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Average Motion Times of Positioners in Random Access Devices

Abstract: An analysis is made of the average motion times of mechanical positioners based on trapezoidal velocity vs time curves. The results are plotted in terms of dimensionless motion parameters. It is shown that average motion time may be optimized by balancing acceleration and velocity in a proper way. The selection of optimum transmission ratios between motor and load is discussed and demonstrated in an example.

Introduction

As the cycle time of digital computers approaches the nanosecond range, the capability of processing large volumes of data in a diminishingly short time period correspondingly increases. Simultaneously, the capacity requirements of storage devices are extending into the billion-character range. Major design problems arise: Core memory in the processor is still too expensive to be used as storage of a large volume of information. On the other hand, large capacity devices such as tape drives, disk files, and card storage units have relatively long access times compared to the speed of processors. It remains a challenging engineering task to reduce these access times, particularly in the area of random access storage devices.

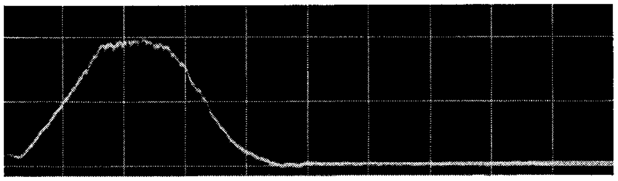
Most of the commonly known random access memories are basically dependent on mechanical systems. Access to particular information in storage is accomplished by physically moving machine parts or storage media over a distance of several inches or more. The distance to be moved varies, depending on the particular storage location at which the motion starts and where it ends. Hence, the minimization of motions or access times cannot be attacked by considering only a single distance within an array of storage elements.

There is a wide range of actuator designs, power sources, and transducers already in use for random access purposes. However, there seems to be no general rule of how to apply these actuators or positioners most efficiently in order to obtain minimum average motion times. In the literature some concern has been given to the optimization of positioning servos subjected to random or statistically representable control signals.[1] The presented analytical methods, however, are rather general and apply to more or less continuous perturbations, e.g., those of radar antennas under wind loads. In addition, the principle of matching motor and load inertias has been considered for maximum power transfer.[2] Finally, as a last resource, the experimental approach was taken and positioners were "tuned" for shortest average times.

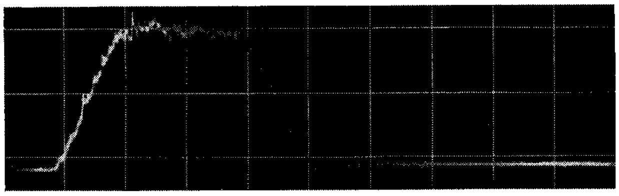
None of these methods, however, is practical or satisfying from a good engineering point of view. A more general analytical way of predicting the average motion times as a function of design parameters such as inertia, driving force, maximum distance, and terminal speed must be provided. An analysis is desirable which correlates these parameters and which shows how a positioning system may be built with minimum average motion time as its objective.

A representation of average motion times as a function of complicated transient response curves would hardly yield any general and practical results. There are literally thousands of possible combinations among force or torque characteristics of motors, load inertias, and control networks that influence the motion of any access system. The analysis has to be restricted, therefore, to a mathematical model which describes motion control in simplified terms, yet closely represents reality. The model has to show, at least on a comparative basis, the mutual influence of characteristic motion parameters on average time. The model should also provide design limits for optimization of a positioner device.

In a search for an adequate motion model, it may be found that most positioners have velocity vs time curves which may be closely represented by triangles or trape-



(a)



(b)

Figure 1 (a) Velocity versus time curve of an access servo system using magnetic powder clutches (IBM 1405 Disk File). (b) Velocity vs time curve of a hydraulic servo system (IBM 2321 Data Cell Drive).

zoids. In Fig. 1, two measured velocity traces illustrate this form. "Velocity trapezoids," therefore, appear as the convenient tools to describe motion. They include, as extremes, rectangles as well as triangles with time as the base line.

Analysis

• *Definition of basic motion parameters*

In our basic positioning model we consider a linear array of storage locations within a total length L. The number of evenly distributed positions Y_n in the array should be sufficiently large that it is possible to smooth the discrete distances of travel by a continuous and variable length $X \leq L$. For practical purposes, the previous assumption yields satisfying results if the number of positions is greater than ten.

The velocity triangles and trapezoids shown in Fig. 2 represent the motion characteristics of our model.* The individual lines and the time axis enclose areas which represent the following distances of travel:

Line 1-1: Maximum distance L in the storage array
Line 1-2: Any random distance $X \leq L$
Line 1-3: Largest distance b travelled in the "triangular mode."

Note that any random distance X can be expressed by the integral

$$X = \int_{t=0}^{t\,x} v(t)\, dt. \tag{1}$$

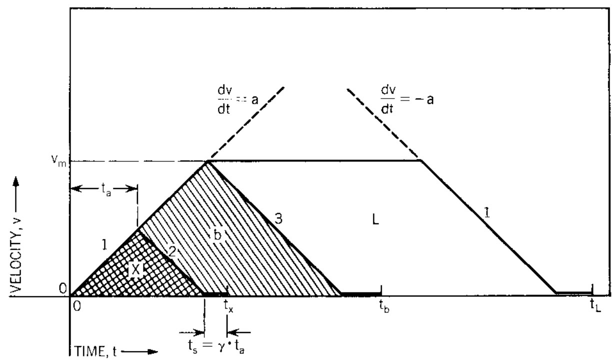
In order to treat the velocity curves in Fig. 2 as generally as possible, the following four parameters are defined.

Control factor γ

In practical cases it cannot be expected that acceleration and deceleration of a positioning system have equal magnitudes. In addition, different settling times at the target points have to be encountered, depending on the type of motion control.

* A list of nomenclature appears at the end of this article.

Figure 2 Mathematical model of velocity versus time curves.



Therefore, we choose a variable factor γ which modifies the motion times in accordance with overshoots and sluggishness encountered in the actual system. As Fig. 2 indicates, the control factor γ is multiplied by the corresponding acceleration times $t_a = f(x)$. The resulting product γt_a is added to the net motion time obtained with the theoretical triangles or trapezoids. Practical experience proves that the latter assumption is a good approximation from two points of view:

(1) Long acceleration periods, i.e., small driving forces, generally result in a "sluggish" time response and longer settling periods at the target point.

(2) High velocities which are reached in extended acceleration times usually cause difficulties in obtaining perfect control over the deceleration period. The resulting "overshoots" are not likely to occur within the tolerable range. In addition, a higher velocity implies a larger power source which inherently has longer time constants compared with small sources.

Speed factor $\beta = b/L$

The particular type of motion model may be described by a dimensionless speed factor:

$$\beta = b/L = v_m^2/aL, \tag{2}$$

where $0 \leq \beta \leq 1$.

This parameter represents the largest distance travelled in the triangular mode divided by the maximum distance L of the file array. Figure 2 illustrates β as the ratio of the two areas b and L. The usefulness of such a definition is demonstrated by the fact that in all possible motion trapezoids, β ties the terminal velocity v_m of the positioning device to the acceleration a and the maximum distance L, in the form of a dimensionless quantity. Thus the following analysis will be greatly simplified.

Relative distance $\xi = X/L$ and *storage location* $\eta = Y/L$

In order to achieve dependency from the widely varying maximum distance L of different file arrays, the relative distance

$$\xi = X/L, \quad \text{where} \quad 0 \leq \xi \leq 1, \tag{3}$$

and the relative storage location

$$\eta = Y/L, \quad \text{where} \quad 0 \leq \eta \leq 1 \tag{4}$$

are introduced. In these expressions, X represents the distance between two randomly picked storage locations Y_n and Y_{n+1}. As previously mentioned, X and Y, and hence ξ and η, will be treated as continuous variables.

- *Probability of moving distances in storage array with random reference activity*

Any storage location in a file with truly random reference activity has the same probability of being accessed. Hence, the probability to move to any location $\eta = Y/L$ can be defined as

$$P(\eta) = 1 \quad \text{for} \quad 0 \leq \eta \leq 1. \tag{5}$$

In a linear storage array the probability P_ξ for moving any distance ξ between two discrete locations η_n and η_{n+1} may now be evaluated as the "convolution" of $P(\eta)$ with itself,[3] which in our case may be defined as

$$P_\xi = P(\eta) * P(\eta) = \int_{-\infty}^{+\infty} P(\eta)P(\eta - \xi)\, d\eta \tag{6}$$

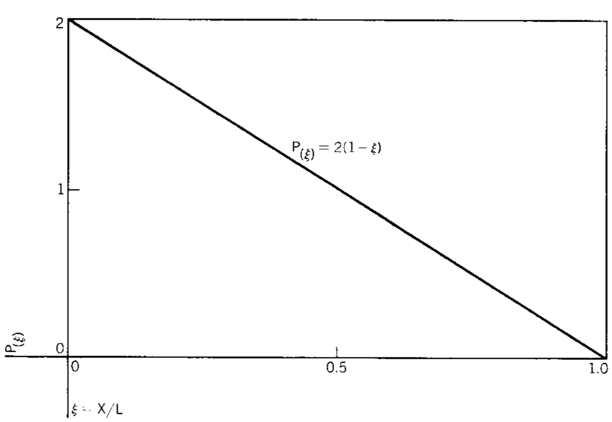
The probability $P(\eta)$ of storage locations is zero outside the positioning range. In addition, only absolute values of distances ξ are accepted. Hence, for $0 \leq \eta \leq 1$

$$0 \leq \xi \leq 1$$

$$0 \leq (\eta - \xi) \leq 1$$

$$0 \leq (\eta + \xi) \leq 1.$$

Figure 3 Probability P to move a variable distance $\xi = X/L$ in a storage array.



Equation (6) may be solved in the form:

$$P_\xi = \int_\xi^1 P(\eta)P(\eta - \xi)\, d\eta + \int_0^{(1-\xi)} P(\eta)P(\eta + \xi)\, d\eta \tag{7}$$

and with Equation (6)

$$P_\xi = 2(1 - \xi). \tag{8}$$

Obviously, the probability to move any distance in the array is a linear relationship in ξ. In Fig. 3 this relationship is drawn. We note that the probability to move short distances is highest, while the probability to move the maximum distance L ($\xi = 1$) approaches zero.

From the foregoing analysis we may easily obtain, as an additional result, the average distance moved by the positioner. We find

$$\xi_{av} = \int_0^1 P_\xi\, \xi\, d\xi = 2\int_0^1 (1 - \xi)\xi\, d\xi = 1/3. \tag{9}$$

Hence, the average distance of motion with random reference activity is exactly 1/3 of the maximum distance.

- *Derivation of average motion time*

In general, average motion time of any series of accesses may be precisely defined as

$$t_{av} = \int_0^1 P_\xi t(\xi)\, d\xi, \tag{10}$$

where again the dimensionless relative distance $\xi = X/L$ is used instead of the absolute distance of motion X.

For our basic motion model in Fig. 2 we obtain the following kinematic laws:

For $0 \leq X \leq b$ or $0 \leq \xi \leq \beta$,

$$t(X) = (2 + \gamma)\sqrt{\frac{X}{a}} \qquad (0 \leq X \leq b), \tag{11}$$

or with Eq. (3)

$$t(\xi) = (2 + \gamma)\sqrt{\xi}\sqrt{\frac{L}{a}} \qquad (0 \leq \xi \leq \beta), \tag{12}$$

where the previously defined variable control factor γ takes the influence of deceleration control and settling time into account. Similarly we obtain:

For $b \leq X \leq L$ or $\beta \leq \xi \leq 1$,

$$t(X) = \frac{b(1 + \gamma) + X}{\sqrt{ba}} \qquad (b \leq X \leq L). \tag{13}$$

Or with Eqs. (2) and (3)

$$t(\xi) = \frac{\beta(1 + \gamma) + \xi}{\sqrt{\beta}}\sqrt{\frac{L}{a}} \qquad (\beta \leq \xi \leq 1). \tag{14}$$

Substitution of Eqs. (8), (12), and (14) into Eq. (10) results in the integral equation for average time:

$$t_{av} = \left[2(2+\gamma) \int_0^{\beta} \sqrt{\xi}(1-\xi)\, d\xi + \frac{2}{\sqrt{\beta}} \int_{\beta}^{1} [\beta(1+\gamma)+\xi](1-\xi)\, d\xi \right] \sqrt{\frac{L}{a}}. \quad (15)$$

The above equation may be easily integrated for any speed factor β and control factor γ that are considered to be constant within a given positioning device. We obtain:

$$t_{av} = \frac{1}{15\sqrt{\beta}} [\beta^3(1+3\gamma) - 5\beta^2(1+2\gamma) + 15\beta(1+\gamma) + 5] \sqrt{\frac{L}{a}} \quad (16)$$

as average motion time.

In the next sections, Eq. (16) is discussed from several practical points of view. In particular, the influence of terminal velocity (see Fig. 2) at constant acceleration and the optimum utilization of a power source are investigated.

Effect of terminal velocity at constant acceleration

Equation (16) may be abbreviated in the form

$$t_{av} = \tau_a \sqrt{L/a} \quad (17)$$

where, as a dimensionless "time constant," the expression

$$\tau_a = \frac{1}{15\sqrt{\beta}} [\beta^3(1+3\gamma) - 5\beta^2(1+2\gamma) + 15\beta(1+\gamma) + 5] \quad (18)$$

is defined.

In the practical case now under consideration, the maximum distance L as well as the acceleration a will be constant. Hence, the average time, according to (17), becomes strictly proportional to the time constant $\tau_a = f(\beta, \gamma)$. In addition, we find according to Eq. (2) that the speed factor β is proportional to the second power of the peak velocity v_m^2. Hence

$$t_{av} = f(v_m^2, \gamma). \quad (19)$$

In Fig. 4 the time constant τ_a is plotted versus the speed factor β with the control factor γ as parameter. Note that according to Eq. (2) the range of β is fixed for all practical cases between 0 and 1. For $\beta = 1$ we have triangular velocity vs time curves throughout the positioning range. For $\beta < 1$, instead, the peak velocity is restricted. We obtain trapezoidal velocity curves where the reduction of peak velocity follows the proportionality

$$v_m \propto \sqrt{\beta}. \quad (20)$$

Fig. 4 shows an interesting result: The average time constant τ_a is almost unaffected by a variation of the speed factor β in the range $0.4 < \beta < 1$. For a control factor $\gamma = 0.5$, the time constant shows even a minimum at $\beta = 0.4$. Hence if the sum of deceleration and settling time is approximately 50% larger than the acceleration time, it is advantageous to select a motion program characterized by $\beta = 0.4$.

An illustrative picture of the physical meaning of the speed factor β in terms of peak velocity is given in Fig. 5.

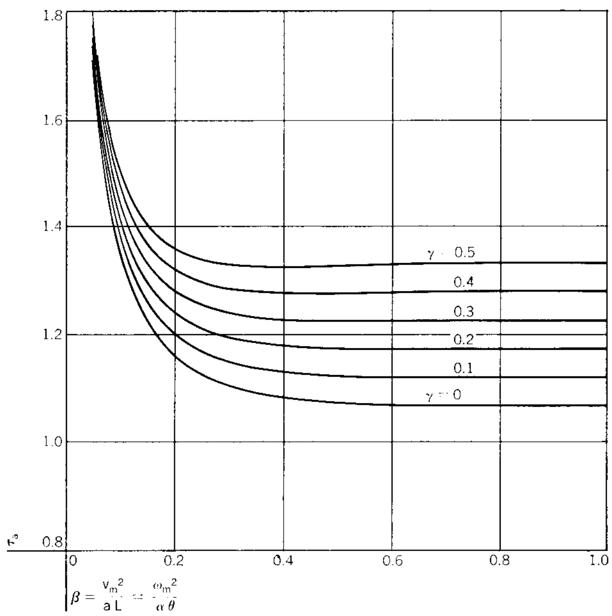
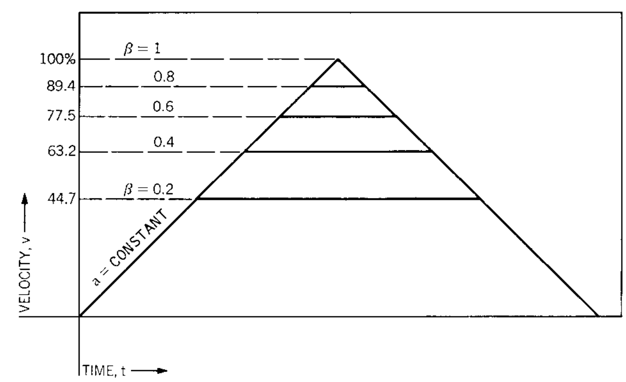


Figure 4 Time constant τ_a versus speed factor β

$$t_{av} = \tau_a \sqrt{\frac{L}{a}} = \tau_a \sqrt{\frac{\theta}{\alpha}}.$$

Figure 5 Illustration of trapezoidal velocity versus time curves for various speed factors β in case of maximum travels L.



Each velocity vs time curve depicts the motion over corresponding maximum distances L in file arrays. The individual peak velocities are expressed in percentages of the extreme velocity obtained with $\beta = 1$ (= triangular profile). Utilizing Eq. (20) the percentage is simply

$$p_v = 100\sqrt{\beta}. \tag{21}$$

Accordingly, if we choose $\beta = 0.4$, the velocity is reduced to 63.2% of the peak velocity which we would obtain with the triangular velocity program. On the other hand, the kinetic energy involved in access is reduced to 40%, which means a saving of 60% compared with the extreme case of triangular velocity curves throughout the positioning range.

The practical implications of the foregoing results are significant. Figures 4 and 5 prove that it is possible to reduce peak velocity and thus kinetic energy of the system without any appreciable change in the average motion time. For a control factor of $\gamma = 0.5$ the average time even slightly decreases if the speed factor is reduced to $\beta = 0.4$.

Of course, such a concept stands in contrast to the engineer's intuitive conclusions. How can it be explained? First of all, there is the linearly decreasing probability to move large distances. Hence, for most of the (shorter) strokes the reduction of terminal velocity remains unnoticed. In addition, a high peak velocity may involve deceleration and settling time problems, which were reflected in the analysis by the control factor γ. The probability terms as well as the control factor are essentially responsible for the results.

Let us consider now a positioning system which follows a velocity program characterized by a speed factor $\beta = 0.35$ and a control factor $\gamma = 0.4$. From looking at Fig. 4 we almost immediately may reject a proposal to decrease average motion time by increasing the peak velocity of the positioner. The gain in terms of random access performance would be negligible.

In another example we may be inclined to select a hydraulic servo valve which supplies sufficient flow such that the maximum possible velocity is obtained by the servo drive. Here we may also question whether a valve with a lower flow rate may not perform as well, if not better. Practical experience with a hydraulic servo system indeed proved that a valve with lower flow rate and subsequently lower peak velocity resulted in better over-all performance, in terms of better positioning accuracy, fewer stability problems (shorter settling times) and shorter average motion times.

Optimum utilization of power

A drive source used for positioning purposes may be applied in several ways: (1) for maximum velocity, (2) for maximum acceleration, and (3) for combinations of acceleration and velocity within a limited range of magnitudes. Usually, a certain combination is intentionally obtained by selecting a transmission ratio between motor and load shaft. Another known method applies the so-called "hotshot" technique, which results in higher force or torque output but not necessarily in an increase of peak velocity. It may even be mandatory to decrease the peak velocity in order to stay within the power and/or heat dissipation design range of the drive source.

In the following analysis the above considerations are taken into account by limiting the peak power of the prime mover. First, for a better understanding, the assumption is made that the total mass or inertia of the system remains unchanged. Later, different load and motor inertias are introduced and the average time is studied in terms of variable transmission ratios.

- *Average motion time with constant peak power and constant inertia load*

By definition we leave the total reflected mass M_{tot} on the positioner output shaft a constant. Then the peak power P is determined by the product

$$P = v_m a M_{\text{tot}} \tag{22}$$

From Eq. (2) we obtain

$$v_m = \sqrt{La\beta} \tag{23}$$

and by combining Eqs. (22) and (23)

$$a = \sqrt[3]{\frac{P^2}{M_{\text{tot}}^2 L\,\beta}}. \tag{24}$$

Substitution of the last expression into Eq. (17) yields:

$$t_{\text{av}} = \tau_a \sqrt[6]{\beta}\sqrt[3]{\frac{M_{\text{tot}}L^2}{P}}. \tag{25}$$

Again, we may define a time constant

$$\tau_p = \tau_a\sqrt[6]{\beta} \qquad [\tau_a \text{ according to Eq. (18)}], \tag{26}$$

which is proportional to the average motion time in a positioning system with a given peak power (P = Force $\times$ Terminal Velocity) and a given inertia load M_{tot}. Combining Eqs. (25) and (26) results in:

$$t_{\text{av}} = \tau_p\sqrt[3]{\frac{M_{\text{tot}}L^2}{P}}. \tag{27}$$

It is interesting now to realize that the average motion time is proportional to the cube root of the total mass and of the maximum distance squared. Simultaneously, the average time is also inversely proportional to the cube root of the power. Hence, a reduction of average time by one-half can only be done by applying 8 times (!) the power. In many practical cases such a large increase would already put us at the limits of available power sources for positioning purposes—a fact which clearly demonstrates the handicap of mechanical positioners in conjunction with high speed computers.

Equation (27) also shows that a reduction of mass to be moved or, even better, a simultaneous decrease of the maximum travel must be a basic goal in reducing motion time. Such considerations are not new to the engineer, except perhaps the functional relationship as stated by Eq. (27). However, there is still another factor: the time constant τ_p which may be modified by selecting a proper velocity vs time curve in line with the basic motion-model (Fig. 2) chosen for this analysis. A plot of the time constant τ_p in Fig. 6 proves that there exist speed factors β in the range $0.13 < \beta < 0.2$, where the time constant reaches a minimum. By operating a positioner at these speed factors, we obtain optimum utilization of the available power. Compared with the pure triangular velocity program, where $\beta = 1$, the reduction of average motion time may be as much as 30%. The same reduction, on the other hand, could be achieved only with an increase of power by factors 2.0 to 2.2, as Eq. (27) states. Thus, an engineer who has to speed up a given positioning system for minimum average motion time may have a good chance to accomplish a portion of his task by designing for the proper speed factor β (compare Fig. 6).

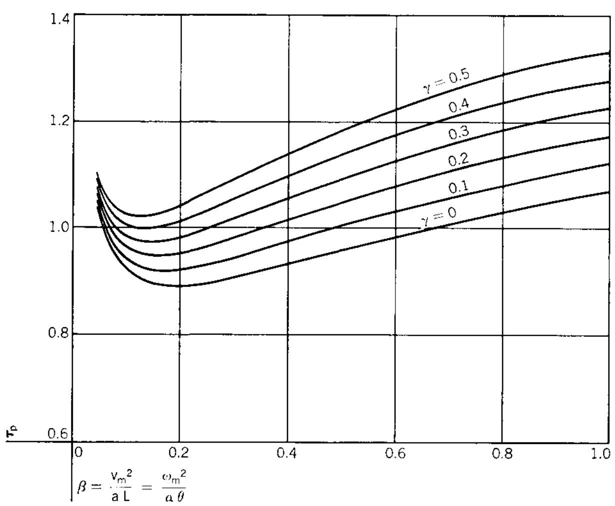
We may ask now what the mechanical implications are in case the speed factor of a system is modified. The question may be answered by inspecting Eq. (2) and Eq. (22). According to definition, we left the peak power P and the total mass M_{tot} a constant. Hence, according to Eq. (22), the product of peak velocity and acceleration stays constant. Any increase of acceleration must necessarily coincide with a proportional decrease of peak velocity. Accordingly, the speed factor $\beta = v_m^2/(a\ L)$ changes.

From a machine design point of view, a variation of acceleration and terminal velocity is usually accomplished by changing the transmission ratio i between the prime mover and the actual load to be positioned. A second method would consist of increasing the force output of an actuator by "hotshot" techniques while decreasing acceleration time and peak velocity such that the power peak and the average heat dissipation remains unchanged within the permissible limits.

Of course, a change of transmission ratio between a motor and a load generally affects the total inertia of the moving parts, a phenomenon which is studied in detail in the following section.

Figure 6 Time constant τ_p versus speed factor β

$$t_{\text{av}} = \tau_p \sqrt[3]{\frac{M_{\text{tot}} L^2}{P}} = \tau_p \sqrt[3]{\frac{I_{\text{tot}} \theta^2}{P}}.$$



• *Average motion time with constant peak power and variable transmission ratio between motor and load*

In concurrence with the preceding analysis, we choose again a linear array of discrete positions. The positioned load as well as the motor or actuator shall perform linear motions. We shall see later how a rotary system may be interpreted within the terminology of such a model.

In order to generalize as well as simplify the analysis, the following dimensionless ratios are defined:

$$\lambda = \frac{M_M}{M_L} = \frac{\text{Mass on motor shaft}}{\text{Mass on load shaft}} \tag{28}$$

$$= \text{ratio of motor and load inertias}$$

$$i = \frac{v_M}{v_m} = \frac{\text{Peak velocity of motor}}{\text{Peak velocity of load}} \tag{29}$$

$$= \text{transmission ratio}$$

Note that the term "load" is used for the actual inertia of the parts being positioned within the range $0 \leq X \leq L$. Consequently, the motor will move through a distance $0 \leq X_M \leq i\,L$.

With F_M as average driving force of the motor, the acceleration of the load may now be written in the form:

$$a = \frac{F_M i}{M_L(1 + \lambda i^2)}. \tag{30}$$

Substituting Eq. (30) in definition Eq. (2) and considering Eq. (29) yields:

$$\beta = \frac{v_M^2 M_L}{F_M L} \cdot \frac{1 + \lambda i^2}{i^3}. \tag{31}$$

In the above equation the first term contains known parameters of the system. Therefore, we define a new and dimensionless quantity:

$$q = \frac{v_M^2 M_L}{F_M L} = \text{load factor. Hence} \tag{32}$$

$$\beta = q\,\frac{1+\lambda i^2}{i^3}. \tag{33}$$

Now we may substitute Eqs. (30) and (31) in Eq. (16) and obtain for the average motion time the equation

$$t_{av} = \frac{iL}{15 v_M}\,[\beta^3(1+3\gamma) - 5\beta^2(1+2\gamma) + 15\beta(1+\gamma) + 5]. \tag{34}$$

In Eq. (34) the only variable not yet known or defined is the optimum transmission ratio i^*. Hence a possible solution would be to plot the average time as a function of the transmission ratio and to find out which ratio minimizes time. We may arrive analytically at the same result, however, by solving the condition

$$\frac{dt_{av}}{di} = 0. \tag{35}$$

We obtain

$$\beta^3(1+3\gamma) - 5\beta^2(1+2\gamma) + 15\beta(1+\gamma) + 5 + i\,\frac{d\beta}{di}\,[3\beta^2(1+3\gamma) - 10\beta(1+2\gamma) + 15(1+\gamma)] = 0 \tag{36}$$

where, according to Eq. (33),

$$\left.\begin{aligned} \beta &= q\,\frac{1+\lambda i^2}{i^3} \\ \frac{d\beta}{di} &= -q\,\frac{3+\lambda i^2}{i^4} \end{aligned}\right\}. \tag{37}$$

From Eqs. (36) and (37) the optimum transmission ratio i^* may be found for any given ratio λ and load factor q as defined by Eqs. (28) and (32). In our case, an IBM 7090 computer was used to solve the equations. The results are depicted in Fig. 7.

With a known optimum gear ratio i^* the corresponding minimum average time may be computed according to Eqs. (33) and (34). Here we may save much of the numerical work by writing Eq. (34) in an abbreviated form:

$$t_{av} = \tau_{i^*}\,\frac{L}{v_M}, \tag{38}$$

where the new time constant

$$\tau_{i^*} = \frac{i^*}{15}\,[\beta^3(1+3\gamma) - 5\beta^2(1+2\gamma) + 15\beta(1+\gamma) + 5], \text{ and} \tag{39}$$

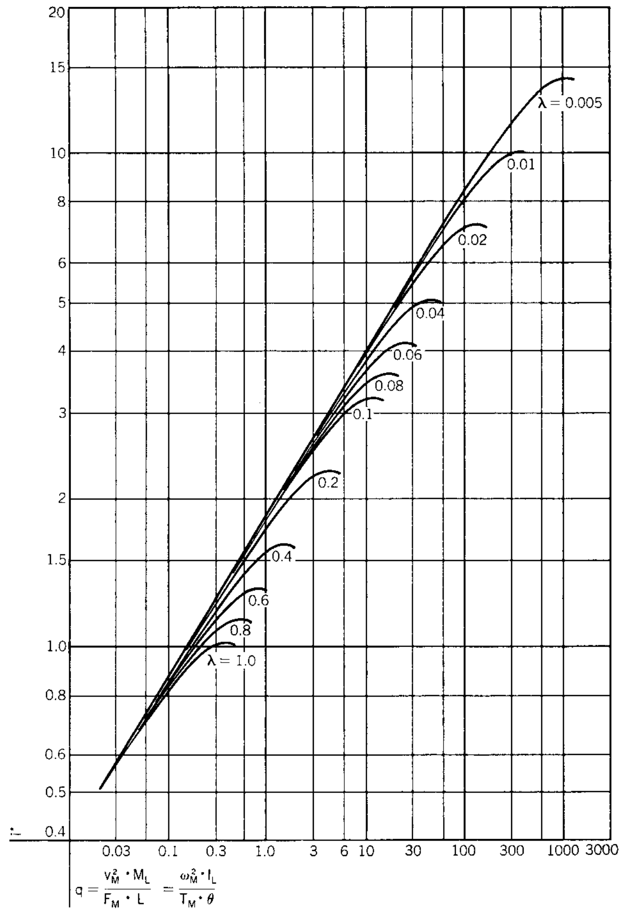


Figure 7 Optimum transmission ratios i^* versus load factor q for $\gamma = 0.25$.

$$\beta = q\,\frac{1+\lambda(i^*)^2}{(i^*)^3} \tag{40}$$

can be demonstrated as a function of the dimensionless quantities λ, q and γ. Figure 8 shows a graph of the defined time constant. Corresponding values may be read off this graph and used for computing the average time by means of the simple relation given in Eq. (38).

Note that Figs. 7 and 8 are precisely valid only for a control factor $\gamma = 0.25$. It was found, however, that control factors of 0 and 0.5 insignificantly change the results. In case of $\gamma = 0$ the optimum gear ratios are only 5% lower and in case of $\gamma = 0.5$ the gear ratios are 5% larger compared with the plotted results for $\gamma = 0.25$. Hence, we are to some degree independent of the control factor—a desirable result, considering the difficulty in assessing its precise magnitude.

Applications of the analysis to rotary systems

In the foregoing analysis a linear positioning system was selected as a basic motion model. Of course, actual positioners do not always comply with this configuration. They contain rotary elements or are built entirely on a rotary incrementing basis scanning a given segment of a circular array. It is, however, quite simple to use the previous analysis by restating the individual parameters in rotary notations. Thus we obtain

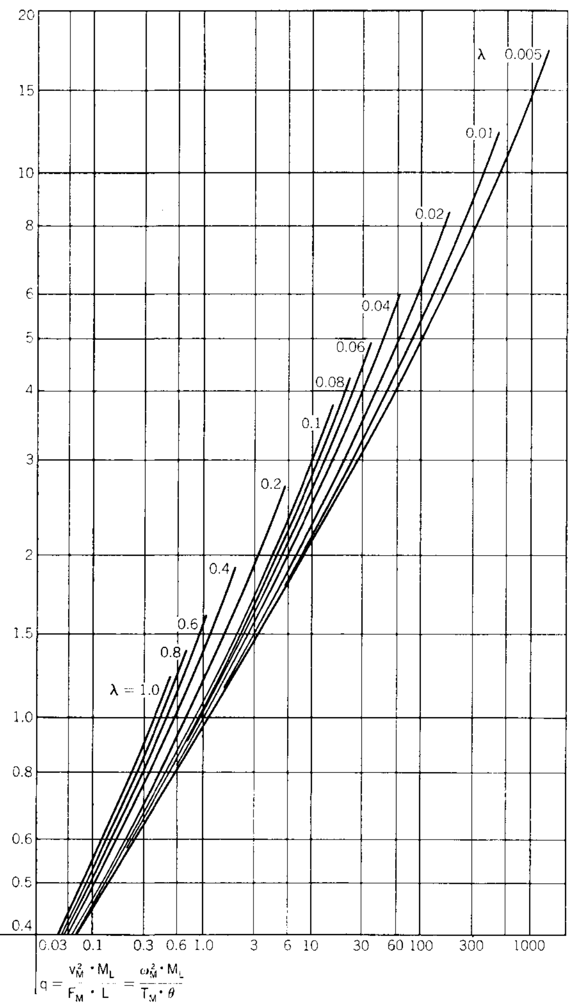
$$\beta = \frac{\phi_b}{\Theta} = \frac{\omega_m^2}{\alpha\Theta}, \tag{41}$$

where

ϕ_b = maximum angle moved in the triangular mode (rad),

Figure 8 Minimum time constants τ_{i^*} versus load factors q for $i = i^*$ and $\gamma = 0.25$.

$t_{av} = \tau_{i^*} L/v_M$.



Θ = maximum angle of the positioning device (rad)
ω_m = maximum angular velocity of the positioned load (rad/sec)
α = angular acceleration of the load (rad/sec^2);

$$\lambda = \frac{I_M}{I_L}, \tag{42}$$

where

I_M = rotary inertia load on motor shaft
I_L = rotary inertia load on load shaft;

$$q = \frac{\omega_M^2 I_L}{T_M \Theta}, \tag{43}$$

where

ω_M = peak angular velocity of the motor (rad/sec)
T_M = average torque of the motor.

By using the foregoing parameters the time constants τ_a, τ_p, and τ_{i^*} remain unchanged. Hence, the average times may be computed according to

$$t_{av} = \tau_a \sqrt{\Theta/\alpha} \tag{44}$$

$$t_{av} = \tau_p \sqrt[3]{(I_{tot}\Theta^2)/P} \tag{45}$$

$$t_{av} = \tau_{i^*}\Theta/\omega_M, \tag{46}$$

depending whether we have constant acceleration, constant power and total inertia, or constant power with variable total inertia, respectively.

It will be remembered that the foregoing analysis was derived for a finite array of positions which may be of a linear as well as a circular order. A slightly different condition exists if we have a closed circular array and the positioner never moves through angles larger than 180 degrees (π). In such a case the probability of moving through any angle would be constant and equal to 1 with random reference activity. Hence, Eq. (15) reduces to

$$t_{av}^0 = \left[(2+\gamma)\int_0^\beta \sqrt{\xi}\, d\xi + \frac{1}{\sqrt{\beta}}\int_\beta^1 [\beta(1+\gamma)+\xi]\, d\xi\right]\sqrt{\frac{\Theta}{\alpha}}. \tag{47}$$

This equation may be easily integrated. We obtain as a *time constant for the closed circular array* similar to Eq. (18)

$$\tau_a^0 = \frac{1}{6\sqrt{\beta}}[3 + 6\beta(1+\gamma) - \beta^2(2\gamma+1)]. \tag{48}$$

The rest of the analysis follows exactly the same path as previously discussed. Note that for all closed circular arrays the maximum angle of travel becomes a constant which is equal to π.

Practical example

As an example, a combination of rotary and linear positioning elements was chosen. Figure 9 shows a scheme of the device. By means of a cable drive, a load of 3.25 lb has to be positioned to random locations within a maximum length of $L = 20$ inches. The problem consists of finding an optimum drive capstan radius R^* which minimizes the average time.

The known quantities of the motor drive are

Torque output	$T_M = 18$ in-lb (average)
Peak velocity	$\omega_M = 105$ rad/sec
Inertia on motor shaft	$I_M = 0.00505$ in-lb-sec^2
Control factor	$\gamma \simeq 0.25$

- *Solution*

The given motor torque T_M and peak velocity ω_M group the example into drive systems with constant peak power ($P = T_M\omega_M$). On the other hand, the selection of any capstan radius R affects the reflected inertia on the motor shaft. Hence, the phenomena discussed in conjunction with variable transmission ratios have to be observed.

An immediate question arising now concerns the definition of transmission ratio in the special case under consideration. A simple trick may answer the question: We define

$$i^* = R_1/R^*, \tag{49}$$

where R_1 is a fictitious or assumed capstan radius and R^* is the optimum radius we are looking for. Thus, we may solve the problem for any fixed radius R_1 and obtain the true optimum radius according to

$$R^* = R_1/i^*. \tag{50}$$

Of course, the choice of the radius R_1 is arbitrary. For convenience, however, it is recommended to use $R_1 = 1$ if the resulting motion parameters are not out of the range of Figs. 7 and 8.

With an assumed radius $R_1 = 1$ inch, the rotary motor may be interpreted as a linear actuator with the characteristics:

$$F_M = \frac{T_M}{1} = 18 \text{ lb}$$

$$M_M = \frac{I_M}{1^2} = 0.00505 \text{ lb-sec}^2/\text{in}$$

$$v_M = \omega_M 1 = 105 \quad \text{in/sec}.$$

Hence, the ratio of motor and load inertia amounts to

$$\lambda = M_M/M_L = 0.00505 \times 386/3.25 = 0.6.$$

The load factor is

$$q = \frac{v_M^2 M_L}{F_M L} = \frac{105^2 \times 3.25}{18 \times 20 \times 386} = 0.2579.$$

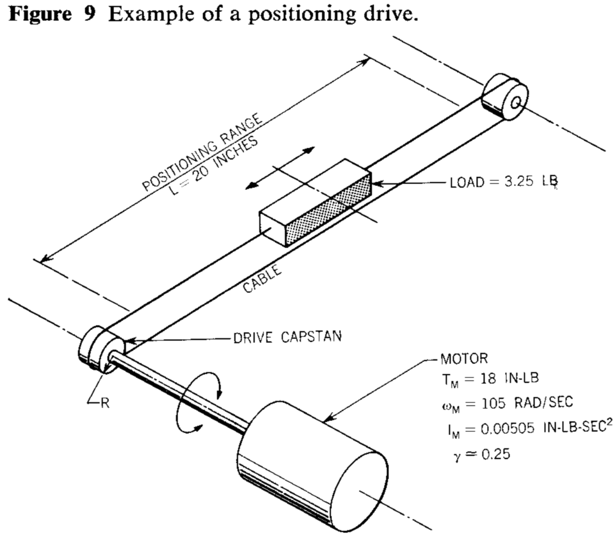
From Fig. 7 we find $i^* = 1.085$. Hence, according to Eq. (48), $R^* = 1/1.085 = 0.922$ inch.

With the optimum radius $R^* = 0.922$ inch, the average motion time is determined by Eq. (38): $t_{av} = \tau_{i*}L/v_M$, where according to Fig. 8, $\tau_{i*} = f(q, \lambda) = 0.758$. Hence,

$$t_{av} = 0.758 \frac{20}{105} = 0.144 \quad \text{sec}.$$

In the above example the inertia on the motor shaft may be slightly affected by the variation of the capstan size. It is possible to take this influence into account by repeating the analysis with a corrected inertia, thus iterating rapidly toward the accurate solution.

Figure 9 Example of a positioning drive.



Summary and conclusions

The design of mechanical positioners for minimum average motion time was attacked by defining a basic motion model and describing the characteristic parameters in terms of dimensionless quantities. It was shown that considerable amounts of power could be saved by balancing load acceleration and peak velocity in a certain way. Graphs were presented which illustrate the dependency of average motion time on defined speed factors and load characteristics. A method of optimizing transmission ratios was discussed by means of a practical example.

The author is aware that the numerical applicability of the results depends largely on the compatibility of the actual design with the chosen motion model. However, from a qualitative point of view the analysis reveals, in general, decisive information on the mutual merit of design quantities such as speed and acceleration.

The statistical approach to average motion time demonstrates that a given power source should be applied more in favor of short motions rather than in favor of the longest motions. The result is also of interest in cases where a random access device is less used in a random mode but more in a skip-sequential mode. There, even more, the optimization of a positioner in favor of short strokes has to be observed.

The presented method of analysis may be applied in case of any reference frequency distribution different from the random, i.e., the uniform distribution selected in this paper. Equation (7) may serve as a basic starting point in such a case.

Nomenclature

Symbol		*Units*
a	Linear acceleration	in/sec^2
b	Maximum distance travelled in triangular mode	in
F_M	Force of linear motor	lb
i	Transmission ratio motor/load	
i^*	Optimum transmission ratio	
I_{tot}	Total rotary inertia of system	in-lb-sec^2
I_M	Rotary inertia on motor shaft	in-lb-sec^2
I_L	Rotary inertia on load shaft	in-lb-sec^2
L	Maximum distance of travel	in
M_{tot}	Total moving mass of system	lb-sec^2/in
M_M	Mass on motor shaft	lb-sec^2/in
M_L	Mass on load shaft	lb-sec^2/in
P_ξ	Probability density of ξ	
P	Power	in-lb/sec
q	Load factor $= (v_M^2 M_L)/(F_M L)$	
R	Radius (example)	in
t	Variable time	sec
t_{av}	Average motion time	sec
T_M	Torque of motor	in-lb
v	Variable velocity	in/sec
v_m	Peak velocity of load	in/sec
v_M	Peak velocity of motor	in/sec
X	Variable distance of motion	in
Y	Variable location to be accessed	in
α	Angular acceleration	rad/sec^2
β	Speed factor $= b/L = v_m^2/(a\,L)$	
γ	Control factor	
η	Relative location to be accessed $= Y/L$	
Θ	Maximum angle of positioned rotary load	rad
λ	Inertia ratio $= M_M/M_L = I_M/I_L$	
ξ	Relative distance of motion $= X/L$	
τ_a	Time constant with constant acceleration	
τ_P	Time constant with constant peak power and inertia load	
τ_{i^*}	Time constant with optimum transmission ratio i^*	
θ_b	Maximum angle travelled in triangular mode	rad
ω_m	Peak angular velocity of load	rad/sec
ω_M	Peak angular velocity of motor	rad/sec

References

1. V. V. Solodovnikov, *Introduction to the Statistical Dynamics of Automatic Control Systems*, Dover Publications, Inc., New York, 1960.
2. Among many textbooks: H. Chestnut and R. W. Mayer, *Servo-mechanisms and Regulating System Design*, Vol. II, John Wiley & Sons, Inc., New York, 1955, pp. 109–114.
3. Convolution is frequently used in statistics for evaluating accumulative density distributions. Among the textbooks we may consider: E. Parzen, *Modern Probability Theory and Its Applications*, John Wiley & Sons, Inc., New York, 1962, p. 404.

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