

On a Queueing Problem Arising in Recirculating Memories

This Communication considers a queueing problem that arises in the design and evaluation of circulating types of computer storage devices, such as magnetic drums. This many-line, many-server queueing problem is treated using approximate methods, and the distribution is determined for the number of items waiting in one of the lines.

The information in this device is stored in units consisting of strings of numbers of equal length. These units are referred to as *records*. The records are written into *fields* such that every field consists of a fixed large number of rows, each of which contains exactly one record. The fields circulate with a constant speed under a small number of adjacent reading heads, that is, they pass by the reading heads continuously one after the other in a fixed order, the last one being followed by the first one. The fields move in a direction parallel to their rows, such that the corresponding rows of every field move always in the same track. Each reading head can be positioned on any single track. Whenever there is a request for reading a record, and at least one of the readers is available when the field containing that record arrives at the readers, a reader is positioned on the required track. The requested record is then read out as it passes by. The requests for reading records are assumed to arrive randomly according to a Poisson distribution with mean rate λ for each field and form a queue at the readers. The service, i.e., reading, is first-come-first-served for requests referring to the same field. It is possible that not all requests can be served during a single passage of a field, and the remaining ones get the next chance on the next arrival of the field. A reader can read only one record from a given field on each passage of that field. Furthermore its availability is determined by the fact that a reader is "dead" for a fixed time after each reading. This time is the access time of the reader, i.e., that needed to switch from one track to any other. If more readers are available than necessary, the ones used are selected at random.

The problem is to find the stationary distribution of the number of requests waiting. The answer to this is obviously necessary in order to know what the minimum number of readers must be to handle a given amount of traffic, and to determine the storage requirements for the queue. We give here an approximate analytical treatment of this problem by reducing it to a form in which a technique similar to the familiar one for many-server systems with constant service times¹ can be applied. Numerical results have been obtained by D. Shah, of the IBM Advanced Systems Development Division, who wrote a machine program based on the formulas developed here.

The following notations will be used: let N be the number of fields, A the access time of the readers, and T the cycle time, that is, the time between two consecutive passages of any given field under the reading heads.

We fix our attention on a given field F . Let $\nu(j)$, $\xi(j)$, and $\mu(j)$ be the following random variables: $\nu(j)$ the number of requests waiting for F just before it arrives at the readers for the j^{th} time, $j = 1, 2, 3, \dots$; $\xi(j)$ the number of free readers at the same time; $\mu(j)$ the number of requests for F arriving between j and $j + 1$. $\nu(j)$ and $\mu(j)$ can take any non-negative integer value, and $\xi(j)$ the values $0, 1, \dots, c$.

The distribution of the $\mu(j)$ is the assumed Poisson distribution:

$$P\{\mu(j) = m\} = e^{-\lambda T} \frac{(\lambda T)^m}{m!}, \quad m = 0, 1, \dots \quad (1)$$

For the sought stationary distribution of the $\nu(j)$ we use the notation

$$P\{\nu(j) = n\} = P_n, \quad n = 0, 1, \dots \quad (2)$$

The stationary distribution of $\xi(j)$ can be obtained by the following consideration: The total number of records read in unit time is on the average $N\lambda$. The total time the readers are dead is therefore $N\lambda A$. Since they are used

with equal probability, each one of them is dead, on the average, for a time

$$q = N\lambda A/c \quad (3)$$

out of a time interval of unit length. Thus q is obviously the probability of finding a given reader dead at any one time if the whole system is in stationary operation. It seems to be a very good approximation to assume that the occupancies of the readers are independent of each other. Then using the notation $p = 1 - q$ we get

$$p_k = P\{\xi(j) = k\} = \binom{c}{k} p^k q^{c-k} \quad \text{for } k = 0, 1, \dots, c. \quad (4)$$

To obtain the distribution (2) let us consider how $\nu(j+1) = n$ can arise from different states at the time j . The probability of $\nu(j+1) = n$ has to be the sum of the probabilities of all mutually exclusive events which lead to this one, thus we get

$$\begin{aligned} P\{\nu(j+1) = n\} &= \sum_{k=0}^{c-1} \sum_{m=k+1}^c P\{\nu(j) = k, \xi(j) = m, \mu(j) = n\} \\ &+ \sum_{m=0}^n \sum_{k=0}^c P\{\nu(j) = n+k-m, \xi(j) = k, \mu(j) = m\}, \\ &n = 0, 1, \dots \quad (5) \end{aligned}$$

This set of equations is rigorously valid, but insufficient without further assumptions for the determination of the P_n . We may, however, make the approximation that the variables $\nu(j)$, $\xi(j)$, and $\mu(j)$ are independent of each other. This is permissible because first the number of arrivals is independent of the state of the system if it is Poissonian as already assumed, and second the state $\xi(j)$ of the servers is determined by all the fields, so the effect of the selected one F may be neglected if there is a great number of them, that is, N is large. The set (5) of equations then becomes by using (1), (2), and (4)

$$\begin{aligned} P_n &= \sum_{k=0}^{c-1} \sum_{m=k+1}^c P_k p_m e^{-\lambda T} \frac{(\lambda T)^n}{n!} \\ &+ \sum_{m=0}^n \sum_{k=0}^c P_{n+k-m} p_k e^{-\lambda T} \frac{(\lambda T)^m}{m!}, \quad n = 0, 1, \dots, (6) \end{aligned}$$

This is a set of infinitely many equations for the infinite number of unknowns P_n . The best way to solve them is by calculating the generating function

$$G(x) = \sum_{n=0}^{\infty} P_n x^n. \quad (7)$$

Multiplying (6) by x^n and summing over n from zero to infinity, we obtain after some manipulation

$$\begin{aligned} G(x) &= e^{\lambda T(x-1)} \sum_{k=0}^{c-1} \sum_{m=k+1}^c P_k p_m \\ &+ \sum_{k=0}^c \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} (x^{n+k-m} P_{n+k-m}) (x^{-k} p_k) e^{-\lambda T} \frac{(\lambda T x)^m}{m!} \quad (8) \end{aligned}$$

Performing the n and m summations in the second term this gives

$$\begin{aligned} G(x) &= e^{\lambda T(x-1)} \left[\sum_{k=0}^{c-1} \sum_{m=k+1}^c P_k p_m \right. \\ &\left. + \sum_{k=0}^c \left(G(x) - \sum_{j=0}^{k-1} P_j x^j \right) x^{-k} p_k \right] \quad (9) \end{aligned}$$

Solving this for $G(x)$ we get

$$\begin{aligned} G(x) &= \frac{x^c \sum_{k=0}^{c-1} \sum_{m=k+1}^c P_k p_m - \sum_{k=0}^c \sum_{j=0}^{k-1} P_j p_k x^{c+j-k}}{x^c e^{-\lambda T(x-1)} - \sum_{k=0}^c p_k x^{c-k}} \\ &= \frac{f(x)}{g(x)} \quad (10) \end{aligned}$$

$G(x)$ has to satisfy two further conditions. First,

$$G(1) = 1 \quad (11)$$

must hold, because $G(1) = \sum_{n=0}^{\infty} P_n$. Second, $G(x)$ as a generating function must be analytic. This implies that the numerator $f(x)$ of (10) has to vanish at each of the c complex roots of the denominator $g(x)$. Equations (10) and (11) give by l'Hospital's rule

$$\begin{aligned} \{[p_1 + 2p_2 + \dots + cp_c]P_0 + [p_2 + 2p_3 + \dots \\ + (c-1)p_c]P_1 + \dots + p_c P_{c-1}\} / \{-\lambda T + [p_1 \\ + 2p_2 + \dots + cp_c]\} = 1 \quad (12) \end{aligned}$$

The roots of the denominator $g(x)$ of (10) can be obtained as follows. Let us use the values of the p_k given by (4), and abbreviate $e^{-\lambda T}$ as

$$a = e^{-\lambda T}. \quad (13)$$

The equation for the k' th root x_k then becomes

$$g(x) = x^c a^{x-1} e^{k'2\pi i} - (p + qx)^c = 0. \quad (14)$$

Let us substitute into this

$$x = re^{i\phi} \quad (15)$$

and separate the real and imaginary parts. Then (14) becomes

$$\begin{aligned} a^{(r/c)(\cos \phi)} \frac{r}{c\sqrt{a}} \left[\cos \phi \cos \left(\frac{r}{c} \lambda T \sin \phi \right) \right. \\ \left. + \sin \phi \sin \left(\frac{r}{c} \lambda T \sin \phi \right) \right] \\ = (p + qr \cos \phi) \cos \frac{2\pi k}{c} + qr \sin \phi \sin \frac{2\pi k}{c} \quad (16) \end{aligned}$$

$$\begin{aligned}
& \text{and } a^{(r/c)(\cos \phi)} \frac{r}{c\sqrt{a}} \left[\sin \phi \cos \left(\frac{r}{c} \lambda T \sin \phi \right) \right. \\
& \quad \left. - \cos \phi \sin \left(\frac{r}{c} \lambda T \sin \phi \right) \right] \\
& = qr \sin \phi \cos \frac{2\pi k}{c} - (p + qr \cos \phi) \sin \frac{2\pi k}{c} \quad (17)
\end{aligned}$$

(16) and (17) are transcendental equations for the unknowns r and ϕ . They can be solved by iteration. Let (r_k, ϕ_k) be the solution for any given value of k , and

$$x_k = r_k \exp(i\phi_k) \quad k = 0, 1, \dots, c-1. \quad (18)$$

Then our requirement that $f(x)$ has to vanish for all roots of $g(x)$ gives

$$f(x) = K \prod_{k=0}^{c-1} (x - x_k). \quad (19)$$

Here K is a constant, unknown for the moment. Since we know now the x_k , a comparison of the coefficients in the two expressions for $f(x)$, that is, (10) and (19), gives the probabilities $P_0, P_1, \dots, P_{c-1}$ as functions of K . The value of K can be determined then by substituting these into Eq. (12). The remaining P_n 's can now be obtained by simple successive elimination.

References

1. See, e.g., J. Riordan, *Stochastic Service Systems*, John Wiley and Sons, New York, 1962.

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