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Stability Criteria for Large Networks*

Abstract: An arbitrarily large network of bistable tunnel diode switching circuits is analyzed for stability. One condition derived indicates that increasing the total "fan" of each circuit might tend to make the whole network unstable. This condition is independent of the tunnel-diode characteristic. Another condition is also derived which depends on this characteristic but does not involve the total "fan". Finally, two general theorems which were proved in another paper are stated and discussed in terms of their applicability to certain classes of large networks and of the types of conditions for stability that can be obtained.

Introduction

Methods for analyzing the global stability of electrical networks have recently been developed[1,2] which lead to the possibility of treating large networks such as those used in computers. Previously, to the best of this author's knowledge, it was usually possible to analyze only a small section of the computer network, e.g., the so-called switching circuit, with the rest of the network being replaced by some simulated load.

In this paper a large network is considered which is composed of many arbitrarily interconnected copies of a particular bistable switching circuit; specifically, the circuit is one containing a negative-resistance device such as a tunnel diode. An important question, and one unanswered so far, is how the size or "dimension" of a network made up of many copies of a single circuit might affect its stability or reliability. For instance, suppose it is known that a particular switching circuit with some simulated load is stable if some stability criterion holds. Will the same criterion guarantee stability of the entire network? If not, how should such a criterion be altered to guarantee stability? In this paper we attempt to answer these questions for this special switching circuit and to indicate how they might be answered for other large networks.

In the example considered, two sufficient conditions for stability are derived. The first is independent of the nonlinearity of the negative-resistance device but depends on the maximum number of connections to a switching circuit (sometimes called the total fan). In other words, the dimension of the network is in some sense represented by the total fan. This result indicates that increasing the dimension might tend to cause instability. It is also indicated how such stability conditions, which depend on the dimension but are independent of the nonlinearity, may be derived for other networks by applying Theorem A, below.

The second condition, which is stated but not proved here, is independent of the "dimension" of the network but depends on the nonlinearity, and it is indicated how such statements of this nature might be obtained for other networks by applying Theorem B, below.

The network

We consider the switching circuit shown in Fig. 1. Here $f(v)$ is a nonlinear function which gives the current through the box in the direction shown. Both E and R are chosen so that the circuit is bistable. The network considered is made of a multiplicity of such circuits interconnected at a node, labeled "A" in Fig. 1a, by a series RL combination such as shown in Fig. 1b. To illustrate a section of the network we denote the switching circuit by the symbol shown in Fig. 1c, and the connecting RL line by a single line. These symbols are used in Fig. 2 to show a section of the large network. No restriction is placed on the number of switching circuits in the network. However, we shall require that the number of connections to any one circuit (the maximum total fan) be bounded and

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denote the maximum number by n. In Fig. 2, for example, n is 5.

Stability theorems

For any network of switching circuits as described we now state a condition which ensures that the network is stable.

• *Theorem 1*

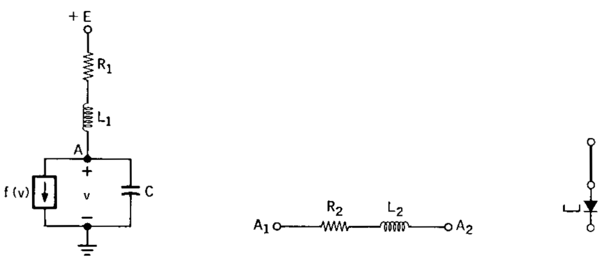
If $\int_0^v f(v)\,dv \to \infty$ *as* $|v| \to \infty$

and

$L_1/(R_1^2C) + 2nL_2/(R_2^2C) < 1,$

where n is the maximum number of connections to any circuit, then every solution of the differential equations of the network approaches an equilibrium solution as $t \to \infty$.

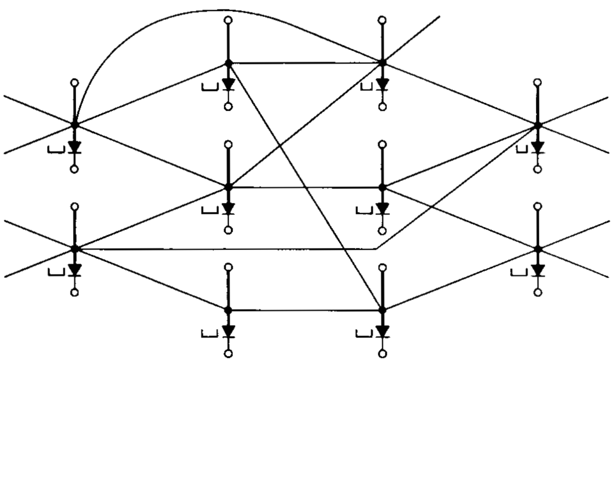
We note that the network has many equilibrium states. The above condition guarantees that any solution approaches one of these states as $t \to \infty$. The condition of Theorem 1 is independent of the number of circuits in the network and the manner in which the circuits are actually connected with each other. We also note that the nonlinearity $f(v)$ does not enter. Furthermore, if $n = 0$, the criterion reduces to that for the single circuit with no load.



(a) (b) (c)

Figure 1 **(a) Tunnel-diode switching circuit; (b) equivalent circuit of line connecting two switching circuits; (c) symbol denoting switching circuit of Figure 1(a).**

Figure 2 **Section of large network with switching circuits and interconnecting lines shown symbolically.**



The differential equations governing such large networks can, in principle, be derived in the usual manner from Kirchhoff's laws, etc. However, it is essential for the proof of our statement to express these equations in a form which reflects the particular physical situation in a clear and compact way. For this purpose we find it appropriate to use the theory developed in Ref. 1. There, the differential equations are derived from a "potential function" which is built up additively from the different components and which contains a connection matrix describing their connection. A summary of this theory can be found in Ref. 2. Although our derivation was motivated by this theory, the following proof is self-contained.

• *Proof of Theorem 1*

The differential equations for the network can be written in the form

$$\left.\begin{aligned} L_1\frac{di_\rho}{dt} &= \frac{\partial P}{\partial i_\rho}, & \rho &= 1, \cdots, r \\ L_2\frac{di_{r+\sigma}}{dt} &= \frac{\partial P}{\partial i_{r+\sigma}}, & \sigma &= 1, \cdots, s \\ C\frac{dv_\rho}{dt} &= -\frac{\partial P}{\partial v_\rho}, & \rho &= 1, \cdots, r \end{aligned}\right\} \tag{1}$$

where $i_1, \cdots, i_r$ are the currents through the inductors with inductance L_1; $i_{r+1}, \cdots, i_{r+s}$ are the currents through the inductors with inductance L_2; and $v_1, \cdots, v_r$ are the voltages across the capacitors. The potential function P is

$$\begin{aligned} P(i, v) = &-\tfrac{1}{2}R_1 \sum_{\rho=1}^{r} i_\rho^2 \\ &- \tfrac{1}{2}R_2 \sum_{\sigma=1}^{s} i_{r+\sigma}^2 + \sum_{\rho=1}^{r} \int_0^{v_\rho} f(u)\,du \\ &+ \sum_{\rho=1}^{r} i_\rho(E - v_\rho) + \sum_{\sigma=1}^{s} i_{r+\sigma}(v_{\nu(\sigma)} - v_{\mu(\sigma)}), \end{aligned}$$

where $\nu(\sigma)$ gives the smaller of the indices of the circuits to which the σ^{th} branch is connected and $\mu(\sigma)$ gives the larger index. The validity of Eq. (1) can be verified directly by simply differentiating $P(i, v)$ and substituting into Eq. (1). The existence of such functions in general for reciprocal networks is derived in Ref. 1.

We now construct the following function

$$\begin{aligned} P^* = P &+ \frac{1}{R_1} \sum_{\rho=1}^{r} (E - v_\rho - R_1 i_\rho)^2 \\ &+ \frac{1}{R_2} \sum_{\sigma=1}^{s} (-R_2 i_{r+\sigma} + v_{\nu(\sigma)} - v_{\mu(\sigma)})^2 \end{aligned}$$

and we will show that:

(a) For any solution $i(t)$, $v(t)$ of Eq. (1) except an equilibrium solution, $P^*(i(t), v(t))$ is a decreasing function of time;

(b) $P^* \geq b > -\infty$ and $P^*(i, v) \to \infty$ if $|i| + |v| \to \infty$.[3] Then the complete stability of this network will follow from the theory of Liapunov functions (see Ref. 4).

To show (a), we differentiate $P(i(t), v(t))$

$$\begin{aligned}\frac{dP}{dt} &= \sum_{\rho=1}^{r} \frac{\partial P}{\partial i_\rho}\frac{di_\rho}{dt} + \sum_{\sigma=1}^{s} \frac{\partial P}{\partial i_{r+\sigma}}\frac{di_{r+\sigma}}{dt} + \sum_{\rho=1}^{r} \frac{\partial P}{\partial v_\rho}\frac{dv_\rho}{dt} \\ &= L_1 \sum_{\rho=1}^{r}\left(\frac{di_\rho}{dt}\right)^2 + L_2 \sum_{\sigma=1}^{s}\left(\frac{di_{r+\sigma}}{dt}\right)^2 \\ &\quad - C\sum_{\rho=1}^{r}\left(\frac{dv_\rho}{dt}\right)^2. \qquad (2)\end{aligned}$$

Now

$$\begin{aligned}\frac{dP^*}{dt} &= \frac{dP}{dt} + \frac{2}{R_1}\sum_{\rho=1}^{r}(E - v_\rho - R_1 i_\rho)\left(-\frac{dv_\rho}{dt} - R_1\frac{di_\rho}{dt}\right) \\ &\quad + \frac{2}{R_2}\sum_{\sigma=1}^{s}(-R_2 i_{r+\sigma} + v_{\nu(\sigma)} - v_{\mu(\sigma)}) \\ &\qquad \left(-R_2\frac{di_{r+\sigma}}{dt} + \frac{dv_{\nu(\sigma)}}{dt} - \frac{dv_{\mu(\sigma)}}{dt}\right).\end{aligned}$$

However,

$$E - v_\rho - R_1 i_\rho = \frac{\partial P}{\partial i_\rho} = L_1\frac{di_\rho}{dt},$$

and

$$-R_2 i_{r+\sigma} + v_{\nu(\sigma)} - v_{\mu(\sigma)} = \frac{\partial P}{\partial i_{r+\sigma}} = L_2\frac{di_{r+\sigma}}{dt}.$$

Therefore,

$$\begin{aligned}\frac{dP^*}{dt} &= \frac{dP}{dt} + \frac{2L_1}{R_1}\sum_{\rho=1}^{r}\frac{di_\rho}{dt}\left(-\frac{dv_\rho}{dt} - R_1\frac{di_\rho}{dt}\right) \\ &\quad + \frac{2L_2}{R_2}\sum_{\sigma=1}^{s}\frac{di_{r+\sigma}}{dt}\left(-R_2\frac{di_{r+\sigma}}{dt} + \frac{dv_{\nu(\sigma)}}{dt} - \frac{dv_{\mu(\sigma)}}{dt}\right).\end{aligned}$$

Combining the above with Eq. (2), we have

$$\begin{aligned}\frac{dP^*}{dt} &= -L_1\sum_{\rho=1}^{r}\left(\frac{di_\rho}{dt}\right)^2 - L_2\sum_{\sigma=1}^{s}\left(\frac{di_{r+\sigma}}{dt}\right)^2 \\ &\quad - C\sum_{\rho=1}^{r}\left(\frac{dv_\rho}{dt}\right)^2 - \frac{2L_1}{R_1}\sum_{\rho=1}^{r}\frac{di_\rho}{dt}\frac{dv_\rho}{dt} \\ &\quad + \frac{2L_2}{R_2}\sum_{\sigma=1}^{s}\frac{di_{r+\sigma}}{dt}\left(\frac{dv_{\nu(\sigma)}}{dt} - \frac{dv_{\mu(\sigma)}}{dt}\right). \qquad (3)\end{aligned}$$

With

$$x_\rho = L_1^{1/2}\frac{di_\rho}{dt},$$

$$y_\sigma = L_2^{1/2}\frac{di_{r+\sigma}}{dt},$$

$$z_\rho = C^{1/2}\frac{dv_\rho}{dt},$$

Eq. (3) becomes

$$\begin{aligned}\frac{dP^*}{dt} &= -\sum_{\rho=1}^{r}(x_\rho^2 + z_\rho^2) - \sum_{\sigma=1}^{s} y_\sigma^2 - \frac{2L^{1/2}}{R_1 C^{1/2}}\sum_{\rho=1}^{r} x_\rho z_\rho \\ &\quad + \frac{2L_2^{1/2}}{R_2 C^{1/2}}\sum_{\sigma=1}^{s} y_\sigma(z_{\nu(\sigma)} - z_{\mu(\sigma)}) \\ &= -\sum_{\rho=1}^{r}\left(x_\rho + \frac{L^{1/2}}{R_1 C^{1/2}} z_\rho\right)^2 \\ &\quad - \sum_{\sigma=1}^{s}\left(y_\sigma - \frac{L_2^{1/2}}{R_2 C^{1/2}}(z_{\nu(\sigma)} - z_{\mu(\sigma)})\right)^2 \\ &\quad + \frac{L_1}{R_1^2 C}\sum_{\rho=1}^{r} z_\rho^2 + \frac{L_2}{R_2^2 C}\sum_{\sigma=1}^{s}(z_{\nu(\sigma)} - z_{\mu(\sigma)})^2 - \sum_{\rho=1}^{r} z_\rho^2.\end{aligned}$$

Thus if

$$\frac{L_1}{R_1^2 C}\sum_{\rho=1}^{r} z_\rho^2 + \frac{L_2}{R_2^2 C}\sum_{\sigma=1}^{s}(z_{\nu(\sigma)} - z_{\mu(\sigma)})^2 < \sum_{\rho=1}^{r} z_\rho^2, \qquad (4)$$

then $P^*(i(t), v(t))$ is a decreasing function of t. However, note that

$$\sum_{\sigma=1}^{s}(z_{\nu(\sigma)} - z_{\mu(\sigma)})^2 \leq 2\sum_{\sigma=1}^{s} z_{\nu(\sigma)}^2 + z_{\mu(\sigma)}^2 = 2\sum_{\rho=1}^{r} n_\rho z_\rho^2$$

where n_ρ is the number of connections to the ρ^{th} circuit. Thus, we have (since $n_\rho \leq n$)

$$\begin{aligned}&\frac{L_1}{R_1^2 C}\sum_{\rho=1}^{r} z_\rho^2 + \frac{L_2}{R_2^2 C}\sum_{\sigma=1}^{s}(z_{\nu(\sigma)} - z_{\mu(\sigma)})^2 \\ &\qquad \leq \left(\frac{L_1}{R_1^2 C} + \frac{2L_2}{R_2^2 C} n\right)\sum_{\rho=1}^{r} z_\rho^2,\end{aligned}$$

which gives the inequality of (4) if

$$\frac{L_1}{R_1^2 C} + \frac{2L_2}{R_2^2 C} n < 1. \qquad (5)$$

To show (b), above, we rewrite P^* in a positive form

$$\begin{aligned}P^* &= \frac{1}{2R_1}\sum_{\rho=1}^{r}(E - v_\rho - R_1 i_\rho)^2 + (E - v_\rho)^2 \\ &\quad + \sum_{\rho=1}^{r}\int_0^{v_\rho} f(v)\,dv + \frac{1}{2R_2} \\ &\quad \times \sum_{\sigma=1}^{s}(-R_2 i_{r+\sigma} + v_{\nu(\sigma)} - v_{\mu(\sigma)})^2 \\ &\quad + (v_{\nu(\sigma)} - v_{\mu(\sigma)})^2.\end{aligned}$$

Therefore, $P^* \geq 0$, and it is also obvious from the above form that $P^*(i, v) \to \infty$ if $|i| + |v| \to \infty$. Applying the Liapunov theorem found in Ref. 4, this concludes the proof of the theorem.

Another way of proving this same theorem is to apply Theorem 3 of Ref. 1, which is restated below as Theorem A.

• *Theorem A*

If the potential function for a network has the form[5]

$$P(i, v) = -\tfrac{1}{2}(i, Ai) + B(v) + (i, \gamma v - a)$$

and the equations are given by[6]

$$L(i)\frac{di}{dt} = \frac{\partial P}{\partial i}$$

$$C(v)\frac{dv}{dt} = -\frac{\partial P}{\partial v},$$

then assuming $L(i)$, $C(v)$ are symmetric, positive definite matrices, A is a positive definite matrix, $B(v) + |\gamma v| \to \infty$ as $|v| \to \infty$ and

$$||L^{1/2}(i) A^{-1}\gamma C^{-1/2}(v)|| < 1, ^{7} \tag{6}$$

the corresponding circuit is completely stable.

Thus, the proof of the complete stability of the network which we have considered reduces, through the use of the above theorem, to showing that condition (5) implies the inequality (6). This theorem is applicable to a large class of networks where all the nonlinearities are voltage controlled. For example, a network in which all the nonlinear elements are tunnel diodes would have the correct form for the potential function. The problem of stability reduces to finding the conditions under which the inequality (6) is satisfied.

A different type of stability statement may be obtained for this network by applying Theorem 5 of Ref. 1, which is restated below as Theorem B.

• *Theorem B*

Suppose the potential function for a network has the form

$$P(i, v) = -A(i) + B(v) + (i, \gamma v - a),$$

where the equations are $L\, di/dt = \partial P/\partial i$, and $C\, dv/dt = -\partial P/\partial v$, and L, C are constant symmetric positive definite matrices. Also assume $P^*(i, v) \to \infty$ as $|i| + |v| \to \infty$ where

$$P^*(i, v) = \left(\frac{\mu_1 - \mu_2}{2}\right)P(i, v) + \frac{1}{2}\left(\frac{\partial P}{\partial i}, L^{-1}\frac{\partial P}{\partial i}\right) + \frac{1}{2}\left(\frac{\partial P}{\partial v}, C^{-1}\frac{\partial P}{\partial v}\right)$$

and μ_1, μ_2 are the smallest eigenvalues of $L^{-\frac{1}{2}}(\partial^2 A/\partial i^2)L^{-\frac{1}{2}}$ and $C^{-\frac{1}{2}}(\partial^2 B/\partial v^2)C^{-\frac{1}{2}}$, respectively. Then, if

$$\mu_1 + \mu_2 > 0, \tag{7}$$

the corresponding network is completely stable.

For this theorem it is not required that all the nonlinearities be voltage controlled. However, note that condition (7) will depend on the nonlinearities. Theorem B applied to the special network being considered yields the following theorem.

• *Theorem 2*

If

$$\min\left(\frac{R_1}{L_1}, \frac{R_2}{L_2}\right) + \min_{v} \frac{f'(v)}{C} > 0, \tag{8}$$

where $f'(v) = df(v)/dr$, then the network is completely stable.

Although this theorem gives a stability condition (8) which depends on $f'(v)$, it is independent of the total fan n. Thus, if n were large, condition (8) would be preferable to condition (5).

Necessity of the stability conditions

The stability conditions derived for this network are sufficient but not necessary. The conditions (5) and (8) have been derived from the general theorems, A and B, and it is to be expected that stronger conditions should be available. However, these will probably have to be obtained by use of more specific knowledge about the network, and one would expect that they might be very hard to obtain. On the other hand, there are some networks where Theorems A and B yield necessary and sufficient conditions for stability (see Sections 9 and 10 of Ref. 1) so that it cannot be expected that the conditions given in these theorems can be improved.

Conclusions

In this paper two general theorems have been stated and applied to a particular large network. The intent has been to show how these theorems might be applied and to furnish some insight about the kind of network to which they are applicable and the kind of stability results they yield. Theorem A applies to networks in which the nonlinearities are voltage controlled and yields a stability criterion which is independent of the nonlinearity. However, because the matrix γ, which is related to the manner in which the network is interconnected, appears in the stability criterion, it is to be expected that this criterion will depend on some "dimension" of the network. Theorem B does not require that the nonlinearities be voltage controlled. The stability criterion will depend on the nonlinearities but, since the condition of (7) does not contain the "connection" matrix γ, one might expect that the resulting stability criterion is independent of "dimension."

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References and footnotes

1. R. Brayton and J. Moser, "A Theory of Nonlinear Networks, I", *Quarterly of Applied Math.* **XXII**, 1, 1–33 (April 1964).

2. R. Brayton and J. Moser, "Some Results on the Stability of Nonlinear Networks Containing Negative Resistances," *IEEE Transactions on Circuit Theory* **CT-11**, 1, 165–167 (March 1964).
3. The notation $|u|$ of an r-vector u represents the Euclidean norm of the vector, i.e.,

$$|u|^2 = \sum_{i=1}^{r} u_i^2.$$

4. J. LaSalle and S. Lefschetz, *Stability by Liapunov's Direct Method with Applications*, Academic Press, New York and London, 1961. p. 66.
5. Here, $(u, v) = \sum^r{}_{i=1} u_i v_i$ stands for the inner product of two r-vectors.
6. $\frac{\partial P}{\partial i} = \left(\frac{\partial P}{\partial i_1}, \cdots, \frac{\partial P}{\partial i_r}\right)$ etc.
7. The norm of a matrix K is meant as

$$||K||^2 = \max \frac{(Kx, Kx)}{(x, x)}.$$

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