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A Numerical Analysis of the Transient Behavior of a Transistor Circuit*

Abstract: This paper describes some difficulties encountered in the numerical solution of nonlinear circuit equations. A particular transistor circuit is analyzed to illustrate the nature of the difficulties and how they may be resolved. In this circuit it is possible, without sacrificing accuracy of the physical model, to eliminate unimportant stray parameters whose presence destroys the efficiency of most integration routines. A method based on a potential function is used for deriving the circuit equations and it is shown how these equations can be systematically reduced upon removing the stray parameters. Application of such techniques to the circuit considered reduced the calculation time (on an IBM 7094) from 30 minutes to 7 seconds.

Introduction

The purpose of this paper is to describe some results and observations obtained during the investigation of the transient behavior of a transistor circuit. Although only one particular circuit was analyzed, it is felt that understanding this example will illuminate many of the difficulties encountered in the general case.

In the circuit considered, a nonlinear model is used for the transistor. The problem is to numerically integrate the nonlinear ordinary differential equations in a reasonable amount of time and with reasonable accuracy. If the problem is approached in a straightforward manner, certain difficulties arise. One is that it takes too long to do the integration (about thirty minutes on the IBM 7094); the second is that, in some cases, the calculated response of the simulated circuit would apparently reach an equilibrium point which was not substantiated by experimental evidence.

A thorough understanding of the cause of these difficulties and of methods for overcoming them allowed computation time on an IBM 7094 to be reduced from 30 minutes to 7 seconds, with an accuracy of about 1%.

Thus, others involved in circuit analysis are likely to find it useful to become aware of these kinds of computational difficulties and their resolution. It will be seen that certain of the mathematical ideas used in the study represent an application of previous theoretical work on nonlinear circuits (see, for example, Ref. 1). During this study, also, some slight extensions of the theory of nonlinear networks were obtained; these make it possible (a) to derive the differential equations of transistor circuits from a potential function and (b) to give a uniform method of reducing the circuit equations when some resistors are set equal to zero.

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The circuit and the circuit equations

The specific circuit under investigation is shown in Figure 1, and a detailed description of the components and their numerical values is given in Appendix 1. The equivalent circuit model of the transistor is also shown in Fig. 1, enclosed in dashed lines. Now, although it is certainly possible to derive the circuit equations using customary methods, we choose to use the potential function approach in order to illustrate its utility. For this purpose we define a function P as follows:

$$\begin{aligned} P(v_1, v_2, v_3, v_4, v_5, v_6) &= \tfrac{1}{2}G_L(V_{CC} - v_3 - v_4)^2 + \tfrac{1}{2}G_p(v_3 + v_4)^2 \\ &+ \tfrac{1}{2}G_S(v_3 + v_4 - v_1)^2 + \tfrac{1}{2}G_p(V_{in} - v_2 - v_4)^2 \\ &+ \tfrac{1}{2}G_B v_2^2 + \tfrac{1}{2}G_T v_4^2 + \tfrac{1}{2}G_{CC}(v_3 + v_4 - v_5 - v_6)^2 \\ &+ \tfrac{1}{2}G_{BB}(v_4 - v_6)^2 + \int_0^{v_5} I_S(v)\,dv \\ &+ \int_0^{v_6} I_E(v)\,dv + Iv_6 + I_1 v_5, \end{aligned}$$

where the G's are the reciprocals of the corresponding R's.

The circuit equations are given by*

$$C\dot{v} = -P_v\Big|_{\substack{I=\alpha_i I_S(v_5)\\ I_1=\alpha(v_6)I_E(v_6)\\ G_{BB}=G_{BB}(v_6)}}$$

where

$$C = \begin{bmatrix} C_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_5(v_5) & 0 \\ 0 & 0 & 0 & 0 & 0 & C_6(v_6) \end{bmatrix}, \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix};$$

and where the superscript dot (as in $\dot{v}$) represents differentiation with respect to time. Explicitly we may put the circuit equations in the following form:

$$C\dot{v} = Gv + f,$$

where

$$f = \begin{bmatrix} 0 \\ G_g V_{in} \\ G_L V_{CC} \\ G_g V_{in} + G_L V_{CC} \\ 0 \\ 0 \end{bmatrix},$$

$$G = \begin{bmatrix} L & M \\ M^T & N(v_5, v_6) \end{bmatrix},$$

$$L = \begin{bmatrix} -G_S & 0 & G_S \\ 0 & -(G_g + G_B) & 0 \\ G_S & 0 & -B \end{bmatrix},$$

$$M = \begin{bmatrix} G_S & 0 & 0 \\ -G_g & 0 & 0 \\ -B & G_{CC} & G_{CC} \end{bmatrix},$$

$$N = \begin{bmatrix} -(B + G_g + G_T + G_{BB}(v_6)) & G_{CC} & G_{CC} + G_{BB}(v_6) \\ G_{CC} & -(G_{CC} + G_2(v_5)) & -(G_{CC} + \alpha(v_6)G_1(v_6)) \\ G_{CC} + G_{BB}(v_6) & -(G_{CC} + \alpha_i G_2(v_5)) & -(G_{CC} + G_{BB}(v_6) + G_1(v_6)) \end{bmatrix}.$$

In the above, M^T denotes the transpose of the matrix M and

$$B = G_L + G_g + G_S + G_{CC},$$

$$G_2(v_5) = I_S(v_5)v_5^{-1},$$

$$G_1(v_6) = I_E(v_6)v_6^{-1}.$$

At first, one would be tempted to numerically integrate the above equations. The problem is that some of the eigenvalues of the matrix $C^{-1}G$ are very large and negative (of the order of -500 nsec^{-1}) while others are relatively small. In other words, some time constants in the circuit are very small compared to the most significant ones. The source of these small time constants can be located in the three resistances R_S, R_g, and R_{CC}, which range in value from 5Ω to 25Ω, while the other resistances are 250Ω or larger.

In Section 3 we discuss why it is important, for numerical reasons, to eliminate the small time constants. For the present we simply do so by setting the three resistors R_S, R_g, and R_{CC} equal to zero. Three capacitor loops are thereby introduced and the number of differential equations reduced from six to three. It is possible, of course, to obtain the reduced equations either from the circuit or from the original equations; however, the method for doing this is not always straightforward and systematic. Using the potential function approach, there is a natural way of obtaining this reduction, the basis of which can be found in Appendix 2. Here, we merely state the results and apply them to the system under consideration. The reader not interested in the technique itself can go on to Eq. (2.4) and make a check of its correctness by using Kirchoff's laws; however, we suggest that the reader make use of both methods of deriving Eq. (2.4) in order to appreciate the simplicity of the potential function approach. If one first writes down the six-dimensional equations, it is very difficult then to reduce those equations by letting R_s, R_g, and R_{cc} equal zero. On the other hand, once the potential function has been obtained for R_s, R_g, and R_{cc} not zero, it can be used to derive either the six-dimensional equations or the reduced ones. The potential function method will display even greater advantage in the case of more complicated networks containing many capacitor loops and inductor cut-sets.

* The notation $P_v|_{a=b(v)}$ stands for the gradient of P evaluated at $a = b(v)$, where a is some parameter appearing in $P(v)$.

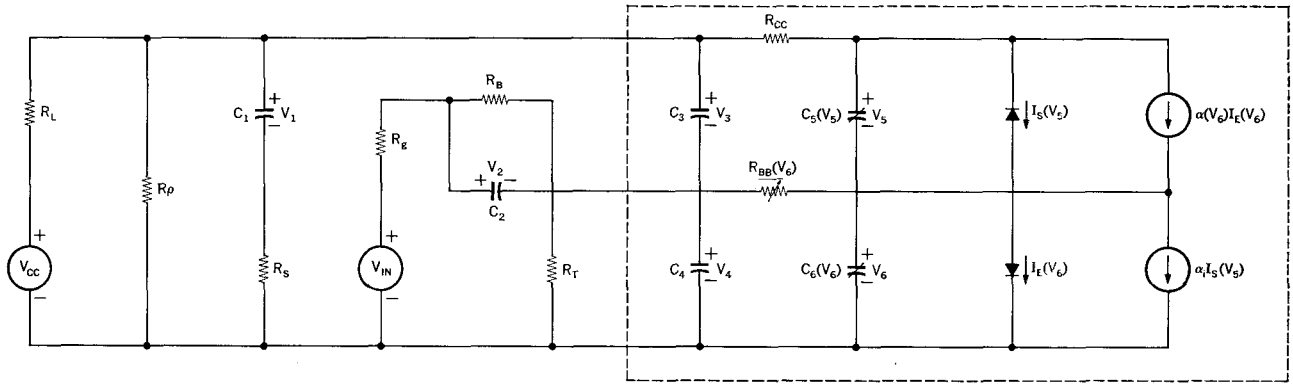


Figure 1 Diagram of the transistor circuit. Components and values are as listed in Appendix 1.

The reduction we seek comes about because the unknown variables are not independent; that is, there is a relation

$$\tilde{v} = A\hat{v} + b(t), \tag{2.1}$$

where $v = \begin{pmatrix} \tilde{v} \\ \hat{v} \end{pmatrix}$. The equations in terms of v have the form $C\dot{v} = -P_v(v)$, where C is a diagonal matrix which we write in block form

$$C = \begin{bmatrix} \tilde{C} & 0 \\ 0 & \hat{C} \end{bmatrix}.$$

However, using Eq. (2.1), the results derived in Appendix 2 give the reduced equations as

$$C^*\dot{\hat{v}} = -\hat{P}_{\hat{v}}(\hat{v}) - A^T\tilde{C}\dot{b}(t),$$

where $\hat{P}(\hat{v}) = P(v)\,|_{\tilde{v}=A\hat{v}+b(t)}$ and $C^* = \hat{C} + A^T\tilde{C}A$.

To obtain this reduction in this case, we write

$$v = \begin{bmatrix} \tilde{v} \\ \hat{v} \end{bmatrix},$$

where

$$\tilde{v} = \begin{bmatrix} v_1 \\ v_2 \\ 3 \end{bmatrix}, \quad \text{and} \quad \hat{v} = \begin{bmatrix} v_4 \\ v_5 \\ v_6 \end{bmatrix}.$$

Since $R_S = R_g = R_{CC} = 0$, we have

$$\tilde{v} = A\hat{v} + b(t), \tag{2.2}$$

where

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}, \quad \text{and} \quad b(t) = \begin{bmatrix} 0 \\ V_{in}(t) \\ 0 \end{bmatrix}.$$

Equation (2.2) is found simply by adding the voltages in the capacitor loops that were formed when the small R's are neglected.

The new circuit equations are

$$C^*\dot{\hat{v}} = -\hat{P}_{\hat{v}}\Big|_{\substack{I=\alpha_i I_S(v_5) \\ I_1=\alpha(v_6)I_E(v_6) \\ G_{BB}=G_{BB}(v_6)}} - A^T\tilde{C}\dot{b}(t), \tag{2.3}$$

where*

$$\begin{aligned}\hat{P}(v_4, v_5, v_6) &= P(v_1, v_2, \cdots, v_6)\,|_{\tilde{v}=A\hat{v}+b(t)} \\ &= \tfrac{1}{2}G_L(V_{CC} - v_5 - v_6)^2 + \tfrac{1}{2}G_\rho(v_5 + v_6)^2 \\ &\quad + \tfrac{1}{2}G_B(V_{in} - v_4)^2 + \tfrac{1}{2}G_T v_4^2 + \tfrac{1}{2}G_{BB}(v_4 - v_6)^2 \\ &\quad + \int_0^{v_5} I_S(v)\,dv + \int_0^{v_6} I_E(v)\,dv + Iv_6 + I_1 v_5,\end{aligned}$$

and

$$C^* = \hat{C} + A^T\tilde{C}A, \quad \text{with}$$

$$\hat{C} = \begin{bmatrix} C_4 & 0 & 0 \\ 0 & C_5(v_5) & 0 \\ 0 & 0 & C_6(v_6) \end{bmatrix},$$

$$\tilde{C} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & C_3 \end{bmatrix}.$$

* In general, $P(v)$ can be written as $P(v) = -\sum_\lambda v_\lambda i_\lambda - \sum_\rho \int_0^{v_\rho} i_\rho\,dv_\rho$ where the sum on ρ is taken over all resistors (including sources) in the network, and the sum on λ is taken over all inductors. Since there are no inductors in this network, $P = -\sum_\rho \int_0^{v_\rho} i_\rho\,dv_\rho$.

Writing Eq. (2.3) explicitly, we have the three-dimensional system which has been investigated numerically:

$$\begin{bmatrix} C_2 + C_3 + C_4 & -C_3 & -C_3 \\ -C_3 & C_1 + C_3 + C_5(v_5) & C_1 + C_3 \\ -C_3 & C_1 + C_3 & C_1 + C_3 + C_6(v_6) \end{bmatrix} \begin{bmatrix} \dot{v}_4 \\ \dot{v}_5 \\ \dot{v}_6 \end{bmatrix} = \begin{bmatrix} G_B V_{in} + C_2 \dot{V}_{in} \\ G_L V_{CC} \\ G_L V_{CC} \end{bmatrix}$$

$$+ \begin{bmatrix} -(G_B + G_T + G_{BB}(v_6)) & 0 & G_{BB}(v_6) \\ 0 & -(G_L + G_\rho + G(v_5)) & -(G_L + G_\rho + \alpha(v_6)G_1(v_6)) \\ G_{BB}(v_6) & -(G_L + G_\rho + \alpha_i G(v_5)) & -(G_L + G_\rho + G_{BB}(v_6) + G_1(v_6)) \end{bmatrix} \begin{bmatrix} v_4 \\ v_5 \\ v_6 \end{bmatrix}. \qquad (2.4)$$

For future reference we write the system of Eq. (2.4) in the matrix form: $C^*\dot{v} = \hat{f} + \hat{G}\hat{v}$, where we will be interested in the eigenvalues of the matrix $C^{*-1}\hat{G}$, which vary continuously with $\hat{v}$.

In any difference scheme, it is necessary to obtain $\dot{v}$ explicitly. In order to obtain $\dot{v}$ from Eq. (2.4), one would have to compute $C^*(\hat{v})^{-1}$; for a 3×3 matrix, this is easy enough to do analytically, but for larger matrices it may be necessary to compute $C^*(\hat{v})^{-1}$ at each time step. In either case, it is advantageous to have C^* symmetric. Therefore, an additional advantage of using the potential function to derive the differential equations is that C^* is automatically a symmetric matrix.

Numerical analysis

If one solves an initial value problem for ordinary differential equations numerically, an error is introduced into the calculation at each integration step due to the inaccuracy of the formula. The magnitude of this so-called local truncation error is a measure of the accuracy of the integration formula. The magnitude of the total error depends upon the magnitude of the local truncation errors and their propagation. Even when the local error at each step is small, the total error may become large due to accumulation and amplification of these local errors. This growth phenomenon is called numerical instability.

It is exactly this phenomenon which makes it necessary to reduce the system of differential equations under investigation. To understand this, consider the following single first-order linear differential equation:

$$\frac{dy}{dt} = -\lambda y, \qquad \lambda > 0. \qquad (3.1)$$

With $y(0) = c$, the solution of this equation is $y(t) = ce^{-\lambda t}$. Suppose, however, we try to solve this equation by some one-step numerical integration method with fixed step size h. In any numerical scheme an error is introduced in the calculation of the solution at each time step. Specifically, let $y_k = y_{k(\text{exact})} + \epsilon_k$ at time $t_k = kh$, ϵ_k being the total truncation error. Let r be the polynomial approximation to $e^{-\lambda h}$ (for small λh) due to replacing the differential equation by a difference equation. Then the computed result of one time step is

$$y_{k+1} = ry_k,$$

while the correct solution is

$$y_{k+1(\text{exact})} = e^{-\lambda h}y_{k(\text{exact})} = e^{-\lambda h}(y_k - \epsilon_k).$$

Subtracting, we obtain,

$$\epsilon_{k+1} = (r - e^{-\lambda h})y_{k(\text{exact})} + r\epsilon_k.$$

Clearly, the error ϵ_k will be amplified if $r > 1$, which is possible for sufficiently large λh; at time $t = (n + k)h$, it will have grown by the factor r^n. Thus, meaningful results can be obtained only for $r < 1$. In this case, errors committed in the past die out exponentially, and the total truncation error will be of the order of the local truncation error, $(r - e^{-\lambda h})y_{k(\text{exact})}$. The local truncation error can be interpreted as the error committed by applying the integration formula to the exact solution at the previous point. In the Runge-Kutta method, this error is of order h^5. More precisely, the Runge-Kutta method approximates $e^{-\lambda h}$ by

$$r = 1 + (-\lambda h) + \tfrac{1}{2}(-\lambda h)^2 + \tfrac{1}{6}(-\lambda h)^3 + \tfrac{1}{24}(-\lambda h)^4.$$

One finds that $r < 1$ only when $-2.78 < -\lambda h < 0$.

We note, also, that as $-\lambda h$ decreases to -2.78, r approaches 1 while $e^{-\lambda h}$ approaches 0.062. This means that, while the Runge-Kutta method is actually stable as $-\lambda h$ approaches -2.78, propagation errors die out more slowly and different asymptotic behavior is obtained for the computed solution in the neighborhood of $-\lambda h = -2.78$. Hence, for reasons of accuracy, one should really have $\lambda h < 1$, say.

Finally, we observe that the above remarks are valid for linear multi-step methods also. For these methods the situation is more complex due to the introduction of so-called parasitic solutions to the difference equation.

In summary, we emphasize that in numerical integration, the total truncation error is a superposition of local errors introduced in the various integration steps. The magnitude of total truncation error thus depends on the magnitude of the local errors and on whether the latter are amplified or damped through propagation.

Results

In describing the results we shall first discuss the three-dimensional system which is a limiting case of the original six-dimensional system. Then, we shall discuss the six-dimensional system, briefly indicating why so much computing time must be allowed and why the results may not always be reliable.

A number of fixed step integrating routines were used on the three-dimensional system. The Runge-Kutta method was used with step sizes of 0.25, 0.50, and 1.0 nsec. Also, several Adams predictor-corrector formulas and a formula due to Hamming[2] were tried with step sizes of 0.25 and 0.50. Stable results were obtained for all three step sizes with the Runge-Kutta method. For $h = 0.25$ and 0.50 the results were in good agreement with experimental evidence whereas with $h = 1.0$ the computed response seemed to level off to a different equilibrium value. For $h = 0.50$ both the Adams and Hamming methods were unstable. However, for $h = 0.25$ excellent results were obtained with both methods.

During one of the Runge-Kutta runs, the eigenvalues of the variable matrix $C^{*-1}\hat{G}$ of the three-dimensional system were calculated at each integration step. The eigenvalues were always negative, and the largest negative value, -3.1 nsec^{-1}, occurred at the initial equilibrium point. By analogy with the analysis of Eq. (3.1), one should choose, for stability with the Runge-Kutta method, $h\,|\lambda|_{\max} < 2.78$. This condition is satisfied for $h = 0.25$ and 0.50, but not for 1.0. In fact, the case $h = 1.0$ is on the borderline of numerical stability, and one can expect either an unstable result or a different asymptotic behavior of the numerical solution. The latter was observed. For both the Adams and Hamming formulas it is necessary to choose $h\,|\lambda|_{\max} < C$ for numerical stability. The value of the constant C depends on the formula used and is less than 1.5. Again, for the more general nonlinear problem, the modified criteria using $h\,|\lambda|_{\max}$ gave excellent results.

It was observed that the largest negative eigenvalue of $C^{*-1}\hat{G}$ increased from -3.1 to -0.55 when the circuit was forced by $V_{in}(t)$. Thus, in the transient region, the various methods would have been stable with larger step sizes. However, during periods of fast variation, the local error increases and the step size must be controlled accordingly. Since the methods used are all quite accurate, it was found that the fixed step size, chosen on the basis of the numerical stability requirements, was small enough to guarantee a reasonably small local error throughout. It is interesting that it was more difficult, in terms of step size, to calculate the solution near the equilibrium value than in the transient region.

For the six-dimensional system, only the Runge-Kutta method was used. For step sizes of $h = 0.5$ and 0.01 the calculation was unstable. For $h = 0.0025$, results were obtained that were within 1% of those for the three-dimensional system. A preliminary analysis indicated that the largest negative eigenvalue of the six-dimensional system was approximately -500 nsec^{-1}. Again, the simple criterion $h\,|\lambda|_{\max} < 2.78$ yielded a good estimate for the step size. It seems clear that it is the aspect of numerical stability that is important for this particular problem.

A variable step integration routine has been used on this same circuit by others.[3] In that routine, an error criterion based on the local truncation error was used in choosing the step size. Essentially, this criterion reduced the step size in rapidly varying regions and increased it in slowly varying ones. Our results indicate that since stability, and not local error, is the limiting factor in this problem, and since $|\lambda|_{\max}$ decreased in the rapidly varying region, it would have been possible, in fact, to increase step size in this region.

To indicate why the three-dimensional system is a good approximation to the six-dimensional system, consider an analogy with linear systems. In passing to the three-dimensional system, we deleted three independent solutions associated with eigenvalues less than -100. The remaining three eigenvalues were negative but greater than -3.1. Since the general solution to a six-dimensional linear system has the form $\sum_{i=1}^{6} C_i e^{\lambda_i t}$, where the λ_i are the eigenvalues, the terms associated with large negative eigenvalues die out quickly and, therefore, can be ignored. Hence, the difference between solutions of the three-dimensional and six-dimensional systems should be negligible. It should be pointed out that there is a difference in the equilibrium solution between the three- and six- dimensional models. However, this error is roughly of the order of the ratio between the resistors neglected and those retained. In any specific problem it would be necessary to determine if this error is acceptable.

The computing time for the three-dimensional system was about 7 seconds while that for the six-dimensional system was about 30 minutes. This reduction is largely accounted for by the 200-fold (0.5/0.0025) increase in step size. Thus, by passing to the three-dimensional system, one can obtain equally good results and compute approximately 250 times faster.

Figures 2 and 3 are graphs of the voltages v_5 and v_6. Superimposed on Fig. 2 is the input voltage $V_{in}(t)$. In each of these figures we have plotted the (graphically identical) output of both the reduced and the full matrix system. The full system was computed only for 80 nsec because the integration time took 11 minutes.

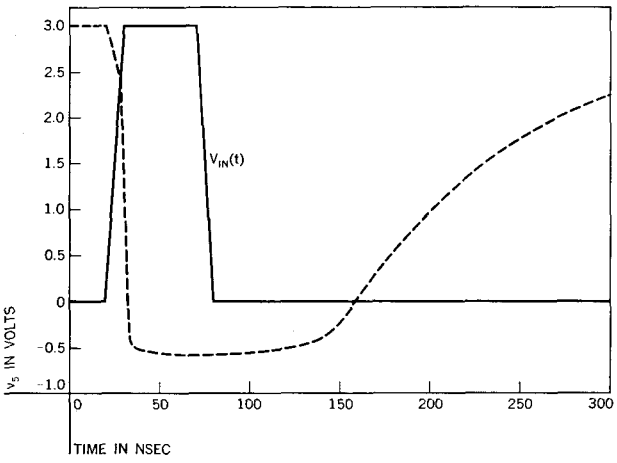
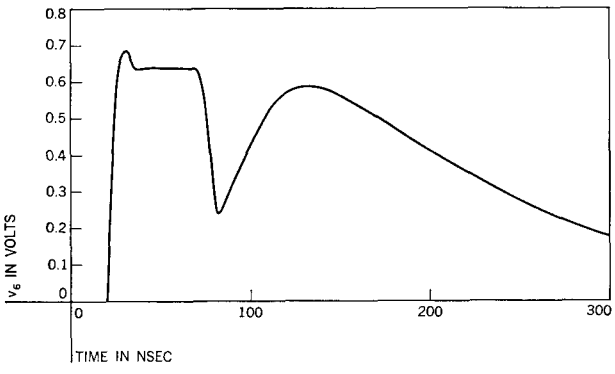


Figure 2 Voltages v_5 and v_{in} versus time.

Figure 3 Voltage v_6 versus time.



Conclusions

The difficulties encountered in any attempt to numerically solve electrical circuit equations can stem from two sources: the electrical circuit itself, and the numerical integration routine. Hence, to overcome these difficulties it will not always be sufficient to be guided purely by physical considerations. For example, numerical computations may be made inefficient by the existence of certain parameters that are unimportant to the functioning of the circuit. Thus, in the circuit analyzed, here, some small resistors were included; some, in fact, were included purposely (apparently to eliminate capacitor loops in the circuit). Their presence forced the choice of a very small time step in the integration procedure and hence a large amount of computation time was needed. By eliminating these small resistors, the computation time was reduced from 30 minutes to 7 seconds with the results agreeing to within 1%. Thus, we conclude that, for numerical reasons: (a) It is bad practice to insert small resistors in order to eliminate capacitor loops; (b) unimportant stray parameters should not be included in the circuit because they may introduce a large increase in the computation time; (c) in many cases, it is possible, by first making an elementary analysis of the circuit equations, to decide which are the unimportant parameters.

Appendix 1

This section presents a detailed description of the circuit elements in Fig. 1. The units are as follows: resistances are given in kilohms (kΩ); capacitances in picofarads (pF); voltages in volts (V); currents in milliamperes (mA); and times in nanoseconds (nsec).

The parameters are as listed below:

$$R_L = 10.1 \qquad R_\rho = 10^4$$

$$R_S = .01 \qquad R_\sigma = .025$$

$$R_B = 5 \qquad R_T = 10^4$$

$$R_{CC} = .007$$

$$C_1 = 7.08 \qquad C_2 = 5.39$$

$$C_3 = 1.3 \qquad C_4 = 7.91$$

$$V_{CC} = 3.0 \qquad V_{OC} = .47$$

$$V_{OE} = .69 \qquad V_{\text{stop}} = .01$$

$$I_{OC} = .32 \times 10^{-6} \qquad I_{OE} = .51 \times 10^{-11}$$

$$\tau_S = 280.0 \qquad \tau_C = .54$$

$$n = .68 \qquad m = .61$$

$$k = .7059 \qquad \gamma_E = 40.6 \text{ volts}^{-1}$$

$$K = .7584 \qquad \gamma_C = 20.7 \text{ volts}^{-1}$$

$$\alpha_i = .2 \qquad \alpha(v_6) = \text{const} = .9985$$

The remaining circuit elements are variable and are given as follows:

$$I_S(v_5) = I_{OC}(1 - e^{-\gamma_C v_5})$$

$$I_E(v_6) = I_{OE}(e^{\gamma_E v_6} - 1)$$

$$C_5(v_5) = C_{TC}(v_5) + C_{DC}(v_5)$$

$$C_{TC}(v_5) = \begin{cases} K/(v_5 + V_{OC})^n & \text{if } (v_5 + V_{OC}) \geq V_{\text{stop}} \\ \text{last computed value of } C_{TC}(v_5) & \\ \quad \text{if } (v_5 + V_{OC}) < V_{\text{stop}} & \end{cases}$$

$$C_{DC}(v_5) = \gamma_C I_{OC} \tau_S e^{-\gamma_C v_5}$$

$$C_6(v_6) = C_{TE}(v_6) + C_{DE}(v_6)$$

$$C_{TE}(v_6) = \begin{cases} k/(v_6 + V_{OE})^m & \text{if } (v_6 + V_{OE}) \geq V_{\text{stop}} \\ \text{last computed value of } C_{TE}(v_6) & \\ \quad \text{if } (v_6 + V_{OC}) < V_{\text{stop}} & \end{cases}$$

$$C_{DC}(v_5) = \gamma_E I_{OE} \alpha \tau_C e^{\gamma_E v_5}$$

R_{BB} is a function of $I_E(v_6)$ and is determined by linear interpolation from the following table:

I_E(mA)	R_{BB}(kΩ)
0	.275
0.3	.215
3.0	.123
10.0	.083
30.0	.067

The input voltage $V_{\text{in}}(t)$ is given in Fig. 2. The initial values used are $t = 0$, $v_1 = 3$, $v_2 = 0$, $v_3 = 3$, $v_4 = 0$, $v_5 = 3$, and $v_6 = 0$.

Appendix 2

In Ref. 1 it was shown that the equations of an electrical network could be written in a standard form

$$L_\rho(i_\rho) \frac{di_\rho}{dt} = \frac{\partial P}{\partial i_\rho}, \qquad \rho = 1, \cdots, r$$

$$C_\sigma(v_\sigma) \frac{dv_\sigma}{dt} = -\frac{\partial P}{\partial v_\sigma}, \qquad \sigma = r+1, \cdots, r+s,$$

where the i_ρ are the currents through the inductors and the v_σ are the voltages across the capacitors. There, it was assumed that the i_ρ, v_σ were independent. It is the purpose of this appendix to derive the form of the equations in the case when the i_ρ, v_σ are not independent.

We therefore assume that the v_σ are related by

$$\tilde{v} = A\hat{v} + a(t), \tag{A.1}$$

where

$$v = \begin{bmatrix} v_{r+1} \\ \vdots \\ v_{r+s} \end{bmatrix} = \begin{bmatrix} \tilde{v} \\ \hat{v} \end{bmatrix}.$$

Similarly, we assume that

$$\tilde{\imath} = B\hat{\imath} + b(t), \tag{A.2}$$

where

$$i = \begin{bmatrix} i_1 \\ \vdots \\ i_r \end{bmatrix} = \begin{bmatrix} \tilde{\imath} \\ \hat{\imath} \end{bmatrix}.$$

In Ref. 1 it was shown that there exists a function $P(i, v)$ depending only on (i, v) such that

$$P(i, v) = -\int_\Gamma \sum_{\rho=1}^{r} V_\rho \, di_\rho + \int_\Gamma \sum_{\sigma=r+1}^{r+s} I_\sigma \, dv_\sigma, \tag{A.3}$$

where Γ is some curve in E_b (b-dimensional Euclidean space, where b is the number of branches of the network) with end point determined by (i, v). Here I_σ is the current through the capacitor with voltage v_σ, and V_ρ is the voltage across the inductor with current i_ρ. Using the notation $(u, w) = \sum_\nu u_\nu w_\nu$, we write Eq. (A.3) as

$$P(i, v) = \int_\Gamma [-(\tilde{V}, d\tilde{\imath}) - (\hat{V}, d\hat{\imath}) + (\tilde{I}, d\tilde{v}) + (\hat{I}, d\hat{v})].$$

Using Eqs. (A.1) and (A.2), this becomes

$$\hat{P}(\hat{\imath}, \hat{v}) = P(i, v)\Big|_{\substack{\tilde{v} = A\hat{v} + a(t) \\ \tilde{\imath} = B\hat{\imath} + b(t)}}$$

$$= \int_\Gamma -(\tilde{V}, B\, d\hat{\imath}) - (\hat{V}, d\hat{\imath}) + (\tilde{I}, A d\hat{v}) + (\hat{I}, d\hat{v}).$$

From this we read off

$$\frac{\partial \hat{P}}{\partial \hat{\imath}} = -B^T \tilde{V} - \hat{V},$$

$$\frac{\partial \hat{P}}{d\hat{v}} = A^T \tilde{I} + \hat{I}. \tag{A.4}$$

However, we also have, using the elementary laws for inductors and capacitors,

$$V = -L(i) \frac{di}{dt},$$

$$I = -C(v) \frac{dv}{dt},$$

where

$$L(i) = \text{diag}\,(L_1(i_1), \cdots, L_r(i_r))$$

and

$$C(v) = \text{diag}\,(C_{r+1}(v_{r+1}), \cdots, C_{r+s}(v_{r+s})).$$

We partition these matrices as follows:

$$L(i) = \begin{bmatrix} \tilde{L}(\tilde{\imath}) & 0 \\ 0 & \hat{L}(\hat{\imath}) \end{bmatrix}, \qquad C(v) = \begin{bmatrix} \tilde{C}(\tilde{v}) & 0 \\ 0 & \hat{C}(\hat{v}) \end{bmatrix},$$

which gives

$$\tilde{V} = -\tilde{L}(\tilde{\imath}) \frac{d\tilde{\imath}}{dt} = -\tilde{L}\left(B \frac{d\hat{\imath}}{dt} + \frac{db}{dt}\right), \qquad \hat{V} = -\hat{L} \frac{d\hat{\imath}}{dt}$$

$$\tilde{I} = -\tilde{C}(\tilde{\imath}) \frac{d\tilde{v}}{dt} = -\tilde{C}\left(A \frac{d\hat{v}}{dt} + \frac{da}{dt}\right), \quad I = -\hat{C} \frac{d\hat{v}}{dt}.$$

Substituting these relations into Eq. (A.4) yields

$$\frac{\partial \hat{P}}{\partial \hat{\imath}} = B^T \tilde{L}\left(B\frac{d\hat{\imath}}{dt} + \frac{db}{dt}\right) + \hat{L}\frac{d\hat{\imath}}{dt}$$

$$\frac{\partial \hat{P}}{\partial \hat{v}} = -A^T \tilde{C}\left(A\frac{d\hat{v}}{dt} + \frac{da}{dt}\right) - \hat{C}\frac{d\hat{v}}{dt},$$

or

$$(\hat{L} + B^T \tilde{L} B)\frac{d\hat{\imath}}{dt} = \frac{\partial \hat{P}}{\partial \hat{\imath}} - B^T \tilde{L}\frac{db}{dt}$$

$$(\hat{C} + A^T \tilde{C} A)\frac{d\hat{v}}{dt} = -\frac{\partial \hat{P}}{\partial \hat{v}} - A^T \tilde{C}\frac{da}{dt}.$$

We shall take this to be the standard form for the equations when there is dependence in the unknown variables. Note that the matrices

$$L^* = \hat{L} + B^T \tilde{L} B$$
$$C^* = \hat{C} + A^T \tilde{C} A$$

are symmetric. Of course, if there are no time variable elements in the network, then the standard form becomes

$$L^*\frac{d\hat{\imath}}{dt} = \frac{\partial \hat{P}}{\partial \hat{\imath}}$$

$$C^*\frac{d\hat{v}}{dt} = -\frac{\partial \hat{P}}{\partial \hat{v}},$$

which is identical to the original form except that the matrices L, C are no longer diagonal but symmetric.

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