S. V. Parter*

Elliptic Equations†

Abstract: A brief review is given of the literature on numerical methods for elliptic problems as related to the ideas introduced by Courant, Friedrichs and Lewy in the fundamental paper published in *Math. Ann.* 100, 32 (1928). The discussion shows how the finite-difference methods have subsequently been extended and applied to higher order elliptic problems and how the recent literature has reported methods for solving large algebraic systems. In addition, there is a discussion of the relation between the Dirichlet problem and the problem of the random walk.

Introduction

The fundamental paper of Courant, Friedrichs, and Lewy[6]** (which we denote by C-F-L) contained an extensive and thorough discussion of finite-difference methods for elliptic problems. While the actual results of this remarkable paper are of value in themselves, the ideas introduced and developed there have been fundamental to the field ever since. For example, the finite-difference equations are obtained from the *variational problem* (Dirichlet's principle) related to the boundary value problem. Moreover, they anticipated Sobelev's lemma[22] by proving a finite-difference variant for two dimensions.‡

In addition, the connection between the Dirichlet problem and the *problem of the random walk*[24] is discussed.

Boundary-value problems

One of the major problems discussed is the Dirichlet problem for Laplace's equation for a bounded plane region with a smooth boundary. That is,

$$\begin{cases} \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0, \\ u(x, y) = f(x, y), \qquad (x, y) \in \Gamma. \end{cases} \tag{1}$$

It is assumed that $f(x, y) \in C(\bar{G})$ and

$$\iint_G \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right] dxdy < \infty . \tag{2}$$

The C-F-L paper proceeds as follows. The finite-difference analog of Eq. (1) is set down and its relationship to the minimum problem is discussed. The discrete *Green's identities* show that there exists a unique solution of this algebraic problem. Using the basic lemma on the interior equicontinuity of the solutions of this problem (the discrete Sobelev's lemma) and the Ascoli-Arzela compactness theorem, one obtains a sequence $U_{h_n}(x, y)$ which converges (as $h_n \to 0$) to a harmonic function $u(x, y)$. The convergence is uniform on every compact subdomain. Finally,

$$\frac{1}{r} \iint_{S_r} [u - f]^2 \, dx\, dy \to 0 \quad \text{as} \quad r \to 0, \tag{3}$$

where S_r is a "boundary strip" of width r. Moreover, this uniquely characterizes the solution of Eq. (1). Therefore one can dispense with the selection of subsequences and the family of solutions $\{U_h(x, y)\}$ converges to $u(x, y)$ uniformly on every compact subdomain. This argument is then extended to the first boundary-value problem for the biharmonic equation.

In his Göttingen dissertation W. v. Koppenfels[15] extended these results to the general second-order elliptic equation with "smooth" coefficients.

In 1941 I. G. Petrowskii[20] modified the basic C-F-L approach to the Dirichlet problem. In particular, he used the notion of a "barrier" function (see Ref. 7, pp. 306–312) to improve the treatment "near" the boundary. For example: if, at every point $P \in \Gamma$ there is a circle S such that $\bar{S} \cap \bar{G} = \{P\}$, then the convergence of $U_h(x, y)$ to $u(x, y)$ is uniform on $\bar{G}$. Moreover, this approach allows one to dispense with the requirement of (2).

In 1930 S. Gerschgorin[12] gave another treatment of the Dirichlet problem. His approach is easily extended to more general second order equations and more general finite-difference equations. Moreover, if $U(x, y) \in C^4(\bar{G})$, the Gerschgorin treatment gives an estimate of the "rate of convergence."

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** Superscripts refer to the bibliography at the end of the paper.

‡ The continuous version of Sobelev's lemma–although not in its sharpest form–was well known in Göttingen (see, e.g., K. O. Friedrichs' dissertation, *Math. Ann.*, 1926). Already in 1912 Haar had observed that the Rayleigh-Ritz procedure converges in the maximum norm for the plate equation because functions of two variables can be bounded by the L^2 norm of their second partial derivatives.

In 1940 Sobelev himself[23] gave a general finite-difference analog of the Sobelev lemma.

In 1953 L. Bers[2] used the Gerschgorin approach for "mildly nonlinear" elliptic problems. In the same year J. L. Walsh and D. M. Young[27] studied the Dirichlet problem in a rectangle—the so-called "model" problem—and showed that the convergence of $U_h(x, y)$ to $U(x, y)$ can be arbitrarily slow.

In a recent series of papers (see Refs. 3–5) J. Bramble and B. Hubbard have extended, for sufficiently smooth solutions $U(x, y)$, the Gerschgorin treatment of the rate of convergence.

More recently this author and P. Jamet[14] have applied the basic C-F-L approach together with the "barrier" idea to second-order elliptic operators whose coefficients become singular on a portion of the boundary.

W. Littman[17] (1958), V. K. Saulev[21] and V. Thomée[25] have studied higher order elliptic problems making use of many of the C-F-L ideas (in particular, the Sobelev inequalities).

The algebraic problem

While C-F-L established that the system of finite-difference equations possess a unique solution and mentions some early work on relaxation methods, it contains no results on methods for solving these large algebraic systems. As the actual computation of such solutions became important, many authors turned to a more extensive discussion of this problem.

Let A be an n by n nonsingular matrix and consider the problem

$$A\mathbf{X} = \mathbf{Y}. \tag{4}$$

Let $A = P - N$ be a "splitting" of A, where P^{-1} exists and is "easily" obtained. Consider the "direct iterative method"

$$P\mathbf{X}_{\nu+1} = N\mathbf{X}_\nu + \mathbf{Y}. \tag{5}$$

It is well known that the vectors $\mathbf{X}_\nu$ converge to the solution X of Eq. (4) if and only if the spectral radius ρ of $P^{-1}N = K$ satisfies

$$\rho < 1. \tag{6}$$

In 1949 H. Geiringer[11] discussed certain "structural" properties of K which would guarantee (6). In 1950 S. P. Frankel[9] studied the case where Eq. (4) arises from the finite-difference equations for the model problem. He obtained explicit formulas for $\rho = \rho(h)$ when the iterative method was the Jacobi method, the Gauss-Seidel method, and the extrapolated Gauss-Seidel method (Successive Over-Relaxation, or SOR).

In 1954 D. M. Young[30] gave a general theory of the relationship of those methods based on "Property A". Let the matrix A satisfy Property A and be "consistently ordered." Let μ be any eigenvalue of B, the associated Jacobi matrix. Then $-\mu$ is also an eigenvalue of B. Let $\omega > 0$ be the parameter of an SOR method. Let $\lambda \neq 0$. Then λ is an eigenvalue of this SOR method if and only if

$$\frac{\lambda + \omega - 1}{\omega\lambda^{1/2}} = \mu \tag{7}$$

for some μ (eigenvalue of B). The Gauss-Seidel method is given by $\omega = 1$.

This theory was then extended to "block" iterative methods by R. J. Arms, L. D. Gates and B. Zondek[1] and B. Friedman.[10]

This approach to the algebraic convergence problem has been extended by R. S. Varga[26] who related it to the Perron-Frobenius theory of non-negative matrices. Varga has used his theory of "regular splitting" to develop general comparison theorems. He has also obtained estimates for $\rho(h)$ in the model problem.

More recently the present author[19] has used the basic compactness ideas of C-F-L to establish a general asymptotic formula for $\rho(h)$ when the matrix A arises from a self-adjoint, uniformly elliptic, second-order operator L. In fact, the method has also been applied to higher order equations.[18,25] Suppose that $A = A(h)$ and $P = P(h)$ are both positive definite. Suppose the splitting $A = P - N$ satisfies "Property B"—loosely speaking this means that the matrices $N(h)$ are uniformly bounded and there is a smooth function $Q(x, y) \geq Q_0 > 0$ such that

$$\Delta x\ \Delta y(N\hat{\mathbf{\Phi}}, \hat{\mathbf{\Psi}}) \to \iint Q(x, y)\Phi(x, y)\Psi(x, y)\ dx\ dy$$

when $\hat{\mathbf{\Phi}}, \hat{\mathbf{\Psi}}$ are the vectors obtained from point evaluation of the continuous functions $\Phi(x, y)$, $\Psi(x, y)$.

Then, under some minor additional conditions,

$$\rho(h) = 1 - \Lambda\ \Delta x\ \Delta y + \rho(\Delta x\ \Delta y), \tag{8}$$

where Λ is the minimal eigenvalue of

$$\begin{cases} LU + \Lambda Q U = 0, \\ U = 0 \quad \text{on} \quad \Gamma. \end{cases} \tag{9}$$

Concluding remarks

The literature on numerical methods for elliptic problems is now so vast that it proves impossible, in any modest space, to mention all important papers in the field. Thus, the specific papers mentioned in this brief sketch are ones permitted by space limitations, immediate relevance to the subject at hand, and this author's knowledge of the literature. Additionally, we wish to mention, in particular, the "alternating direction iterative methods" for the algebraic problem discussed by Olof Widlund[29] and the work of H. F. Weinberger[28] on the elliptic eigenvalue problem. Further, the interested reader will find excellent bibliographies in Refs. 8, 13, 16, and 31.

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