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Tensor Analysis of Spatial Mechanisms

Abstract: The position analysis of a general four-bar spatial mechanism is developed using tensor notation and operations. To exemplify the convenience of tensors in kinematic analysis the solution is obtained for a mechanism containing two revolute pairs of links and two spherical pairs.

Introduction

Until recently the analysis of spatial mechanisms has not occupied a prominent place in the work of kinematicians. Their reluctance to study these mechanisms was perhaps due to the apparently formidable and tedious tasks of mathematically formulating problems and obtaining solutions. However, recent publications have applied various mathematical tools to the analysis of spatial mechanisms in an effort to simplify the computational process and thus make the undertaking of work in this field more attractive. Denavit and Hartenberg[1] adopted the matrix calculus; Chace,[2] Beyer,[3] and Harrisberger[4] used the vector technique; Yang and Freudenstein[5] chose quaternions; and kinematicians in the USSR, for example, Mangeron and Dregan,[6] and Kalitsin,[7] applied tensor analysis.

The present paper proposes another approach to the use of tensor notation and operations in treating such problems. The first part of the paper reviews the aspects of tensor algebra that are pertinent to this application. Then an analysis of a spatial four-bar linkage is used to demonstrate the technique. The analysis corresponds to Chace's[2,8] solution of the vector tetrahedron equation. As a specific example of applying the technique, a mechanism containing two revolute pairs of links and two spherical pairs (*R-S-S-R*) is analyzed.

The paper intends to establish a basis for future spatial-linkage tensor analysis. It emphasizes the comprehensiveness and brevity of the tensor notation for this type of analysis.

Preliminary mathematics[9]

- *Notational conventions*

A point in three-dimensional space located with respect to a Cartesian coordinate system X_i, may also be located with respect to another Cartesian coordinate system, X'_i, by the equations

$$\begin{aligned} X_1 &= A_{11}X'_1 + A_{21}X'_2 + A_{31}X'_3 + B_1 \\ X_2 &= A_{12}X'_1 + A_{22}X'_2 + A_{32}X'_3 + B_2 \\ X_3 &= A_{13}X'_1 + A_{23}X'_2 + A_{33}X'_3 + B_3, \end{aligned} \tag{1}$$

where the A_{ij}'s are constants of rotation between the axes of the two coordinate systems and the B_i's are constants of translation between the origins of the systems.

Expressions like Eq. (1) may be expressed more concisely by adopting the following notational conventions:

Range convention—When an index (subscript) occurs unrepeated in a term of an expression, it is understood to take, in turn, each value in the range of that index. In this paper the values will always be 1, 2, 3.

Summation convention—When an index is repeated in a term, summation over the range of that index is implied. Using these conventions, Eq. (1) is rewritten as

$$X_i = A_{ji}X'_j + B_i, \tag{2}$$

and the inverse transformation is written

$$X'_i = A_{ij}X_j + B'_i. \tag{3}$$

The range and summation conventions will be used throughout this paper. No confusion should occur if the reader remembers that they are implicitly present in the notation used henceforth.

With the orthogonal property of the coordinate systems, the rotational coefficients have the relation

$$A_{ik}A_{jk} = A_{ki}A_{kj} = \delta_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j, \end{cases} \tag{4}$$

where δ_{ij} is the so-called Kronecker delta.

In the paper the orthogonal transformation is always positive; i.e., $|A_{ij}| = 1$.

• *Definition of a tensor*

Each number, T_i, of a set of quantities associated with a Cartesian coordinate system, X_i, and with a point, P, is said to be a component of a tensor of the first rank* if the quantities transform to any other coordinate system X'_i according to the equation

$$T'_j = A_{ji}T_i. \tag{5}$$

It will be seen that a Cartesian tensor of the first rank is equivalent to an ordinary Cartesian vector.

Tensors of higher rank can also be defined. That is, quantities T_{ab} which transform according to

$$T'_{ij} = A_{ia}A_{jb}T_{ab} \tag{6}$$

are called tensors of the second rank. A tensor of the n^{th} rank has n indices and transforms through its multiplication by n coefficients:

$$T'_{ij\cdots k} = A_{ia}A_{jb}\cdots A_{kc}T_{ab\cdots c}. \tag{7}$$

• *Some tensor properties*

A tensor is said to be symmetric in two indices, j and k, if the value of any component is not changed by interchanging the positions of j and k. That is, if

$$T_{i\cdots jk\cdots m} = T_{i\cdots kj\cdots m},$$

the tensor is symmetric in j and k. The tensor is completely symmetric if its components retain the same value when any two indices are interchanged. Similarly, a tensor is said to be skew-symmetric in the indices j and k if

$$S_{i\cdots jk\cdots m} = -S_{i\cdots kj\cdots m}.$$

The tensor is completely skew-symmetric if its components retain the same value, but are changed in sign when any two indices are interchanged.

The product of any completely symmetric tensor, say T_{ij}, and any completely skew-symmetric tensor, say S_{ij}, is

$$T_{ij}S_{ij} = 0. \tag{8}$$

This property will be used to great advantage in the application of tensor operations discussed in this paper.

The Kronecker delta defined in Eq. (4) is an example of a symmetric tensor. It is also called an isotropic tensor because its components retain the same values in any coordinate system.

Another tensor which will be very useful in the present application is the permutational symbol, ϵ_{ijk}. This tensor is both isotropic and completely skew-symmetric; the values of its components are obtained as follows:

$$\epsilon_{ijk} = \begin{cases} 0 \text{ if any two indices have the same value.} \\ 1 \text{ if the values of the indices } ijk \text{ represent an even permutation* of the sequence } 1, 2, 3. \\ -1 \text{ if the values of the indices } ijk \text{ represent an odd permutation of the sequence } 1, 2, 3. \end{cases} \tag{9}$$

An important relation between δ_{ij} and ϵ_{ijk} is given by

$$\epsilon_{ikj}\epsilon_{mpj} = \delta_{im}\delta_{kp} - \delta_{ip}\delta_{km}. \tag{10}$$

For a further understanding of the use of tensor notation, a brief summary of the operations of vector algebra written in tensor notation is provided in Table 1.

Table 1 Comparison of vector operations in vector and tensor notations.

Operation	*Vector notation*	*Tensor notation*
Denotation	$\mathbf{T}$	T_i
Addition and subtraction	$\mathbf{P}=\mathbf{T}+\mathbf{S}$ $\mathbf{Q}=\mathbf{T}-\mathbf{S}$	$P_i=T_i+S_i$ $Q_i=T_i-S_i$
Multiplication by a scalar	$\mathbf{R}=\phi\mathbf{T}$	$R_i=\phi T_i$
Scalar product	$\phi=\mathbf{T}\cdot\mathbf{S}$	$\phi=T_iS_i$
Vector product	$\mathbf{M}=\mathbf{T}\times\mathbf{S}$	$M_i=\epsilon_{ijk}T_jS_k$
Tensor product		$N_{ij}=T_iS_j$
Triple scalar product	$\phi=\mathbf{P}\cdot\mathbf{M}$ $=\mathbf{P}\cdot(\mathbf{T}\times\mathbf{S})$	$\phi=P_iM_i$ $=P_i\epsilon_{ijk}T_jS_k$
Triple vector product	$\mathbf{L}=\mathbf{P}\times\mathbf{M}$ $=\mathbf{P}\times(\mathbf{T}\times\mathbf{S})$	$L_i=\epsilon_{ijk}P_jM_k$ $=\epsilon_{ijk}P_j\epsilon_{klm}T_lS_m$ $=\epsilon_{ijk}\epsilon_{klm}P_jT_lS_m$ $=(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})P_jT_lS_m$ $=\delta_{il}\delta_{jm}P_jT_lS_m$ $-\delta_{im}\delta_{jl}P_jT_lS_m$ Since in the first term above, the coefficients of $P_jT_lS_m$ will be zero unless $l = i$ and $j = m$, and in the second term, will be zero unless $i = m$ and $j = l$, we may substitute indices and write $L_i=P_jT_iS_j-P_jT_jS_i$.

* In general (i.e., when a Cartesian coordinate system is not specified) a tensor of the first rank, also called a contravariant vector, is defined by the transformation $T'^j = (\partial X'^j/\partial X^i)\ T^i$. The transformation $T_j' = (\partial X^i/\partial X'^j)\ T_i$ defines what is called a covariant vector. However, with Cartesian coordinates, $(\partial X'^j/\partial X^i) = (\partial X^i/\partial X'^j) = A_{ji}$, so the distinction between contravariant and covariant vectors is eliminated. Throughout this paper, the indices of tensor operation are written as subscripts only.

* A permutation of the sequence 1, 2, . . . , n is even if an even number of interchanges of adjacent integers is required to attain the permutation. Similarly, a permutation is odd if an odd number of interchanges is required. Thus,
$\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = 1$, and $\epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1$.

Position analysis of the spatial four-bar linkage

• *General description*

In order to demonstrate the application of the tensor method to the analysis of spatial mechanisms, a closed-loop, four-link spatial linkage has been studied. Figure 1 represents the general diagram of the linkage and relationship of the links to the coordinate frames. It is convenient that each link be determined with respect to its local coordinate frame by a set of spherical polar coordinates. The so-called ground link, Cc_i, has a length of C units and is directed along a vector of unit length, c_i. The components of c_i with respect to the ground coordinate frame are defined by

$$
\begin{aligned}
c_1 &= \sin\phi \cos\theta, \\
c_2 &= \sin\phi \sin\theta, \\
c_3 &= \cos\phi,
\end{aligned} \tag{11}
$$

where ϕ and θ represent the polar and azimuthal angles, respectively.

Likewise, the first link, Rr_i^I, has magnitude R and is directed along a unit vector r_i^I which originates at the first joint. The components of r_i^I with respect to its local coordinate frame are defined by

$$
\begin{aligned}
r_1^I &= \sin\phi^I \cos\theta^I, \\
r_2^I &= \sin\phi^I \sin\theta^I, \\
r_3^I &= \cos\theta^I,
\end{aligned} \tag{12}
$$

where ϕ^I and θ^I represent the polar and azimuthal angles of the link in the first frame. In the same manner, the components of the second and third link vectors, s_i^{II} and t_i^{III} can be expressed in terms of the polar coordinates of the second and the third frames, respectively.

To derive relationships between the link vectors it is desirable to specify an arbitrary point to which all the vectors may be referred. Throughout this paper the ground joint (see Fig. 1) will serve as the reference point.

The ground coordinate system X_1, X_2, X_3 and the first-joint coordinate system X_1^I, X_2^I, X_3^I are both taken to have their origins at the ground joint. The transformation matrix A_{ij}^I can be written as the direction cosines between the axes; that is

$$
A_{ij}^I = \begin{bmatrix} A_{11}^I & A_{12}^I & A_{13}^I \\ A_{21}^I & A_{22}^I & A_{23}^I \\ A_{31}^I & A_{32}^I & A_{33}^I \end{bmatrix}
= \begin{bmatrix} \cos(X_1^I, X_1) & \cos(X_1^I, X_2) & \cos(X_1^I, X_3) \\ \cos(X_2^I, X_1) & \cos(X_2^I, X_2) & \cos(X_2^I, X_3) \\ \cos(X_3^I, X_1) & \cos(X_3^I, X_2) & \cos(X_3^I, X_3) \end{bmatrix}. \tag{13}
$$

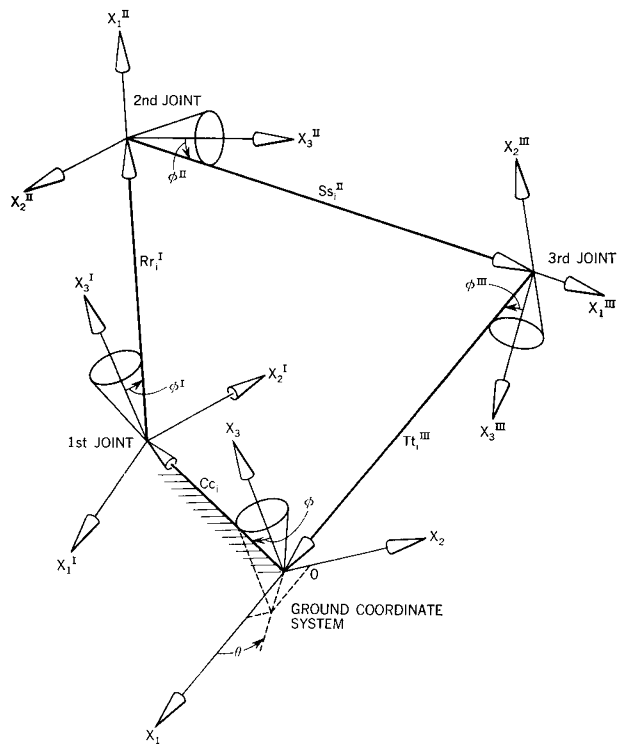

Figure 1 General diagram of four-bar spatial linkage.

• *Categorization of the possible solutions*

In this paper, a single closed-loop vector equation has been used:

$$
Cc_i + Rr_i + Ss_i + Tt_i = 0. \tag{14}
$$

Components of unit vectors c_i, r_i, s_i, and t_i are all taken with respect to the ground frame by the transformations

$$
r_i = A_{mi}^I r_m^I \tag{15a}
$$

$$
s_i = A_{mi}^{II} s_m^{II} \tag{15b}
$$

$$
t_i = A_{mi}^{III} t_i^{III}, \tag{15c}
$$

where A_{mi}^I, A_{mi}^{II}, and A_{mi}^{III} are the transformation matrixes relating the ground frame and the first, second, and third frames, respectively. Equation (14) actually represents three individual equations (for $i = 1, 2, 3$); therefore, it is possible to solve for three arbitrary unknowns. The ground link is Cc_i; its magnitude and direction are always given. Various possible combinations of unknowns for this four-bar linkage can be categorized into the four non-coplanar cases in Table 2. These are the cases used by Chace[8] in his vector solution of Eq. 14.

Case 1. R, S, and T unknown. R can be obtained by multiplying Eq. (14) by $\epsilon_{ijk} s_j t_k$ and eliminating terms

Table 2 Categorization of the solutions.

Case	Unknown quantities	Known quantities	Maximum possible solutions
1	R, S, T	C, c_i, r_i, s_i, t_i	1
2	R, S, θ^{III}	$C, c_i, r_i, s_i, A_{ij}^{III}, T, \phi^{III}$	2
3	$R, \theta^{II}, \theta^{III}$	$C, c_i, r_i, A_{ij}^{II}, A_{ij}^{III}, S, \phi^{II}, T, \phi^{III}$	4
4	$\theta^{I}, \theta^{II}, \theta^{III}$	$C, c_i, A_{ij}^{I}, A_{ij}^{II}, A_{ij}^{III}, R, \phi^{I}, S, \phi^{II}, T, \phi^{III}$	8

containing the product of a symmetric tensor and a skew-symmetric tensor. That is, if Eq. (14) is written

$$C\epsilon_{ijk}c_is_jt_k + R\epsilon_{ijk}r_is_jt_k + S\epsilon_{ijk}s_is_jt_k + T\epsilon_{ijk}t_is_jt_k = 0, \tag{16}$$

the tensor product s_is_j results in a symmetric second-rank tensor Y_{ij}, and t_it_k gives a symmetric tensor Z_{ik}. Recalling that ϵ_{ijk} is a completely skew-symmetric tensor, we have

$$\epsilon_{ijk}s_is_jt_k = \epsilon_{ijk}Y_{ij}t_k = 0$$

$$\epsilon_{ijk}t_is_jt_k = \epsilon_{ijk}Z_{ik}s_j = 0.$$

Equation (16) becomes

$$C\epsilon_{ijk}c_is_jt_k + R\epsilon_{ijk}r_is_jt_k = 0,$$

so that

$$|R| = \frac{C\epsilon_{ijk}c_is_jt_k}{\epsilon_{ijk}r_is_jt_k}. \tag{17a}$$

S and T can be similarly obtained by multiplying Eq. (14) by $\epsilon_{ijk}r_jt_k$ and $\epsilon_{ijk}r_js_k$ and eliminating terms containing the product of symmetrical and skew-symmetrical tensors:

$$|S| = \frac{C\epsilon_{ijk}c_ir_jt_k}{\epsilon_{ijk}r_is_jt_k} \tag{17b}$$

$$|T| = \frac{C\epsilon_{ijk}c_is_jr_k}{\epsilon_{ijk}r_is_jt_k}. \tag{17c}$$

Only one solution is possible.

Case 2. R, S, and θ^{III} unknown. In this case the transformation matrix A_{mi}^{III} between the third frame and the ground frame is known. Then t_i can be obtained from Eq. (15c). Substituting $A_{mi}^{III}t_m^{III}$ for t_i into Eq. (14), we have

$$Cc_i + Rr_i + Ss_i + TA_{mi}^{III}t_m^{III} = 0. \tag{18}$$

Multiplying Eq. (18) by $\epsilon_{ijk}r_js_k$ and using the symmetrical and skew-symmetrical tensor products,

$$C\epsilon_{ijk}c_ir_js_k + T\epsilon_{ijk}r_js_kA_{mi}^{III}t_m^{III} = 0, \tag{19}$$

where $t_m^{III} = (\sin\phi^{III}\cos\theta^{III}, \sin\phi^{III}\sin\theta^{III}, \cos\phi^{III})$. Expanding Eq. (19)

$$\epsilon_{ijk}r_js_k(Cc_i + TA_{3i}^{III}\cos\phi^{III}) + T\sin\phi^{III}\epsilon_{ijk}r_js_k(A_{1i}^{III}\cos\theta + A_{2i}^{III}\sin\theta^{III}) = 0,$$

or

$$a\cos\theta^{III} + b\sin\theta^{III} = c, \tag{20}$$

where a, b, and c are the known quantities

$$a = T\sin\phi^{III}\epsilon_{ijk}r_js_kA_{1i}^{III} \tag{21a}$$

$$b = T\sin\phi^{III}\epsilon_{ijk}r_js_kA_{2i}^{III} \tag{21b}$$

$$c = -\epsilon_{ijk}r_js_k(Cc_i + TA_{3i}^{III}\cos\phi^{III}). \tag{21c}$$

Two solutions are possible for Eq. (20).

Case 3. R, θ^{II}, and θ^{III} unknown. In this case A_{mi}^{II} and A_{mi}^{III} are known. Substituting $A_{mi}^{II}s_m^{II}$ and $A_{mi}^{III}t_m^{III}$ for s_i and t_i, respectively, into Eq. (14) results in

$$Cc_i + Rr_i + SA_{mi}^{II}s_m^{II} + TA_{mi}^{III}t_m^{III} = 0. \tag{22}$$

Multiplying Eq. (22) by $\epsilon_{ijk}r_j$ and, again, eliminating terms containing a product of symmetric and skew-symmetric tensors gives

$$C\epsilon_{ijk}c_ir_j + S\epsilon_{ijk}A_{mi}^{II}r_js_m^{II} + T\epsilon_{ijk}A_{mi}^{III}r_jt_m^{III} = 0. \tag{23}$$

Expanding Eq. (23),

$$\begin{aligned}&\epsilon_{1ij}r_j(Cc_i + SA_{3i}^{II}\cos\phi^{II} + TA_{3i}^{III}\cos\phi^{III})\\ &\quad + S\epsilon_{1ij}r_j\sin\phi^{II}(A_{1i}^{II}\cos\theta^{II} + A_{2i}^{II}\sin\theta^{II})\\ &\quad + T\epsilon_{1ij}r_j\sin\phi^{III}(A_{1i}^{III}\cos\theta^{III} + A_{2i}^{III}\sin\theta^{III}) = 0.\end{aligned} \tag{24a}$$

$$\begin{aligned}&\epsilon_{2ij}r_j(Cc_i + SA_{3i}^{II}\cos\phi^{II} + TA_{3i}^{III}\cos\phi^{III})\\ &\quad + S\epsilon_{2ij}r_j\sin\phi^{II}(A_{1i}^{II}\cos\theta^{II} + A_{2i}^{II}\sin\theta^{III})\\ &\quad + T\epsilon_{2ij}r_j\sin\phi^{III}(A_{1i}^{III}\cos\theta^{III} + A_{2i}^{III}\sin\theta^{III}) = 0.\end{aligned} \tag{24b}$$

Equations (24) can be written as

$$d\cos\theta^{II} + e\sin\theta^{II} + f\cos\theta^{III} + g\sin\theta^{III} = h, \tag{25a}$$

$$d'\cos\theta^{II} + e'\sin\theta^{II} + f'\cos\theta^{III} + g'\sin\theta^{III} = h', \tag{25b}$$

where d, e, f, g, h, are known quantities written explicitly as

$$d = S\epsilon_{1ij}r_j\sin\phi^{II}A_{1i}^{II}, \tag{26a}$$

$$e = S\epsilon_{1ij}r_j\sin\phi^{II}A_{2i}^{II}, \tag{26b}$$

$$f = T\epsilon_{1ij}r_j\sin\phi^{III}A_{1i}^{III}, \tag{26c}$$

$$g = T\epsilon_{1ij}r_j\sin\phi^{III}A_{2i}^{III}, \tag{26d}$$

$$h = -\epsilon_{1ij}r_j(Cc_i + SA_{3i}^{II}\cos\phi^{II} + TA_{3i}^{III}\cos\phi^{III}), \tag{26e}$$

and the respective primed quantities are written by replacing ϵ_{1ij} with ϵ_{2ij}. Four solutions are possible from Eqs. (25).

Case 4. θ^I, θ^{II}, *and* θ^{III} *unknown.* In this case r_i, s_i, and t_i in Eq. (14) are replaced by $A^I_{mi}r^I_m$, $A^{II}_{mi}s^{II}_m$ and $A^{III}_{mi}t^{III}_m$, respectively. Equation (14) then becomes

$$Cc_i + RA^I_{mi}r^I_m + SA^{II}_{mi}s^{II}_m + TA^{III}_{mi}t^{III}_m = 0. \tag{27}$$

Multiplying Eq. (27) by A^I_{3i}, A^{II}_{3i}, and A^{III}_{3i} and recalling from Eq. (4) that $A_{3i}A_{mi} = \delta_{3m} = 0$ if $m \neq 3$ and $\delta_{33} = 1$ if $m = 3$, results in

$$CA^I_{3i}c_i + Rr^I_3 + SA^I_{3i}A^{II}_{mi}s^{II}_m + TA^I_{3i}A^{III}_{mi}t^{III}_m = 0 \tag{28a}$$

$$CA^{II}_{3i}c_i + RA^{II}_{3i}A^I_{mi}r^I_m + Ss^{II}_3 + TA^{II}_{3i}A^{III}_{mi}t^{III}_m = 0 \tag{28b}$$

$$CA^{III}_{3i}c_i + RA^{III}_{3i}A^I_{mi}r^I_m + SA^{III}_{3i}A^{II}_{mi}s^{II}_m + Tt^{III}_3 = 0. \tag{28c}$$

Expanding simultaneous Eqs. (28)

$$\begin{aligned}(CA^I_{3i}c_i + Rr^I_3 + SA^I_{3i} + A^{II}_{3i}\cos\phi^{II} + TA^I_{3i}A^{III}_{3i}\cos\phi^{III}) \\ + SA^I_{3i}\sin\phi^{II}(A^{II}_{1i}\cos\theta^{II} + A^{II}_{2i}\sin\theta^{II}) \\ + TA^I_{3i}\sin\phi^{III}(A^{III}_{1i}\cos\theta^{III} + A^{III}_{2i}\sin\theta^{III}) = 0.\end{aligned} \tag{29a}$$

$$\begin{aligned}(CA^{II}_{3i}c_i + Ss^{II}_3 + RA^{II}_{3i}A^I_{3i}\cos\phi^I + TA^{II}_{3i}A^{III}_{3i}\cos\phi^{III}) \\ + RA^{II}_{3i}\sin\phi^I(A^I_{1i}\cos\theta^I + A^I_{2i}\sin\theta^I) \\ + TA^{II}_{3i}\sin\phi^{III}(A^{III}_{1i}\cos\theta^{III} + A^{III}_{2i}\sin\theta^{III}) = 0.\end{aligned} \tag{29b}$$

$$\begin{aligned}(CA^{III}_{3i}c_i + Tt^{III}_3 + RA^{III}_{3i}A^I_{3i}\cos\phi^I + SA^{III}_{3i}A^{II}_{3i}\cos\phi^{II}) \\ + RA^{III}_{3i}\sin\phi^I(A^I_{3i}\cos\theta^I + A^I_{2i}\sin\theta^I) \\ + SA^{III}_{3i}\sin\phi^{II}(A^{II}_{1i}\cos\theta^{II} + A^{II}_{2i}\sin\theta^{II}) = 0.\end{aligned} \tag{29c}$$

Equations (29) can be written as

$$i\cos\theta^{II} + j\sin\theta^{II} + k\cos\theta^{III} + l\sin\theta^{III} = m, \tag{30a}$$

$$i'\cos\theta^I + j'\sin\theta^I + k'\cos\theta^{III} + l'\sin\theta^{III} = m', \tag{30b}$$

$$i''\cos\theta^I + j''\sin\theta^I + k''\cos\theta^{II} + l''\sin\theta^{II} = m'', \tag{30c}$$

where i, j, k, l, m and their primed and double-primed counterparts are invariants whose explicit expressions can be obtained in the same way that those for Cases 2 and 3 were obtained. Eight solutions are possible from Eqs. (30).

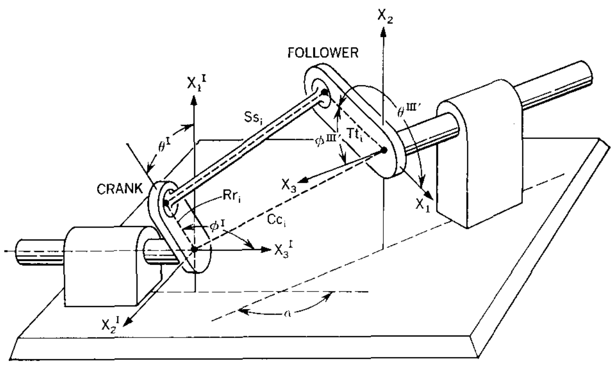

Figure 2 *R-S-S-R* mechanism (after Hartenberg and Denavit[10]).

Application to R-S-S-R four-link mechanism

A recent paper by L. Harrisberger[4] has treated the mobility of an *R-S-S-R* four-link mechanism in detail using the vector method. Hartenberg and Denavit[10] have also analyzed this mechanism with a matrix technique. Here, we wish to demonstrate the simplicity of the tensor method by determining the linkage positions of the same mechanism. Figure 2 is a diagram of the *R-S-S-R* mechanism with revolute pairs on the ground and first joints, and with spherical pairs on the second and third joints. The problem may be stated as follows:

Given

1. The c_i and the magnitude C.
2. Input angle θ^I, the polar angle ϕ^I, and the magnitude R. A^I_{im} and r^I_i are known.
3. Magnitude S.
4. Polar angle $\phi^{III'}$ between the ground frame and t_i, and the magnitude T.

Find

1. Components of unit vector s_i with respect to the ground frame; and the polar and azimuthal angles of the link Ss_i, $\phi^{II'}$ and $\theta^{II'}$, with respect to the ground frame.
2. Output azimuthal angle $\theta^{III'}$ with respect to the ground frame.

From Eqs. (14) and (15), the three unknown quantities can be determined by solving

$$Cc_i + RA^I_{mi}r^I_m + Ss_i + Tt_i = 0, \tag{31}$$

where

$$s_i = (\sin\phi^{II'}\cos\theta^{II'}, \sin\phi^{II'}\sin\theta^{II'}, \cos\phi^{II'})$$

$$t_i = (\sin\phi^{III'}\cos\theta^{III'}, \sin\phi^{III'}\sin\theta^{III'}, \cos\phi^{III'}).$$

Letting $K_i = Cc_i + RA^I_{mi}r^I_m$, a constant vector for any given input crank angle θ^I, Eq. (31) can be expanded as

$$K_1 + S \sin \phi^{II'} \cos \theta^{II'} + T \sin \phi^{III'} \cos \theta^{III'} = 0. \quad (32a)$$

$$K_2 + S \sin \phi^{II'} \sin \theta^{II'} + T \sin \phi^{III'} \sin \theta^{III'} = 0. \quad (32b)$$

$$K_3 + S \cos \phi^{II'} + T \cos \phi^{III'} = 0. \quad (32c)$$

Equations (32) are the general scheme for analyzing an *R-S-S-R* mechanism. In the *R-S-S-R* mechanism shown in Fig. 2 the X_3 axis of the ground frame is the axis of rotation of the follower. The angle, $\phi^{III'}$, between the X_3 axis and follower T is a right angle. The X_2^I axis of the first frame is the axis of rotation of the crank. The angle, ϕ^I, between the X_3^I axis and the crank is also a right angle, so the input crank vector r_i^I may be written in terms of polar angle θ^I and the input azimuth angle ϕ^I of its local frame as

$$r_1^I = \sin \phi^I \cos \theta^I = \cos \theta^I \quad (33a)$$

$$r_2^I = \sin \phi^I \sin \theta^I = \sin \theta^I \quad (33b)$$

$$r_3^I = \cos \phi^I = 0. \quad (33c)$$

The orientation of the coordinate frames as shown in Fig. 2 makes the X_1^I axis and the X_2 axis parallel, so the transformation coefficients have the following values:

$$A_{mi}^I = \begin{bmatrix} 0 & 1 & 0 \\ -\cos \alpha & 0 & \sin \alpha \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix}. \quad (34)$$

Inserting Eqs. (33) and (34) into Eq. (15) gives

$$r_i = (-\cos \alpha \sin \theta^I, \sin \theta^I, \sin \alpha \sin \theta^I). \quad (35)$$

Then the constant vector K_i for given input crank angle θ^I is

$$K_i = (Cc_1 - R \cos \alpha \sin \theta^I, \; Cc_2 + R \sin \theta^I, \; Cc_3 + R \sin \alpha \sin \theta^I). \quad (36)$$

Equations (32) now become

$$K_1 + S \sin \phi^{II'} \cos \theta^{II'} + T \cos \theta^{III'} = 0 \quad (37a)$$

$$K_2 + S \sin \phi^{II'} \sin \theta^{II'} + T \sin \theta^{III'} = 0 \quad (37b)$$

$$K_3 + S \cos \phi^{II'} = 0. \quad (37c)$$

The solution of Eq. (37c) is

$$\phi^{II'} = \text{arc} \cos \frac{K_3}{S} = \text{arc} \sin \frac{\sqrt{S^2 - K_3^2}}{S}. \quad (38)$$

Substituting Eq. (38) into Eqs. (37a) and (37b),

$$K_1 + \sqrt{S^2 - K_3^2} \cos \theta^{II'} + T \cos \theta^{III'} = 0 \quad (39a)$$

$$K_2 + \sqrt{S^2 - K_3^2} \sin \theta^{II'} + T \sin \theta^{III'} = 0, \quad (39b)$$

it becomes clear that the output follower angle $\theta^{III'}$ has two possible solutions. This confirms the statement by Harrisberger[4] that the *R-S-S-R* mechanism has two degrees of freedom.

Conclusions

This paper has shown that tensor notation provides a convenient and compact means for expressing relationships in spatial mechanisms. Some tensor operations that have no counterparts in vector algebra have been demonstrated as being powerful aids to obtaining problem solutions. Further, the tensor transformations have relieved the burdens of the tedious and confusing references of the coordinate frame. It is evident that the tensor notation lends itself well to programming for computer solution of problems in spatial kinematics.

The author hopes that others will also find the tensor method a suitable addition to the existing methods of exploring the spatial domain of linkages, and that this paper will stimulate the engineer's interest in spatial mechanisms.

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