

F. Odeh

An Existence Theorem for the BCS Integral Equation

An important consequence of the Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity is the existence of a gap in the energy spectrum. In the original BCS paper¹ the gap function was found by solving a homogeneous nonlinear integral equation in which the phonon interaction energy was taken to be a constant in some appropriate region and to be zero everywhere else. In that case the gap function is a constant which is determined by a simple algebraic relation. If one considers more general interactions, the existence of a nonzero energy-gap function is then by no means obvious. We must mention, however, that certain numerical calculations employing linearization procedures do produce nontrivial solutions for the cases they consider. (See Ref. 2, where other references are cited).

The purpose of this Communication is to discuss the special case of negative phonon interactions and to show how the questions of existence and uniqueness may be decided by means of descriptive geometrical arguments. Some of the results below overlap with results announced without proofs by Huber,^{3a,b} who has employed iteration techniques. The work in Ref. 3 appears to be mainly concerned with the actual calculation of the gap function, under additional restrictions on the interaction, while this Communication is concerned only with qualitative aspects.

The BCS equation for spherically symmetric gaps at zero temperature has, in appropriate units,² the form

$$f(x) = Tf \equiv \int_0^1 V(x, \xi) \frac{f(\xi)}{\sqrt{\xi^2 + f^2(\xi)}} d\xi, \quad (1)$$

where the real function f is proportional to the energy gap, and V is the negative of the phonon-interaction energy. We assume the following:

(i) There are positive numbers m, M such that

$$\inf_x V(x, \xi) \geq m > 0 \quad (2a)$$

$$\sup_x \int_0^1 V(x, \xi) d\xi < M; \quad (2b)$$

(ii) $\lim_{\alpha \rightarrow 0} \iint |V(x + \alpha, \xi) - V(x, \xi)| dx d\xi = 0. \quad (2c)$

These conditions will be sufficient for the existence theorem below. In discussing uniqueness we assume the stronger condition

$$\lim_{\alpha \rightarrow 0} \int |V(x + \alpha, \xi) - V(x, \xi)| d\xi = 0, \quad (2d)$$

uniformly in x .

At this point we introduce some new notation. Let $L_1(0, 1)$ be the Banach space of integrable functions on $(0, 1)$ with the usual norm which is denoted by $\|\cdot\|$. Let S be the intersection of the sphere consisting of all functions in L_1 such that $\|f\| \leq M$ with the cone defined by $f \geq 0$. Let S_ϵ be the part of S for which $f \geq \epsilon > 0$. Then under the hypotheses expressed by Eqs. (2a, 2b) one has the following:

• *Lemma*

If ϵ is small enough, then $TS_\epsilon \subset S_\epsilon$.

• *Proof*

It is clear that the whole $L_1(0, 1)$ space is contracted by T into the M -sphere. Let $f(x)$ belong to S_ϵ , i.e., $f(x) = \epsilon + \eta(x)$, $\eta \geq 0$; then

$$\begin{aligned} Tf &\geq m \int_0^1 \frac{\epsilon + \eta(\xi)}{\sqrt{\xi^2 + (\epsilon + \eta)^2}} d\xi \\ &\geq m\epsilon \int_0^1 \frac{1}{\sqrt{\xi^2 + \epsilon^2}} d\xi > \epsilon, \end{aligned}$$

if ϵ is small enough.

It should be noted here that ϵ, M may be chosen so as to have TS_ϵ properly included in S_ϵ . This fact is useful in the uniqueness question.

We state now a special version of Schauder's fixed point theorem: *A continuous map in a Banach space which maps a complete set S into a relatively compact set $S' \subset S$ has a fixed point.* This version may be proved by applying the familiar form of Schauder's theorem⁴ to the closed convex hull of $T(S')$. Now we have the following:

• **Theorem**

The spectrum of the operator T includes the whole positive real axis. In particular, the equation $Tf = f$ has nontrivial solutions in S_ϵ .

• **Proof**

The set S_ϵ is a convex complete set in L_1 . [Notice that it will still be convex if one uses a Hilbert space norm.] Since the map T is completely continuous by the assumption expressed by Eq. (2c),⁵ it has a fixed point by Schauder's theorem. Hence, $\lambda = 1$ is in the spectrum of T . Replacing V by $\lambda^{-1} V$, $\lambda > 0$, does not change the validity of Eqs. (2a, b, and c). Therefore, the equation $Tf = \lambda f$ has a nontrivial solution for all $\lambda > 0$, which proves the theorem.

References

1. As seen from the proof, the existence of a fixed point of T is due, geometrically speaking, to the fact that T contracts a large enough sphere but expands a neighborhood of the origin. This latter property of T is seen clearly by calculating its functional derivative at zero, which is logarithmically divergent. If the power of ξ in the quasiparticle energy $(\xi^2 + f^2)^{1/2}$ were reduced—which is physically unacceptable—then the derivative at zero would be finite and there may exist no solutions. For example, the equation

$$f = \int_0^1 V(x, \xi) f(\xi) [\xi^{2-2\epsilon} + f^2]^{-1/2} d\xi$$

has no nontrivial solutions if $|V| < 1$.

2. The temperature-dependent BCS equation has the form

$$f = T_v f = \int_0^1 V(x, \xi) f(\xi^2 + f^2)^{-1/2} \cdot \tanh \left(\frac{(\xi^2 + f^2)^{1/2}}{\theta} \right) d\xi, \quad (3)$$

where θ is some reduced temperature. If θ is small enough, then the proofs of the lemma and the theorem still hold. In fact, a very crude estimate will be to take $\epsilon = \theta$, and the existence of an energy gap for small θ is assured. If, however, θ is large enough, there will be no solution. To see this, assume Eq. (2d) to be true; then the gap will be continuous and has a maximum N . Equation (3) then implies that

$$N \leq MN \frac{2}{\theta}.$$

Hence, N will be zero if $2M/\theta < 1$.

3. The anisotropic gap equation has the form $f(x) = Pf$ where x denotes a three-dimensional vector and P is a nonlinear integral operator. Although P and T have different forms they share the properties stated in Remark 1, namely they expand a neighborhood near zero but contract large enough spheres. Hence an existence theorem for an anisotropic energy gap would follow from reasoning similar to that followed in the theorem proof.

4. To prove the uniqueness of the solution to $f = Tf$ by topological methods is more subtle. One may, however, exploit the classical degree theory of continuous maps as used, for example, by Rothe.⁶ It is more convenient now to consider the set S_ϵ of continuous functions f such that $f \geq \epsilon$ and $\max f \leq M$, that is, to consider S_ϵ in the Banach space $C(0, 1)$ under the maximum norm. We state now sufficient conditions to ensure uniqueness:

Let the interaction V satisfy Eqs. (2a, b, and d). Suppose, for all f in S_ϵ , the linear integral equation

$$z(x) = \int_0^1 \frac{\partial}{\partial f} [V(x, \xi) f(\xi^2 + f^2)^{-1/2}] z(\xi) d\xi$$

has no nontrivial solutions, i.e., that the functional derivative of $I - T$ has a bounded inverse in S_ϵ . Then, the solution of $Tf = f$, which exists by the above theorem, is also unique.

The proof becomes exactly the same as that of Theorem 3 in Ref. 6 when one takes into account the following: (1) the solution of $Tf = f$ is continuous by the assumption expressed in (2d); and (2) the image of the boundary of S_ϵ lies properly in the interior of S_ϵ , as remarked in the proof of the lemma.

5. If the interaction changes its sign, then the existence of a nontrivial solution may, in general, be asserted only under additional conditions which restrict the detailed properties of such an interaction. This topic is to be the subject of a future communication.

References

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