stereographic grid).

n	5	10	20	40
Δt (in minutes)	40	20	10	5

Table 9 Approximate timing on STRETCH for the program sections of a nine-level "dry" hemispherical model with $n = 40$ and $\Delta t = 5$ minutes.

Section of program	*Execution time per time point (in seconds)*
Initializations	114
"Inner loop"	45
Vertical diffusion	28
Horizontal diffusion	33
Integrals	20
Radiation	455

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