H. J. Zweig
D. P. Gaver†

Coincidence Counter Models with Applications to Photographic Detection Theory*

Abstract: Various types of counter models have been treated in the literature over the past twenty years. In all these models the counter mechanism involves a fixed or random dead time following a registered event. In this paper a different type of counter mechanism is introduced in which the occurrence of two or more input events within a relatively short time is required to produce a registered (output) event. This model of an "R-fold coincidence counter" is applied to the development of grains in a photographic emulsion for both low-intensity and high-intensity reciprocity failure.

Symbols

λ	parameter of Poisson process
$\hat{\lambda}$	estimate of the parameter λ
T	time of operation of the detector on the stochastic process
T_i	time to the i^{th} event
X_i	a random variable denoting elapsed time between events $(i - 1)$ and i
t	a point in time, or an amount of elapsed time depending on the context
$F(t)$	inter-arrival time distribution of input stochastic process
$F_o(t)$	inter-arrival time distribution of output stochastic process
p	probability of an event
$E[\,]$	expectation of [ ]
var[ ]	variance of [ ]
$\phi(S)$, $(\phi_o(S))$	Laplace transform of input (output) inter-arrival distribution
μ, (μ_o)	mean of $F(t)$, $(F_o(t))$
σ^2, (σ_0^2)	variance of $F(t)$, $(F_o(t))$
τ	a time interval of fixed or variable duration following an input event to the detector
$G(\tau)$	the distribution function of τ
$\{\ \}^*$	Laplace transform of $\{\ \}$
τ_d	a dead-time interval following input event
S_j, $j = 1, 2, \cdots, R - 1$	the states through which an R-fold coincidence counter must pass before registering an output event
S_0	the ground state (the most stable state) of a coincidence counter and the state at $t = 0$
$p_j(t, \tau)$	the probability density that state is j at time t, and has been for a time τ, and that state R has not occurred in $(0, t)$ and state at $t = 0$ is S_0
$p_j(t)$	probability that state at time t is j and that state R has not occurred in $(0, t)$
$\mu(\tau)$	hazard function or conditional probability density of decay at time τ given survival to time τ
$H_\lambda(T)$	probability that at least one count is registered in time interval $(0, t)$
R	number of photons required to make a photographic grain developable
S	the number of photographic grains involved in a photographic detection problem
T'	that amount of time in $(0, T)$ during which detector is not "dead".

I. Introduction

Various types of counter models have been introduced and discussed in the literature of the last twenty years.[1–8] Type I and Type II counters have received the most extensive treatment. Type I counters are characterized by

*Based on a Ph.D. thesis, "Counter Models and Applications to Detection Problems," submitted by H. J. Zweig to the Department of Statistics, Stanford University, April 1963.

† Mathematics Department and Graduate School of Industrial Administration, Carnegie Institute of Technology.

the fact that a registered event produces a dead time during which no further events can be registered. Type II counters are characterized by the fact that incoming events produce dead time during which further incoming events cannot be registered, although these events are capable of prolonging the dead time, that is, paralyzing the counter. Modifications of Type I and Type II counters have also been introduced and studied in the literature.

Here we shall be concerned with a different type of counter model called a *coincidence counter*. The motivation for such a counter model arose out of a study of the mechanism involved in the photographic detection process. However the coincidence counter model is interesting in its own right and there are other applications of the model in such fields as reliability.

The simple coincidence counter model we now introduce is intended to work as follows: An incoming event, say a photon, is *not* directly recorded by the detector. Instead it changes the state of the detector say from S_0 to S_1. The state S_1 is maintained for a fixed time τ following the incidence of the photon. If during this time interval no further arrival takes place, the detector reverts to state S_0 and *no* output count is recorded. If, however, while the detector is in state S_1 another photon arrives an output count is registered and the detector immediately reverts to state S_0. Thus for example if photons were incident on the detector at time t, $t + 2.0\tau$, $t + 2.1\tau$, $t + 2.5\tau$, and $t + 3\tau$ an output event would be registered at times $t + 2.1\tau$ and at $t + 3\tau$; so in this case five input events have produced two output events. The *R-fold coincidence counter* is a generalization of the simple coincidence counter. Here incoming events cause changes in the state of the detector from S_i to S_{i+1} provided the detector is in state S_i ($i = 0, 1, \cdots, R - 1$) at the time of the arrival of the input event. For $i = 1, 2, \cdots, R - 1$ the counter remains in state i for a time τ_i which is now taken to be a random variable with distribution independent of i. If no further event occurs during time τ_i following the last input the counter reverts to state S_0 without registering a count. Only if state S_R is reached is a count registered with the counter immediately reverting to S_0.

The photographic detector is a counting device of an entirely different type, but the counter models nevertheless play a part in the analysis of this kind of detector. Basically the photographic detector (the emulsion or photographic plate) consists of an ensemble of many counters (the individual photographic grains), each of which is able to register a count *only once*. The fractional number of grains that develop is then a measure directly related to the average number of photons incident on each grain during the time of exposure. The simplest photographic detector consists of grains all of the same size and speed, i.e., each grain requires R photons to make it developable (that is, to register a count). Aside from variations in grain size, which will not be considered, actual photographic detectors consist of an ensemble of grains of varying speed so that R, the number of photons required for registering an output event, is a random variable with distribution dependent on the particular photographic material.

There is a photographic effect which can be explained in terms of the dead time and coincidence counter models introduced above. This effect is known technically as *reciprocity failure* and manifests itself in two different ways: (a) At high intensities λ and short exposure times T the photographic detector registers fewer events *for a given average number of photons per grain* λT than when λ is somewhat smaller (but λT is fixed). This can be explained by postulating a dead time τ following each incident photon so that only those photons which arrive at least τ units apart contribute to produce the necessary number R of input events required for developability of the grain. (b) Low-intensity reciprocity failure denotes the condition in which the response, for a fixed average input λT, is again lowered, this time when the intensity of radiation λ is low; that is, the photons arriving at a grain tend to be spaced far apart. By assuming the type of mechanism discussed in the case of the coincidence counter, this type of reciprocity failure can also be explained. The dead time counter models and coincidence time counter models postulated for photographic detectors can be identified with the times required for the creation of a sublatent image speck and the duration of such a speck.

II. Simple coincidence counter

As a model for a simple coincidence counter, we shall consider the following mechanism: An incoming event impinges upon the counter, which is so constructed as to be able to hold this event in memory for a time τ (a fixed constant). If another event occurs during the time that the first event is in memory, the counter registers an output event and resets its memory to zero content. If no second event occurs during time τ, the first event is lost, that is, produces no count.

The formal definition of a simple coincidence counter can be stated as: *a counter which immediately registers an output count whenever an input event is preceded during a fixed time interval τ by a nonregistered input event.*

It should be noted that the coincidence model suggested above differs from the general counter model proposed by Takacs[8] in that the occurrence of an event within τ units of an earlier one produces a regeneration point (in effect, a new origin) for the output stochastic process in our case, but not in Takacs' case. In his process, an output event could occur after the arrival of both the second and the third input event. In our model, that would be considered physically impossible. To get two output counts, at least four input events are required.

Using an approach similar to that of Feller[2] the Laplace transform of the output inter-arrival time random variable is derived as follows: Beginning at a time when an *output* event has just occurred, let $X_0, X_1, X_2, \cdots$ denote the time between further successive input events. The input stream is assumed to be a Poisson process, so that the X_i have a common exponential distribution. Assuming that the N^{th} event produces the first count, we have $X_1 > \tau, X_2 > \tau, \cdots, X_{N-1} > \tau$, but $X_N \le \tau$ (note that there are no conditions on X_0). Let $T(N) = X_0 + X_1 + \cdots + X_N$ denote the total time between output events, then the conditional Laplace transform of $T(N)$, given N, is

$$E[e^{-sT(N)} \mid N] = E[e^{-sX_0}]E\left[\exp\left\{-s\sum_{i=1}^{N-1} X_i\right\} \,\middle|\, X_i > \tau\right]E[e^{-sX_n} \mid X_n \le \tau]. \tag{2.1}$$

Noting that the conditional probability density that two input events are X units apart, given that they are not more than τ units apart, has density

$$\lambda e^{-\lambda X}/1 - e^{-\lambda\tau}, \tag{2.2}$$

we find the conditional Laplace transform (i.e., the transform of this conditioned random variable) is

$$\begin{aligned} E[e^{-sX} \mid X \le \tau] &= \int_0^\tau e^{-sX}\frac{\lambda e^{-\lambda X}}{1 - e^{-\lambda\tau}}\,dX \\ &= \frac{\lambda(1 - e^{-(s+\lambda)\tau})}{(s+\lambda)(1 - e^{-\lambda\tau})}. \end{aligned} \tag{2.3}$$

Similarly, if we know that two input events differ in time by at least τ units, we obtain the conditional Laplace transform

$$\begin{aligned} E[e^{-sX} \mid X > \tau] &= \int_\tau^\infty e^{-sX}\frac{\lambda e^{-\lambda X}}{e^{-\lambda\tau}}\,dX \\ &= \frac{\lambda e^{-(s+\lambda)\tau}}{(s+\lambda)e^{-\lambda\tau}}. \end{aligned} \tag{2.4}$$

From these results we obtain the conditional Laplace transform of $T(N)$ given N

$$\begin{aligned} &E[e^{-sT(N)} \mid N] \\ &= E[e^{-sX_0}]\{E[e^{-sX} \mid X > \tau]\}^{N-1}E[e^{-sX} \mid X \le \tau] \\ &= \frac{\lambda}{\lambda+s}\left\{\frac{\lambda e^{-(s+\lambda)\tau}}{(s+\lambda)e^{-\lambda\tau}}\right\}^{N-1}\frac{\lambda(1 - e^{-(s+\lambda)\tau})}{(s+\lambda)(1 - e^{-\lambda\tau})} \\ &= \left(\frac{\lambda}{\lambda+s}\right)^{N+1}\frac{e^{-(N-1)(s+\lambda)\tau}}{e^{-(N-1)\lambda\tau}}\cdot\frac{1 - e^{-(s+\lambda)\tau}}{1 - e^{-\lambda\tau}}. \end{aligned} \tag{2.5}$$

To obtain the unconditional transform of $T(N)$ we multiply by the probability that the first output event will occur at the N^{th} input event, namely, $(1 - e^{-\lambda\tau})(e^{-\lambda\tau})^{N-1}$, and sum over N obtaining

$$E[e^{-sT}] = \sum_{N=1}^{\infty} E[e^{-sT} \mid N](1 - e^{-\lambda\tau})(e^{-\lambda\tau})^{N-1}. \tag{2.6}$$

This yields

$$E[e^{-sT}] = \left(\frac{\lambda}{\lambda+s}\right)^2(1 - e^{-(\lambda+s)\tau})\sum_{N=1}^{\infty}\left[\frac{\lambda e^{-(s+\lambda)\tau}}{s+\lambda}\right]^{N-1}, \tag{2.7}$$

and after summing the infinite series, letting $\phi_o(s) = E[e^{-sT}]$,

$$\phi_o(s) = \left(\frac{\lambda}{\lambda+s}\right)^2\frac{1 - e^{-(\lambda+s)\tau}}{1 - \dfrac{\lambda}{\lambda+s}e^{-(\lambda+s)\tau}}. \tag{2.8}$$

In this formula, the term $\lambda^2/(\lambda + s)^2$ represents the transform of the waiting time for the second input event of the Poisson stream. As $\tau \to \infty$ the second input event can be taken as producing an output event, which is what we would expect of an infinite memory device.

III. R-fold coincidence counter

One natural extension of the simple coincidence counter is to permit input events arriving within specified time intervals to change the state or levels of a counter, and to allow the registering of an output event only after a terminal state, say after the R^{th} state, is reached. We term this the R-fold coincidence counter. For $R = 2$, this reduces to the previously described simple coincidence counter which has two states S_0 and S_1. Then an input event changes the counter from S_0 to S_1, and if another event occurs before time τ, an output count is registered and the counter immediately reverts to S_0. The counter also immediately reverts to S_0 at time τ but without registering an output count if the input does not arrive within the time interval τ.

Briefly we can describe the R-fold coincidence counter as being *a counter which immediately registers an output event whenever an input event is preceded by* $(R - 1)$ *non-registered input events none of which are spaced more than* τ *units apart where* τ *is a time interval of random length whose* $(R - 1)$ *values are independently and identically distributed.*

Here we obtain the waiting time distribution (or, rather, its Laplace transform) for the first time occurrence of the critical level R in the case where the waiting time between upward jumps of unit magnitude has negative exponential distribution, $F(t) = 1 - e^{-\lambda t}$. This is equivalent to the assumption that input impulses constitute a Poisson process. We also assume that all the levels S_j, $j = 1, 2, \cdots, R - 1$ can decay only to S_0 and that such decay takes place at the end of an interval of duration τ_j

following the latest incident impulse. We assume that each τ_j is itself a random variable with absolutely continuous distribution function $G(\tau)$ which is independent of S_j and the times of occurrence of the states S_j.

We shall write $p_j(t, \tau)$ for the probability density that jointly:

a) at time t the process is in state S_j;
b) that it reached this state τ time units ago, that is, at $t - \tau$;
c) that the state at $t = 0$ is S_0, and that previous to time t the state R was not reached.

Now the probability that a decay to S_0 takes place during the small interval $(\tau, \tau + \delta)$ after occurrence of the last impulse is $\delta G'(\tau)$, and the probability that no decay takes place during the time interval $(t - \tau, t)$ is $1 - G(\tau)$. Hence the *conditional probability of decay* during the interval $(\tau, \tau + \delta)$ given that no decay has taken place during the elapsed time τ since the last impulse is given by

$$\delta \cdot \mu(\tau) = \frac{\delta \cdot G'(\tau)}{1 - G(\tau)} \tag{3.1}$$

so that

$$G(\tau) = 1 - \exp\left\{-\int_0^\tau \mu(x)\, dx\right\}. \tag{3.2}$$

The function $\mu(\tau)$ is sometimes called the *hazard function*, and has been discussed, for example, in connection with telephone call demands and other queueing problems.

We shall derive here a set of differential equations from which the Laplace transform of the inter-arrival time distribution of output counts can be obtained. In terms of $p_j(t, \tau)$ and the hazard function $\mu(\tau)$, we can write

$$p_j(t + \delta, \tau + \delta) = (1 - \lambda\delta)(1 - \delta\mu(\tau))p_j(t, \tau) \tag{3.3}$$

or, neglecting terms in δ^2 since δ is assumed small, we get

$$p_j(t + \delta, \tau + \delta) = [1 - \lambda\delta - \delta\mu(\tau)]p_j(t, \tau). \tag{3.4}$$

If during the interval $(t, t + \tau)$ an impulse occurs, we have, for the probability that the state is S_j at $t + \delta$ and that the time τ' since the last impulse was less than δ [denote this by $p_j(t + \delta, \tau' < \delta)$],

$$p_j(t + \delta, \tau' < \delta) = \lambda\delta \int_0^\infty (1 - \delta\mu(\tau))p_{j-1}(t, \tau)\, d\tau \tag{3.5}$$

where the integrand on the right represents the probability that the state was S_{j-1} at time t, had been attained τ time units earlier; and did *not* spontaneously decay in the time span $(\tau, \tau + \delta)$ following the moment it was attained. Since all possible times, τ, of attainment of the state S_{j-1} must be considered, the probability of no decay from state S_{j-1} during $(t, t + \delta)$ is $\int_0^\infty (1 - \delta\mu(\tau))p_{j-1}(t, \tau)\delta\tau$.

Again neglecting terms in δ^2 this becomes

$$p_j(t + \delta, \tau' < \delta) = \lambda\delta \int_0^\infty p_{j-1}(t, \tau)\, d\tau. \tag{3.6}$$

These considerations hold for $j = 1, 2, \cdots, R - 1$. For state S_0, we have

$$p_0(t + \delta) = (1 - \lambda\delta)p_0(t) + \sum_{j=1}^{R-1} (1 - \lambda\delta) \int_0^\infty \delta\mu(\tau)p_j(t, \tau)\, d\tau, \tag{3.7}$$

i.e., the probability of being in state S_0 at time $t + \delta$ is the probability of being in that state at time t and having no incident events in $(t, t + \delta)$ plus the probability that a decay occurs in the period $(t, t + \delta)$, when the process is in state S_j $(j = 1, 2, \cdots, R - 1)$ at time t and no incident events occur during $(t, t + \delta)$. Note that the integral under the summation represents the probability of a decay in the period $(t, t + \delta)$ when the process is in state S_j $(j = 1, 2, \cdots, R - 1)$ at time t and this probability is independent of how long ago the state S_j was attained previous to time t. Similarly for $p_0(t + \delta, \tau' < \delta)$ we have, by (3.5),

$$p_0(t + \delta, \tau' < \delta) = \lambda\delta \int_0^\infty (1 - \delta\mu(\tau))p_{R-1}(t, \tau)\, d\tau. \tag{3.8}$$

In the period $(t, t + \delta)$ a transition from state S_{R-1} to state S_R may occur thus causing the process to revert *immediately* to state S_0 and register an output count, or we may write

$$p_0(t + \delta) = (1 - \lambda\delta)p_0(t) + \sum_{j=1}^{R-1} \int_0^\infty \lambda\mu(\tau)p_j(t, \tau)\, d\tau \tag{3.9}$$

and

$$p_0(t + \delta, \tau' < \delta) = \lambda\delta \int_0^\infty p_{R-1}(t, \tau)\, d\tau, \tag{3.10}$$

if terms in δ^2 are neglected. If in (3.4) we transpose $p_j(t, \tau)$, then divide each side of the equation by δ and pass to the limit, we obtain

$$\lim_{\delta \to 0} \frac{p_j(t + \delta, \tau + \delta) - p_j(\tau, t)}{\delta} = \lim_{\delta \to 0} - \frac{(\lambda\delta + \delta\mu(\tau))}{\delta} p_j(t, \tau)$$

or

$$\frac{\partial p_j}{\partial t} + \frac{\partial p_j}{\partial \tau} = -(\lambda + \mu(\tau))p_j(t, \tau). \tag{3.11}$$

This is so for $j = 1, 2, \cdots, R - 1$ and similarly from (3.9) we get for $j = 0$

$$\frac{dp_0}{dt} = -\lambda p_0(t) + \sum_{j=1}^{R-1} \int_0^\infty \mu(\tau)p_j(t, \tau)\, d\tau. \tag{3.12}$$

These are the differential equations whose solutions will provide the inter-arrival time distribution of the output counts.

Letting * denote Laplace transformation using $g(\tau) = 1 - G(\tau)$ and $G'^*(\tau) = (dG/d\tau)^*$ it can be shown that

$$E[e^{-st_R}] = \frac{\lambda^R\{g^*(s+\lambda)\}^{R-1}}{s + \lambda - \lambda G'^*(s+\lambda)\left[\dfrac{1 - \{\lambda g^*(s+\lambda)\}^{R-1}}{1 - \lambda g^*(s+\lambda)}\right]} \tag{3.13}$$

is the Laplace transform of the inter-arrival time distribution of output counts from the general R-fold coincidence counter.[12]

For $R = 2$; $G(\tau) = 0$ for $\tau < \tau_o$; and $G(\tau) = 1$ for $\tau \geq \tau_o$, which is the case treated previously in Section 2 we have

$$G'^*(s + \lambda) = \int_0^{\tau_0} e^{-(s+\lambda)\tau}\, d\tau = e^{-(s+\lambda)\tau_0} \tag{3.14}$$

and

$$E[e^{-st_2}] = \left(\frac{\lambda}{\lambda + s}\right)^2 \frac{1 - e^{-(s+\lambda)\tau_0}}{1 - \left(\dfrac{\lambda}{\lambda + s}\right)e^{-(s+\lambda)\tau_0}}, \tag{3.15}$$

which is the result previously derived for this case by the conditional probability approach used in that section.

Another interesting case which is mathematically tractable is that in which G is a negative exponential distribution, $G(\tau) = 1 - e^{-\xi\tau}$. In this case we find

$$\begin{aligned} G'^*(s + \lambda) &= \int_0^\infty e^{-(s+\lambda)\tau}\xi e^{-\xi\tau}\, d\tau \\ &= \xi/(s + \lambda + \xi) \end{aligned} \tag{3.16}$$

and

$$E[e^{-st_2}] = \lambda^2/(s^2 + (2\lambda + \xi)s + \lambda^2). \tag{3.17}$$

IV. Application of counter models to photographic detectors

We shall consider a (simple) photographic detector as an ensemble of go-no-go detectors, each of which is capable of responding just once immediately following the arrival of the R^{th} event. Consequently further arrivals at a detector having already received R "hits" are wasted, and cannot be transferred to another detector or registered in the output. The photographic detector is unique in that it can operate simultaneously on a large set of Poisson processes,* namely, on all those sources in space which are imaged on the face of the photographic plate.[11] Also, because of imperfect imaging and scatter within the emulsion, a Poisson point source is imaged, not on one, but on a set S of detectors (or photographic grains). Thus, in comparing the intensity of two point sources, the output from $2S$ detectors must be compared. An ideal non-photographic ensemble of S detectors exposed to a Poisson source of intensity λ for a time T would provide an estimate $\hat{\lambda}$ of λ with a variance of λ/ST, assuming the set of S detectors to operate independently of each other on the same Poisson source.

If we suppose that each photographic detector requires exactly R hits to become developable and *retains its developability indefinitely*, then we can calculate that the mean number of detectors responding to an intensity λ (per grain per unit time) after an exposure time T will be

$$\begin{aligned} S \cdot H(\lambda T) &= S\left[\sum_{j=R}^{\infty} e^{-\lambda T}(\lambda T)^j/j!\right] \\ &= S\left[1 - e^{-\lambda T}\sum_{j=0}^{R-1}(\lambda T)^j/j!\right]. \end{aligned} \tag{4.1}$$

Now if H is the probability that a detector will respond, and if S independent experiments are performed, the probability that exactly k responses are obtained is

$$p(k) = \binom{S}{k} H^k(1 - H)^{S-k} \tag{4.2}$$

and hence the variance of the number of responses is

$$\sigma_S^2 = S \cdot H(\lambda T)[1 - H(\lambda T)]. \tag{4.3}$$

The use of dead time and coincidence counter models occurs in connection with the photographic phenomenon known as *reciprocity failure.* Reciprocity failure means that the photographic detector responds not just to the total number of photons, λT, incident during the time of exposure, but reacts differently, depending on whether for λT = constant, it is the time of exposure or the strength of radiation which is large. There are two types of reciprocity failure: high-intensity failure, which can be attributed to a type of dead time phenomenon as in a Type I counter, and low-intensity failure, which can be thought of as being due to the finite memory of a coincidence type of counter mechanism. Diverse explanations of this phenomenon have been given[9] but the crucial experiments to determine the precise mechanism whereby reciprocity failure is produced have not yet been attained. Consequently any model whose consequences are in reasonable

* Throughout we assume that the light fluctuations which, strictly speaking, obey Bose-Einstein statistics, can be approximated by Maxwell-Boltzmann statistics so that the spatio-temporal distribution of photons incident on our detector constitutes a Poisson stream.

accord with existing experimental data can provide a step forward.

We assume here that for photons in the visible region there is not sufficient energy to produce a developable grain. As is well known, at shorter wavelengths there is enough energy in a particle to trigger one or more photographic grains, and for such particles the problem of reciprocity failure does not arise.

For low-intensity reciprocity failure, we assume that a photon can produce a sublatent image speck which can persist for a time τ. Another photon incident during this time will cause this speck to grow to a stable silver speck. Such a speck may, or may not, in itself be developable. If it is, we have a two-photon photographic detector with low-intensity reciprocity failure. If τ is infinite, this reduces to the type of photographic detector discussed above.

It may be necessary to have a larger speck of silver to produce development than one obtained from two photons. If we assume that the two-photon speck is nondevelopable but one twice as large is developable, then two further photon hits within an interval τ are required to produce either another speck or to enlarge the one already formed. Since actual emulsions are a mixture of grains of varying sensitivity, we would have to combine various models to simulate an actual photographic material. Here we shall content ourselves with some discussion of the two- and four-photon photographic detectors as even these present considerable difficulty. To arrive at the response curve of a two-photon photographic detector with low-intensity reciprocity failure, we should proceed as follows.

Beginning with the Laplace transform of the inter-arrival time distribution of the coincidence counter output (2.8), we find the inverse Laplace transform and integrate this function from zero to T. The result, $F_o(\lambda, T, \tau)$, indicates the probability of one or more coincidence events, which is the probability of a grain becoming developable during the time of exposure T, that is, the probability that *at least one* silver speck is formed in a grain irradiated by an average of λT photons, (and one such speck is sufficient for developability). If there are S photographic grains in the area under consideration, the expected number of grains which will contain a developable speck after exposure time T will then be $S \cdot F_o(\lambda, T, \tau)$. A plot of $F_o(\lambda, T, \tau)$ versus λ or T indicates the average fractional number of grains which become developable as λ or T increases, the other variable being held constant. In photographic technology, when studying reciprocity failure, it is customary to hold constant $F_o(\lambda, T, \tau)$ (which corresponds to the developed optical density) and plot for various values of λ or T the value of λT needed to produce a fixed $F_o(\lambda, T, \tau)$.

The problem of carrying out this procedure arises right at the start in trying to obtain an explicit closed form for the inverse Laplace transform. To find the inverse transform of the coincidence counter inter-arrival time,

$$\phi_o(s) = \left(\frac{\lambda}{\lambda + s}\right)^2 \frac{1 - e^{-(\lambda+s)\tau}}{1 - \dfrac{\lambda}{\lambda + s} e^{-(\lambda+s)\tau}}, \tag{4.4}$$

one should take a contour integral over the left half plane. (From the fact that the distribution function is zero for $t < 0$, we know that all the poles of $\phi(s)$ must have negative real part.[10]) Hence we must find the roots of $s + \lambda(1 - e^{-(\lambda+s)\tau})$.

Even though it is possible to obtain an indication of the location of these roots, some direct attempts at a solution by numerical methods indicate that large scale computer programming is necessary.

Thus a closed form approximation to the distribution function $F_o(\lambda, T, \tau)$ would be helpful. One possibility is to fit a gamma distribution, i.e., a density function of the form

$$\Gamma(x; u, k) = ue^{-ux}(ux)^{k-1}/(k - 1)! \tag{4.5}$$

by fitting the first few moments of this distribution to those obtainable by differentiating $\phi_o(s)$. Since the first two moments of $\Gamma(x; u, k)$ are

$$\mu_o = \frac{k}{u}, \qquad \sigma_o^2 = \frac{k}{u^2} \tag{4.6}$$

and these determine the particular gamma distribution completely, we could set

$$\phi_o'(s = 0) = \frac{k}{\mu} \tag{4.7}$$

and

$$\phi_o''(s = 0) = \frac{k}{\mu^2} + \left(\frac{k}{\mu}\right)^2. \tag{4.8}$$

If τ is large $\phi_o(s)$ tends to $\lambda^2/(\lambda + s)^2$, that is, the Laplace transform of a gamma distribution with $k = 2$ and $\lambda = u$. This is as it should be, for that corresponds to the waiting time distribution for the second input event in the case of a Poisson process. For large enough values of τ, we are justified in approximating $F_o(\lambda, T, \tau)$ near the origin by a gamma distribution with $k = 2$. (Some trial approximations easily show that this parameter is not very sensitive to variations in k.) Thus to a crude approximation we will need only u and this parameter is found from

$$\frac{2}{u} = \frac{1}{\lambda} + \frac{1}{\lambda(1 - e^{-\lambda\tau})} \tag{4.9}$$

or

$$u = \frac{2\lambda(1 - e^{-\lambda\tau})}{2 - e^{-\lambda\tau}} \tag{4.10}$$

so that we shall take

$$F_o(\lambda, T, \tau) = \int_0^T \left[\frac{2\lambda(1 - e^{-\lambda\tau})}{2 - e^{-\lambda\tau}}\right]^2 t \times \exp\left\{-\left[\frac{2\lambda(1 - e^{-\lambda\tau})}{2 - e^{-\lambda\tau}}\right]\right\} \delta t \tag{4.11}$$

as response function in the case that two photons within time τ can produce developability. We shall take

$$F_o^{(2)}(\lambda, T, \tau) = \int_0^T \left[\frac{2\lambda(1 - e^{-\lambda\tau})}{2 - e^{-\lambda\tau}}\right]^4 \frac{t^3}{3!} \times \exp\left\{-\left[\frac{2\lambda(1 - e^{-\lambda\tau})}{2 - e^{-\lambda\tau}}\right]\right\} \delta t \tag{4.12}$$

if twice as large a silver speck is required.

Formula (4.11) can be integrated to yield

$$F_o(\lambda, T, \tau) = 1 - e^{-uT} - uT\, e^{-uT}, \tag{4.13}$$

which indicates that the response function of the two-photon photographic detector with low-intensity reciprocity failure is identical in shape (to this crude approximation) to that of the two-photon detector without reciprocity failure but stretched along the λ-axis by the factor λ/u.

High-intensity reciprocity failure can be handled in much the same way as low-intensity reciprocity failure. In this case we postulate a *dead time*, τ_d, so that only those photons contribute to making an R-photon photographic detector respond which arrive at least τ units apart. In this case the probability that a grain will become developable in time T is obtained by calculating the probability of R or more output events from a Type I counter. Here again we encounter the apparent difficulty that as a first step we need the inverse Laplace transform of the R^{th} power of the inter-arrival time transform of the Type I counter[2]

$$[\phi_o(s)]^R = [\lambda/(\lambda + s)e^{-\tau s}]^R, \tag{4.14}$$

which represents the waiting time distribution for the R^{th} output event from a Type I counter. Subsequently we need the integral from 0 to T which is the probability that the R^{th} event will occur prior to time T and corresponds to the probability that a grain exhibiting high-intensity reciprocity failure will become developable.

However, in the present case, we can make use of the following device. We replace T by $T' = T - (R - 1)\tau$, a contracted time interval. It is easily shown that the output process from a Type I counter is again a Poisson process with parameter λ in *contracted time*. Hence high-intensity reciprocity failure is equivalent, under the present model, to shortening of the exposure time for an R-photon simple photographic detector by an amount $(R - 1)\tau$—the accumulated dead time arising from the first $(R - 1)$ input photons which are incident on a photographic grain. Hence the response of an R-photon photographic emulsion with high-intensity reciprocity failure and deadtime τ_d is obtained by using the relation

$$H(\lambda, T, \tau_d, R) = H(\lambda T', R), \tag{4.15}$$

which indicates that on a part of H vs T (not log T) high intensity reciprocity failure should correspond approximately to a shift of the characteristic curve.

V. Conclusion

A mathematical model of a coincidence time counter has been constructed which is based on the idea that two or more input events are required within a relatively short time to register an output count. Earlier (dead time) counter models were based on the idea that closely spaced input events may not be effectively detected due to the dead time generated by the arrival of an input event, whereas in the model introduced here the initial arrival of an event serves to cock or activate the counter mechanism for some time (the coincidence time) so that the next input event, if it arrives during the coincidence time, produces an output count.

The counter models were applied to a photographic detection problem, and the behavior of the characteristic curve of a photographic material exhibiting reciprocity failure was related to the time constants of dead time and coincidence time counter mechanisms for high- and low-intensity reciprocity failure, respectively. It is shown that to a first approximation the density-exposure curve is displaced parallel to itself toward higher exposure values, the amount of displacement being given by the time constants of reciprocity failure and the number of photons required for grain developability.

References

1. S. C. Curran and J. D. Craggs, *Counting Tubes, Theory and Application*, Academic Press, New York (1949).
2. W. Feller, "On Probability Problems in the Theory of Counters," in *Courant Anniversary Volume*, Interscience, New York (1948).
3. C. Levert and W. Scheen, "Probability Fluctuation of Discharges in a Geiger-Muller Counter Produced by Cosmic Radiation," *Physica's Grav.*, **10**, 225–238 (1943).
4. S. Malmquist, "A Statistical Problem Connected with the Counting of Radioactive Particles," *Am. Math. Stat.*, **18**, 255–264 (1947).
5. E. Parzen, *Stochastic Processes*, Holden Day, San Francisco, 1962.
6. A. E. Ruark and F. E. Brammer, "The Efficiency of Counters and Counter Circuits," *Phys. Rev.*, **52**, 322–324 (1937).

7. W. L. Smith, "Renewal Theory and Its Ramifications," *J. Roy. Stat. Soc. Ser. B*, **20**, No. 2, 243–284 (1958).
8. L. Takacs, "On the Sequence of Events, Selected by a Counter from a Recurrent Process of Events," *Th. of Prob. and Its Appl.*, **1**, 81–91 (1956).
9. C. E. K. Mees, *The Theory of the Photographic Process*, MacMillan, New York, 1954.
10. D. V. Widder, *The Laplace Transform*, Oxford University Press, London, 1941.
11. H. J. Zweig, "Theoretical Considerations on the Quantum Efficiency of Photographic Detectors," *J. Opt. Soc. Am.*, **51**, 310–319 (1961).
12. H. J. Zweig, "Counter Models and Applications to Detection Problems," Stanford University Applied Mathematics and Statistics Laboratories, Technical Report No. 87, April 12, 1963.

Received January 15, 1965.