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Stability of Flexible Tapes in Parallel Flow

Abstract: A stability analysis by the method of normal modes is carried out for a plane Poiseuille flow in a channel with a free-flying elastic tape in midstream. The effects of transverse and rotary inertia and flexural rigidity of the tape on the stability criteria are investigated. An increase of transverse inertia due to heavier tapes tends to decrease the critical Reynold's number while the addition of flexural rigidity improves the stability criteria by increasing the critical Reynold's number.

Introduction

Hydrodynamic stability in the presence of moving flexible boundaries has become an important concern with the increasing use of thin flexible tapes. This paper addresses one aspect of that concern by seeking to establish the instability conditions for the flow in a channel (plane Poiseuille flow) for the case where there is a "free-flying" elastic tape in the center of the channel. The tape moves with the maximum stream velocity and experiences no shear drag from the fluid. It is also our aim to investigate the effect of density, rotary inertia, and flexural rigidity of the tape upon the critical Reynolds number of the flow above which instability may occur. A similar problem when the boundary is a non-rigid wall possessing bulk damping and flexibility has been considered by Hains and Price [1] in connection with the control of boundary layer and prevention of turbulence.

For a better understanding of the interaction between fluid and a moving elastic tape, we recall the basic equations governing such interactions and properly characterize the primary (time independent) and perturbation solutions. The analysis of stability is then made by the method of normal modes as described by Lin [2].

The resulting equations are, of course, the Orr-Sommerfeld equation with boundary conditions which reflect the interaction between tape and the fluid. These equations are solved by the finite difference method developed by L. H. Thomas [3] and by a combination of iteration and searching process. The results generally show an increase in critical Reynolds number with an increase in flexural rigidity and/or a decrease in density of the tape; they also include neutral stability curves for various values of the tape parameters.

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Formulation of the problem

Let D be a plane region occupied by a viscous incompressible fluid whose component velocities in the direction of the Cartesian coordinates $x_i (i = 1, 2)$ will be denoted by $u_i(x_j, t)$ where t is the time. Let the boundary Γ of D comprise two portions, one consisting of rigid walls and the other consisting of an inextensible elastic tape; the two are denoted, respectively, by Γ_R and Γ_F. The flexible portion of the boundary may be presented in the parametric form

$$\Gamma_F : x_i = y_i(s, t), \tag{1}$$

where s is the length of arc along Γ_F, as in Fig. 1. The

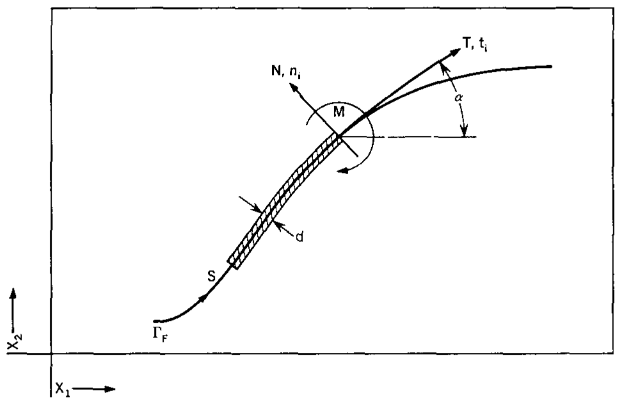


Figure 1 A tape segment: geometry and forces.

equations of fluid motion and the boundary conditions are

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\mu}{\rho}\frac{\partial^2 u_i}{\partial x_j\, \partial x_j}, \quad \text{in } D; \tag{2}$$

$$\frac{\partial u_i}{\partial x_i} = 0, \quad \text{in } D; \tag{3}$$

$$u_i = 0, \quad \text{on } \Gamma_R; \tag{4}$$

$$u_i(y_j(s, t), t) = \frac{\partial y_i}{\partial t}. \tag{5}$$

The equations of motion of the inextensible tape are

$$\frac{\partial R_i}{\partial s} + f_i = \rho_0 d \frac{\partial^2 y_i}{\partial t^2}, \tag{6}$$

$$\frac{\partial M}{\partial s} - N + \rho_0 I \frac{\partial^2 \alpha}{\partial t^2} = 0, \quad \text{and} \tag{7}$$

$$\frac{\partial y_i}{\partial s}\frac{\partial y_i}{\partial s} = 1, \tag{8}$$

where

$$R_i = Tt_i + Nn_i, \tag{9}$$

$$M = -EI(k - k_0), \tag{10}$$

T and N are the tangential and normal forces acting on the tape, and α is the angular inclination of the tangent (see Fig. 1). The vector f_i represents the force per unit area acting on the tape due to the fluid motion and is given by

$$f_i = -(\tau_{ij})_{\mathbf{x}=\mathbf{y}}\, n_j, \tag{11}$$

where

$$\tau_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{12}$$

and where n_i is the unit normal vector and is related to t_i (that is, to the unit tangential vector) by the equations

$$n_i = -e_{ij}t_j, \qquad t_i = \frac{\partial y_i}{\partial s}; \tag{13}$$

$$e_{11} = e_{22} = 0, \quad e_{12} = -e_{21} = 1. \tag{14}$$

The curvature k and the time rate of change of α may also be expressed in terms of y_i by

$$k \equiv \frac{\partial \alpha}{\partial s} = -t_i \frac{\partial n_i}{\partial s} = e_{ij}\frac{\partial^2 y_j}{\partial s^2}\frac{\partial y_i}{\partial s}; \tag{15}$$

$$\frac{\partial \alpha}{\partial t} = e_{ij}\frac{\partial^2 y_j}{\partial s\, \partial t}\frac{\partial y_i}{\partial s}. \tag{16}$$

The undeformed curvature is represented by k_0 and the constants ρ_0, d, E, and $I = d^3/12$ are, respectively, the density, thickness, Young's modulus, and moment of inertia per unit width of the tape. From the equations of conservation of angular momentum, Eqs. (7) and (10), one can solve for N in terms of k and α. The result may then be substituted into Eq. (9), and the result into Eq. (6), to obtain

$$\frac{\partial}{\partial s}\left[Tt_i - E\frac{\partial}{\partial s}(k - k_0)n_i\right] - (\tau_{ij})_{\mathbf{x}=\mathbf{y}}n_j = \rho_0 d\frac{\partial^2 y_i}{\partial t^2} + \rho_0 I \frac{\partial}{\partial s}\left[\frac{\partial^2 \alpha}{\partial t^2} n_i\right]. \tag{17}$$

When τ_{ij} is known, Eq. (17) becomes, with the aid of Eqs. (13) and (15)–(16), two equations involving three unknowns, T and y_i. The necessary third equation is the inextensibility condition, Eq. (8). The formulation of the problem is, therefore, complete if we adopt, in addition to Eqs. (2)–(5), Eqs. (8) and (17).

When the fluid is in a state of steady motion the flexible boundary assumes a shape which does not change with time. Therefore, there exists a function $F(y_i)$ such that the coordinates y_i satisfy $F(y_i) = \text{Const.}$ Differentiation yields

$$\frac{\partial F}{\partial y_i}\frac{\partial y_i}{\partial s} = \frac{\partial F}{\partial y_i}\frac{\partial y_i}{\partial t} = 0, \tag{18}$$

and since $\partial F/\partial y_i \neq 0$, the two plane vectors $\partial y_i/\partial t$ and $\partial y_i/\partial s$ are proportional,

$$\frac{\partial y_i}{\partial t} = V\frac{\partial y_i}{\partial s}. \tag{19}$$

The proportionality factor V is, of course, the speed and it can be shown that inextensibility condition of Eq. (7) implies that V is at most a function of t (independent of s). For steady state solutions, $V = \text{Const.}$, Eq. (16) implies that $y_i = y_i(s + V_t)$ and u and p are functions of x only.

Let us now turn to the specific problem, that of hydrodynamic stability of Poiseuille flow with a free-flying elastic tape located in midstream. The free-flying condition of the tape implies that there is no shear stress acting on the tape due to fluid motion. In order that this be true, there must exist the steady state solution

$$\left.\begin{aligned} \bar{u}_1 &= V[1 - (x_2/h)^2] \\ \bar{p} &= -\frac{2\mu V}{h}(x_1/h) + \bar{P} \\ \bar{y}_1 &= s + Vt \\ \bar{u}_2 &= \bar{y}_2 = \bar{T} = 0 \end{aligned}\right\}, \tag{20}$$

where V is the maximum or the mid-stream velocity which is also the tape velocity, $2h$ is the channel width, and $\bar{P}$ is a constant reference pressure. To investigate stability, we introduce—in the usual fashion—the perturbations

$$\left.\begin{aligned} u_i &= \bar{u}_i(x_j) + \epsilon u_i'(x_j, t) + \cdots \\ p &= \bar{p}(x_j) + \epsilon p'(x_j, t) + \cdots \\ y_i &= \bar{y}_i(s + Vt) + \epsilon y_i'(s, t) + \cdots \\ T &= \bar{T}(s + Vt) + \epsilon T'(s, t) + \cdots \end{aligned}\right\}. \tag{21}$$

Growth or decay of the perturbed quantities u_i', p', etc., represents the instability or the stability of the primary flow of the equations of (20).

Adopting Lin's approach by the theory of normal modes, the perturbed solutions may be written in the form

$$\{u_1', u_2', p'\} = \{V\psi'(\theta), i\omega V\psi(\theta), P(\theta)\} \times \exp\left[-i\omega\left(\frac{x_1}{h} - \frac{cVt}{h}\right)\right], \tag{22}$$

where $\theta = x_2/h$, ψ, and P are functions of θ only, and $\psi'(\theta) = d\psi/d\theta$. Here ω and c represent the wave number and speed of the disturbance traveling in the x_1-direction. Im $(c) > 0$ implies stability, and Im $(c) < 0$ instability. The continuity equation, Eq. (3), is satisfied automatically by Eq. (22). The equations of motion of Eq. (2) yield the Orr-Sommerfeld equation for $\psi(\theta)$, $-1 < \theta < 1$,

$$(D^2 - \omega^2)^2\psi = -i\omega R\{(1 - \theta^2 - c)(D^2 - \omega^2)\psi + 2\psi\}, \tag{23}$$

where $R = hV/\nu$ is the Reynolds number and $D^n = d^n/d\theta^n$. The boundary conditions of Eqs. (4) and (5) give

$$\left.\begin{aligned} \psi(\pm 1) &= \psi'(\pm 1) = 0 \\ \psi(0) &= -(1 - c) \\ \psi'(0) &= 0 \end{aligned}\right\}. \tag{24}$$

The above boundary conditions allow $\psi(\theta)$ to have symmetric solutions with respect to θ. Therefore, one may consider the smaller interval $\theta \epsilon$ (0, 1), taking due account of the forces acting on the tape from the fluid in the region $-1 < \theta < 0$. Calculation of Eq. (17) in terms of ψ completes the description of the eigenvalue problem $\psi(\theta)$ satisfying the Orr-Sommerfeld equation (23) for $0 < \theta < 1$, subject to the following boundary conditions:

$$\left.\begin{aligned} \psi(1) &= \psi'(1) = 0 \\ \psi'(0) &= 0 \\ \psi'''(0) &- K\psi(0) = 0 \end{aligned}\right\}, \tag{25}$$

where

$$K = -\tfrac{1}{2}i\omega^3\left\{(K_1 + K_2\omega^2)(1 - c)R - \frac{\omega^2}{(1 - c)R}K_3\right\}, \tag{26}$$

and

$$\left.\begin{aligned} K_1 &= \frac{\rho_0}{\rho}\frac{d}{h} \\ K_2 &= \frac{\rho_0}{\rho}\frac{I}{h^3} \\ K_3 &= \frac{EI\rho}{h\mu^2} \end{aligned}\right\}. \tag{27}$$

The last boundary condition in Eq. (25) is obtained from Eq. (17) by assuming that y_i' and T' are waves which travel in the positive s-direction and which are in phase with the wave forms, Eq. (22).

Numerical solution and results

The numerical finite difference method developed by L. H. Thomas [3] is used to solve the Orr-Sommerfeld equation (23) and the associated boundary conditions. The method has been used by Hains and Price in their investigation of stability of Poiseuille flow between flexible walls [1]. Rather than describe the details of the finite difference method, we refer the reader to the above two references.

The homogeneous boundary value problem, Eqs. (23) and (25), has a solution if and only if the constants ω, c, R, K_1, K_2, K_3, satisfy a characteristic equation of the form

$$\Delta(\omega, c, R, K_1, K_2, K_3) = 0. \tag{28}$$

The function Δ is regarded as a complex-valued function of the complex variable c.

Of particular interest in the stability analysis is the condition of neutral stability for which Im $(c) = 0$. This condition, for fixed K_1, K_2, K_3, represents a curve in the ω, R plane. In order to plot this curve we set Im $(c) = 0$, and fix the values of the constants K_1, K_2, K_3 and R. Setting the real and imaginary parts of Eq. (1) equal to zero we obtain two equations of the form

$$\begin{aligned} \Delta_i(\omega, c_r) &= 0, \\ \Delta_r(\omega, c_r) &= 0, \end{aligned} \tag{29}$$

where subscripts r and i refer to real and imaginary parts. The equations of (29) were solved by a searching method and, for sufficiently large R, yielded two solutions (ω_1, c_{r_1}) and (ω_2, c_{r_2}) which correspond to the intersections of the upper and lower branches of the neutral stability curve with the line $R =$ Const. From these known values of ω and c_r and with slight changes in R, it proved a simple matter to trace the neutral stability curve. Actually, at the beginning we used $K_1 = K_2 = K_3 = 0$ (no tape) and $R = 10000$, and constructed the neutral stability curve for the Thomas problem up to the value of $R = 250 \times 10.^3$ This proved the feasibility of our method and, of course, yielded confirmation of Thomas's results, as well as an extension of them. Then we proceeded to construct Shen's [4] amplification rates for the plane Poiseuille flow with rigid walls. These represent curves in ω, R plane for which $c_i =$ Const. < 0. In Fig. 2, the outermost curve is the neutral stability curve, and the closed curve for which $c_i = -10^{-2}$ is the first amplification rate curve which closed on itself for $R < 240 \times 10^3$. The intermediate curve represents an amplification rate of $c_i = -2 \times 10^{-3}$. The results are plotted against $R^{1/3}$ to allow direct comparison with Fig. 3.1 of Reference 2. We obtain $R_{cr} = 5775$ for

Thomas's 5780 but we disagree somewhat with Shen's results.

Once the case of rigid walls was dispensed with, the effect of the tape parameters was introduced gradually. From Eq. (26) it appears that the effect of the rotatory

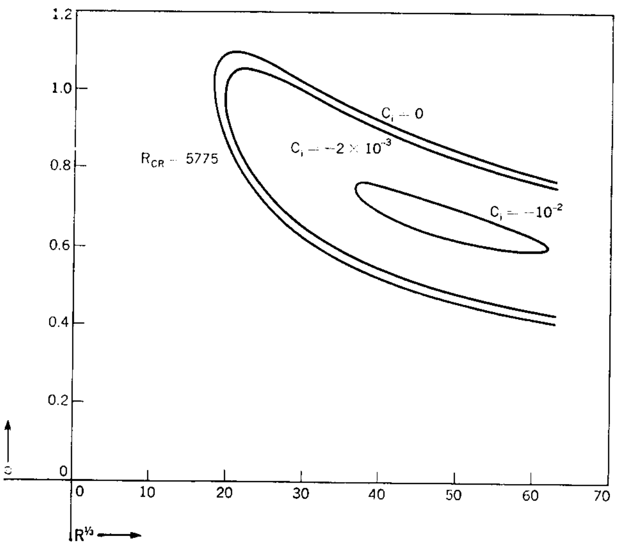
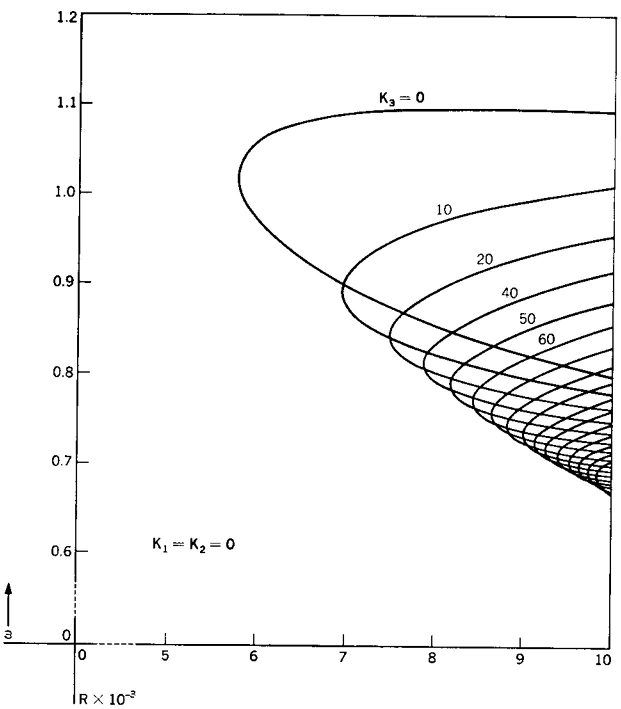


Figure 2 Stability characteristics of plane Poiseuille motion.

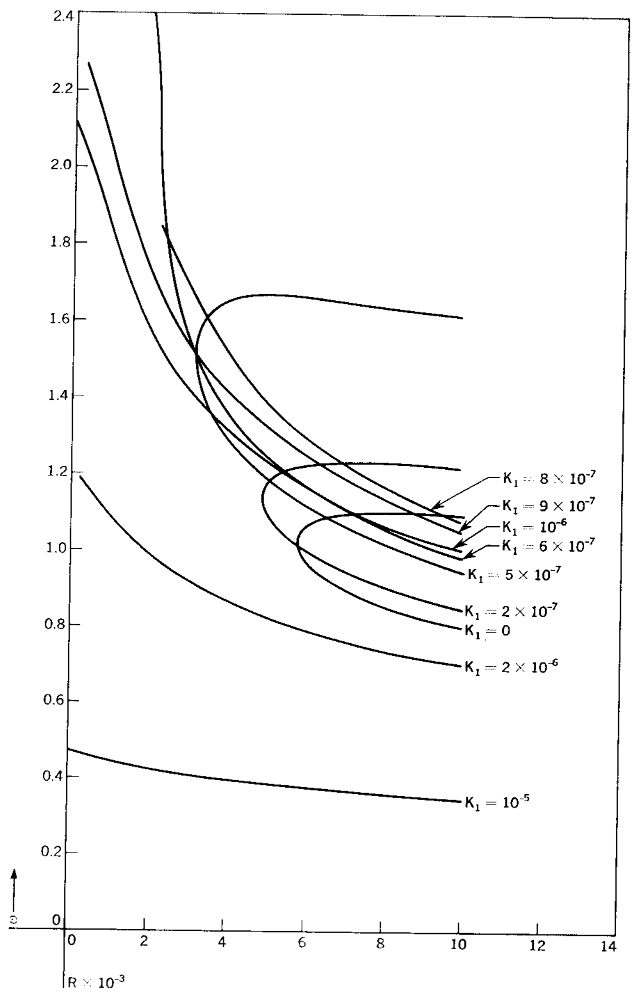
Figure 3 Effect of flexural rigidity of the tape on the stability of flow.



inertia parameter K_2 is very small compared to K_1 for $K_2 \sim K_1(d/h)^2$ and for a thin tape $d \ll h$. It was decided, therefore, to set $K_2 = 0$ in all subsequent computations. The effect of flexural rigidity parameter K_3 is shown in Fig. 3. Here we chose $K_1 = 0$ and chose for K_3 the sequence of values 0, 10, 20 $\cdots$ 170. The left-most curve is again Thomas's neutral stability curve and the other curves show the gradual increase in stability with increase in K_3. There appears to be no tendency to an asymptotic value as $K_3 \rightarrow 0$ other than $R_{cr} \rightarrow \infty$. This seems to show that plane parallel flow between two rigid walls is unconditionally stable for the case in which one wall is stationary and the other moves at the maximum fluid velocity and experiences no drag.

The effect of increasing transverse inertia K_1 when the other parameters K_2, K_3 are set equal to zero is shown in Fig. 4. It is seen that, generally, heavier tapes appear to

Figure 4 Effect of transverse inertia of the tape on the stability of flow.



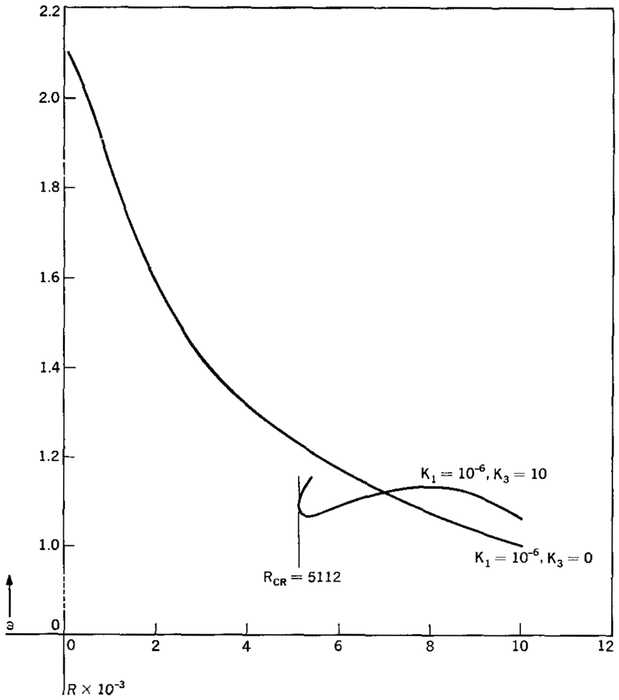


Figure 5 Neutral stability curve for $K_1 = 10^{-6}$ and $K_3 = 10$.

be less stable. Also, the upper branch of neutral stability curve opens up with increasing K_1 to the point where, at $K_1 = 6 \times 10^{-7}$, it is almost vertical and for larger values of K_1 there is no critical R below which the flow is stable. With the addition of some flexural rigidity, however, such an unconditionally unstable flow may become stable for sufficiently small R. This is shown in Fig. 5, where the neutral stability curve corresponding to the values of $K_1 = 10^{-6}$, $K_2 = K_3 = 0$ is modified by choosing $K_3 = 10$. The resulting neutral stability curve indicates instability only for $R > 5112$.

References

1. F. D. Hains and J. F. Price, "Stability of Plane Poiseuille Flow Between Flexible Walls," *Proc. National Congress Appl. Mech.* (1962).
2. C. C. Lin, *The Theory of Hydrodynamic Stability*, Cambridge, 1955.
3. L. H. Thomas, "The Stability of Plane Poiseuille Flow," *Phys. Rev.* **91**, 4, 780 (August 1953).
4. S. F. Shen, "Calculated Amplified Oscillations in Plane Poiseuille and Blasius Flows," *J. Aero. Sci.* 21, 62–64 (1954).

Received January 19, 1966