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On a Circular Crack in a Transversely Isotropic Elastic Material Under Prescribed Shear Stress

In a recent paper the elasticity problem of a planar displacement discontinuity in a transversely isotropic elastic material was studied.[1] This note is concerned with the analogous, mixed boundary value problem of a penny-shaped crack under prescribed shear stress at its surface. The analysis is partly based upon the treatment of a similar problem in isotropic elastic materials.[2] The problem is formulated for the quasistatic situation, i.e., when the crack is moving at a constant velocity along a direction perpendicular to its surfaces; the solution to the static case is then obtained by making the magnitude of the velocity vanish. Strain energy associated with the crack is calculated.

Basic equations

In the linear theory of elasticity the fundamental system of field equations is: the linearized strain-displacement equations, the linear stress-strain relations, and the stress equations of motion. We shall restrict ourselves to transversely isotropic, homogeneous media with the axis of symmetry of the material taken to be the z axis.

The stress-strain relations are:

$$\begin{aligned}
\sigma_{rr} &= c_{11}e_{rr} + c_{12}e_{\theta\theta} + c_{13}e_{zz} \\
\sigma_{\theta\theta} &= c_{12}e_{rr} + c_{11}e_{\theta\theta} + c_{13}e_{zz} \\
\sigma_{zz} &= c_{13}(e_{rr} + e_{\theta\theta}) + c_{33}e_{zz} \\
\sigma_{r\theta} &= \tfrac{1}{2}(c_{11} - c_{12})e_{r\theta} \\
\sigma_{\theta z} &= c_{44}e_{\theta z} \\
\sigma_{rz} &= c_{44}e_{rz}\ ,
\end{aligned} \tag{1}$$

where the five c_{ij}'s are the elastic constants.

It is assumed that the crack moves along the z axis at a constant velocity v for a long time so that to an observer moving at the same velocity, the displacement field appears to be always the same. We use a cylindrical coordinate system (r, θ, z) moving with the crack, so that the plane of the crack always coincides with the $z = 0$ plane.

It can be shown[3] that all the field equations are satisfied if the displacement components u_r, u_θ, u_z are expressed in terms of three "harmonic" functions ϕ_1, ϕ_2, ψ which are solutions of

$$\left(\nabla_1^2 + \frac{\partial^2}{\partial z_j^2}\right)\phi_j = 0 \qquad (j = 1, 2) \tag{2}$$

$$\left(\nabla_1^2 + \frac{\partial^2}{\partial z_3^2}\right)\psi = 0, \tag{3}$$

where

$$z_i = z/\sqrt{\nu_i} \qquad (i = 1, 2, 3) \tag{4}$$

$$\nabla_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{5}$$

and ν_1, ν_2 are roots of the equation

$$\begin{aligned}
&c_{11}c_{44}\nu^2 - [(c_{44} - \rho v^2)c_{44} + (c_{33} - \rho v^2)c_{11} \\
&- (c_{13} + c_{44})^2]\nu + (c_{33} - \rho v^2)(c_{44} - \rho v^2) = 0,
\end{aligned} \tag{6}$$

and also

$$\nu_3 = 2(c_{44} - \rho v^2)/(c_{11} - c_{12}). \tag{7}$$

In Eqs. (6) and (7), ρ is the density of the material.

The displacement components are

$$\begin{aligned}
u_r &= \frac{\partial}{\partial r}(\phi_1 + \phi_2) + \frac{1}{r}\frac{\partial \psi}{\partial \theta} \\
u_\theta &= \frac{1}{r}\frac{\partial}{\partial \theta}(\phi_1 + \phi_2) - \frac{\partial \psi}{\partial r} \\
u_z &= \frac{k_1}{\sqrt{\nu_1}}\frac{\partial \phi_1}{\partial z_1} + \frac{k_2}{\sqrt{\nu_2}}\frac{\partial \phi_2}{\partial z_2},
\end{aligned} \tag{8}$$

where k_1 and k_2 are given by

$$\nu_j = \frac{k_j(c_{33} - \rho v^2)}{k_j c_{44} + (c_{13} + c_{44})} = \frac{k_j(c_{13} + c_{44}) + (c_{44} - \rho v^2)}{c_{11}}. \quad (9)$$

From the above equations the three stress components of immediate interest are

$$\sigma_{zz}/c_{44} = [(1 + k_1) + k_1\beta/\nu_1] \frac{\partial^2\phi_1}{\partial z_1^2} + [(1 + k_2) + k_2\beta/\nu_2] \frac{\partial^2\phi_2}{\partial z_2^2} \quad (10)$$

$$\sigma_{\theta z}/c_{44} = [(1 + k_1)/\sqrt{\nu_1}] \frac{1}{r} \frac{\partial^2\phi_1}{\partial\theta\,\partial z_1} + [(1 + k_2)/\sqrt{\nu_2}] \frac{1}{r} \frac{\partial^2\phi_2}{\partial\theta\,\partial z_2} - \frac{1}{\sqrt{\nu_3}} \frac{\partial^2\psi}{\partial r\,\partial z_3} \quad (11)$$

$$\sigma_{rz}/c_{44} = [(1 + k_1)/\sqrt{\nu_1}] \frac{\partial^2\phi_1}{\partial r\,\partial z_1} + [(1 + k_2)/\sqrt{\nu_2}] \frac{\partial^2\phi_2}{\partial r\,\partial z_2} + \frac{1}{\sqrt{\nu_3}} \frac{1}{r} \frac{\partial^2\psi}{\partial\theta\,\partial z_3}. \quad (12)$$

In Eq. (10), $\beta = \rho v^2/c_{44}$. (13)

It is assumed that the roots in Eq. (6), ν_1 and ν_2, will be unequal, and may be positive real, or complex conjugates. The root ν_3 is always assumed to be real and positive. It is specified that in the case of complex roots, $\sqrt{\nu_1}$ and $\sqrt{\nu_2}$ will have positive real parts. Physically we restrict v to be less than any of the propagation velocities of the material.

Boundary conditions

As mentioned previously, the crack is situated at the plane, $z = 0$. It is assumed to be circular and defined by the circle $r = 1$. The boundary conditions may be stated as:

(1) The functions $(\sigma_{zz} - \rho v^2 \partial u_z/\partial z)$, $(\sigma_{\theta z} - \rho v^2 \partial u_\theta/\partial z)$, and $(\sigma_{rz} - \rho v^2 \partial u_r/\partial z)$ are continuous everywhere across the $z = 0$ plane. Within the unit circle, σ_{rz} and $\sigma_{\theta z}$ are specified.

(2) The displacement components u_z, u_r, and u_θ are continuous across the $z = 0$ plane except perhaps within the unit circle.

(3) Stress components must vanish at infinity.

Let plain and prime quantities be associated with half spaces defined by $z \geq 0$ and $z \leq 0$, respectively. Boundary conditions become:

At $z = 0$

$$\left(\sigma_{rz} - \rho v^2 \frac{\partial u_r}{\partial z}\right) = \left(\sigma'_{rz} - \rho v^2 \frac{\partial u'_r}{\partial z}\right) \quad (14)$$

$$\left(\sigma_{\theta z} - \rho v^2 \frac{\partial u_\theta}{\partial z}\right) = \left(\sigma'_{\theta z} - \rho v^2 \frac{\partial u'_\theta}{\partial z}\right) \quad (15)$$

$$\left(\sigma_{zz} - \rho v^2 \frac{\partial u_z}{\partial z}\right) = \left(\sigma'_{zz} - \rho v^2 \frac{\partial u'_z}{\partial z}\right). \quad (16)$$

At $z = 0$, $r > 1$

$$u_r = u'_r,\ u_\theta = u'_\theta \quad (17)$$

$$u_z = u'_z. \quad (18)$$

At $z = 0$, $r < 1$

$$\sigma_{rz} = \sum_n M_n(r) \cos n\theta \quad (19)$$

$$-\sigma_{\theta z} = \sum_n N_n(r) \sin n\theta. \quad (20)$$

Solution

We shall be concerned with the two half spaces $z \geq 0$ and $z \leq 0$ to which are assigned the potential functions (ϕ_1, ϕ_2, ψ) and $(\phi'_1, \phi'_2, \psi')$, respectively. We try the following substitutions:

$$\phi_1(r, \theta, z_1) = -\phi'_1(r, \theta, -z_1) = \frac{2}{(1 + k_1)c_{44}} H(r, \theta, z_1)$$

$$\phi_2(r, \theta, z_2) = -\phi'_2(r, \theta, -z_2) = -\frac{2}{(1 + k_2)c_{44}} H(r, \theta, z_2) \quad (21)$$

$$\psi(r, \theta, z_3) = -\psi'(r, \theta, -z_3) = \frac{2\sqrt{\nu_3}}{c_{44}} G(r, \theta, z_3).$$

$H(r, \theta, z)$ and $G(r, \theta, z)$ are both harmonic functions in (r, θ, z) space. It is observed that Eqs. (14), (15), (16), and (18) are now satisfied.

The functions H and G are assumed to be of the form:

$$H(r, \theta, z) = \sum_{n=0}^{\infty} \cos n\theta \int_0^\infty u^{-1} h_n(u) e^{-uz} J_n(ru)\, du$$

$$G(r, \theta, z) = \sum_{n=1}^{\infty} \sin n\theta \int_0^\infty u^{-1} g_n(u) e^{-uz} J_n(ru)\, du. \quad (22)$$

$J_n(u)$ is a Bessel function of order n; $h_n(u)$ and $g_n(u)$ are yet to be determined. After Eq. (22) is substituted into Eqs. (21), (8), (11), and (12), the conditions in Eqs. (17), (19), and (20) become for $n = 0$:

$$2B \int_0^\infty u h_0(u)\, J_1(ur)\, du = M_0(r) \qquad (r < 1)$$

$$\int_0^\infty h_0(u)\, J_1(ur)\, du = 0. \qquad (r > 1) \tag{23a}$$

For $n \geq 1$:

$$\int_0^\infty [Ah_n(u) + g_n(u)]\, J_{n-1}(ur)\, du = 0 \qquad (r > 1)$$

$$\int_0^\infty [Ah_n(u) - g_n(u)]\, J_{n+1}(ur)\, du = 0 \qquad (r > 1)$$

$$\int_0^\infty u[Bh_n(u) + g_n(u)]\, J_{n-1}(ur)\, du \tag{23b}$$

$$= -\tfrac{1}{2}[M_n(r) + N_n(r)] \qquad (r < 1)$$

$$\int_0^\infty u[Bh_n(u) - g_n(u)]\, J_{n+1}(ur)\, du$$

$$= \tfrac{1}{2}[M_n(r) - N_n(r)], \qquad (r < 1)$$

where

$$A = \frac{1}{\sqrt{\nu_3}}\left[\frac{1}{1 + k_1} - \frac{1}{1 + k_2}\right]$$

$$B = \left[\frac{1}{\sqrt{\nu_1}} - \frac{1}{\sqrt{\nu_2}}\right]. \tag{24}$$

Pairs of simultaneous dual integral equations of the above type with $n = 1$, $M_1(r) = N_1(r) = 1$, have been solved by Westmann,[2] whose work leads us to try to solve the case where $M_n(r) = N_n(r) = r^{n-1}$. Setting

$$g_n(u) = Ah_n(u), \text{ and } (A + B)h_n(u) = -p_n(u) \tag{25}$$

we obtain

$$\int_0^\infty u p_n(u)\, J_{n-1}(ur)\, du = r^{n-1} \qquad (r < 1) \tag{26}$$

$$\int_0^\infty p_n(u)\, J_{n-1}(ur)\, du = 0 \qquad (r > 1) \tag{27}$$

$$\int_0^\infty u p_n(u)\, J_{n+1}(ur)\, du = 0. \qquad (r < 1) \tag{28}$$

Solution to the pair formed by Eqs. (26) and (27) is given by [see Luke,[4] Eq. 13.4.8 (10)]:

$$p_n(u) = \frac{\Gamma(n)}{\sqrt{2u}\, \Gamma(n + \frac{1}{2})} J_{1/2+n}(u). \tag{29}$$

From the well-known relation,

$$\int_0^\infty u^{m+1-n} J_n(au)\, J_m(bu)\, du = 0 \qquad (0 < a < b)$$

$$= \frac{b^m(a^2 - b^2)^{n-m-1}}{2^{n-m-1} a^n \Gamma(n - m)}, \qquad (0 < b < a) \tag{30}$$

$\operatorname{Re}(n - m) > 0$, $\operatorname{Re}(m) > -1$.

It can be verified that Eq. (29) also satisfies Eq. (28) [see Luke,[4] Eq. 13.4.2 (4)]. Hence, Eqs. (25) and (29) form the solution to the particular pair of simultaneous dual integral equations, Eqs. (26) to (28).

With the knowledge of the solution to the general pair of dual integral equations, Eqs. (23b), the elasticity problem as stated between Eqs. (14) and (20) is solved. We shall now examine two simple physically interesting examples utilizing the general solution.

Examples

- *Uniform shear stress*

Suppose there is a unit uniform shear stress acting along the direction $\theta = 0$, on the surface of this crack of unit radius. In Eqs. (19) and (20) we shall have $M_1(r) = N_1(r) = 1$, all other $M_n(r)$ and $N_n(r)$ vanish. From Eq. (29) we have

$$p_1(u) = \left(\frac{2}{\pi u}\right)^{1/2} J_{3/2}(u) \tag{31}$$

and all other $p_n(u)$ vanish. Equations (21), (22), (25), and (31) form the solution to this particular elasticity problem.

We shall evaluate the strain energy associated with this crack of unit radius under unit shear stress at its surfaces. We shall assume also that it is stationary, i.e., $\rho v^2 = 0$. We find that the energy is

$$W = \int_0^{2\pi} \int_0^1 (\sigma_{\theta z} u_\theta + \sigma_{rz} u_r) \Big|_{z=0^-} r\, dr\, d\theta$$

$$= \frac{8A}{3c_{44}(A + B)}.$$

From Eq. (24), with $\beta = 0$,

$$W = \{8c_{11} \sqrt{\nu_1\nu_2}\, (\sqrt{\nu_1} + \sqrt{\nu_2})\}$$
$$\div \{3c_{44}[\sqrt{\nu_1\nu_2}\, (\sqrt{\nu_1} + \sqrt{\nu_2})c_{11}$$
$$+ (c_{13} + c_{44})(1 + k_1)(1 + k_2) \sqrt{\nu_3}]\}. \tag{32}$$

In the case of isotropy, this expression reduces to

$$W_I = \frac{8(2\mu + \lambda)}{3\mu(4\mu + 3\lambda)}, \tag{33}$$

where λ and μ are Lamé's constants. This result may be shown to agree with the value of strain energy calculated from Westmann's results.[2]

- *Linearly varying shear stress*

In this example the magnitude of the shear stress varies linearly with the distance along the $\theta = 0$ direction (say, the x direction). Hence at $z = 0$, $r < 1$,

$$\sigma_{xz} = x \tag{34}$$

or

$$\begin{aligned} \sigma_{rz} &= \tfrac{1}{2}r(1 + \cos 2\theta) \\ -\sigma_{\theta z} &= \tfrac{1}{2}r \sin 2\theta. \end{aligned} \tag{35}$$

From the previous results contained between Eqs. (23) and (30), we find that

$$h_0(u) = \frac{1}{3B\sqrt{2\pi u}} J_{5/2}(u) \tag{36}$$

and

$$p_2(u) = \frac{1}{3}\left(\frac{2}{\pi u}\right)^{1/2} J_{5/2}(u) \tag{37}$$

and all other $p_n(u)$ vanish for $n \geq 1$.

Equations (36) and (37) thus form the solution to the elasticity problem specified by Eq. (34).

When $z = 0^-$, $r < 1$, the displacement components u_θ, u_r, are

$$\begin{aligned} u_\theta &= -\frac{4\sqrt{\nu_3}\, Ar(1 - r^2)^{1/2} \sin 2\theta}{3\pi(A + B)c_{44}} \\ u_r &= \frac{2\sqrt{\nu_3}\, Ar(1 - r^2)^{1/2}[(A + B) + 2B\cos 2\theta]}{3\pi(A + B)Bc_{44}}. \end{aligned} \tag{38}$$

The strain energy is found to be

$$W = \frac{8A\sqrt{\nu_3}\,(A + 3B)}{45c_{44}B(A + B)}. \tag{39}$$

From Eq. (24) with $\beta = 0$, the strain energy is

$$\begin{aligned} W = \{&8c_{11}\sqrt{\nu_1\nu_2}\,(\sqrt{\nu_1} + \sqrt{\nu_2})[c_{11}(\sqrt{\nu_1} + \sqrt{\nu_2}) \\ &\cdot \sqrt{\nu_1\nu_2} + 3(1 + k_1)(1 + k_2)(c_{13} + c_{44})\sqrt{\nu_3}]\} \\ &\div \{45c_{44}(c_{13} + c_{44})(1 + k_1)(1 + k_2) \\ &\cdot [c_{11}(\sqrt{\nu_1} + \sqrt{\nu_2})\sqrt{\nu_1\nu_2} \\ &+ (1 + k_1)(1 + k_2)(c_{13} + c_{44})\sqrt{\nu_3}]\}. \end{aligned} \tag{40}$$

In the case of isotropy, this expression reduces to

$$W = \frac{4(8\mu + 7\lambda)(2\mu + \lambda)}{45\mu(\mu + \lambda)(4\mu + 3\lambda)}. \tag{41}$$

The solution to this elasticity problem seems to be new even for the isotropic case.

Conclusion

The analysis provides the solution for the elasticity problem of a circular flat crack in a transversely isotropic elastic solid under uniform or linear shear stress distribution at the crack surface. This analysis, together with the published works[5,6] on circular cracks under normal stress, solves the problem of a circular crack under any uniform stress distribution at large distance from the crack. The result should be of interest in fracture mechanics. It may be of interest to remark that the solution presented here has provided valuable insight to solving the corresponding problem for the elliptical crack.

References

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