E. S. Barrekette
H. Freitag

Diffraction by a Finite Sinusoidal Phase Grating

In recent years several processes, such as Eidophor[1] and thermoplastic recording, have been developed for recording information in the form of deformations in the surface of a transparent medium. The phase of the light passed through such a medium is modulated according to the information stored. The information is retrieved by means of spatial filtering[2] in the Fraunhofer diffraction domain. In order to design useful spatial filters it is desirable to know the diffraction patterns formed by the storage medium.

In this paper we derive the diffraction pattern produced by a single sinusoidal groove and by a finite sinusoidal phase grating. We assume that the grooves are long, that the source is a long uniform line parallel to the grooves, and that a one-dimensional treatment is permitted, i.e., variations of light amplitude in the direction of the grooves can be neglected. Our approach is based on the usual linearizing assumptions made in the "communications theory of optics." A quantitatively rigorous analysis is beyond the scope of the present discussion. Such a treatment would necessitate a vector rather than a scalar representation of the electromagnetic field.

It is shown that for an infinite grating the power spectrum consists of discrete lines at the integral orders, where the amplitude is proportional to the Bessel function of the first kind, whose order is a linear function of the spatial frequency at the point of observation and whose argument is proportional to the product of the phase deviation and the depth of the surface deformation of the storage medium. It is also shown that if the grating is finite, particularly if only a few grooves are to be examined, as is the case in color recording, then the power spectrum cannot be represented simply as a product of a Bessel function with the interference function corresponding to the number of grooves present.

Fraunhofer diffraction by phase modulating objects

Consider the configuration shown in Fig. 1. Let P_c be a generic point in the source and let P be a generic point in the Fraunhofer diffraction field. Let the diffracting object be defined by the surface $z(x, y)$ which consists of: 1) a region A of uniform unit transmittance, and 2) an opaque screen everywhere outside A. The source and field points are in a pair of conjugate planes of an aberration-free lens whose aperture is larger than the region A.

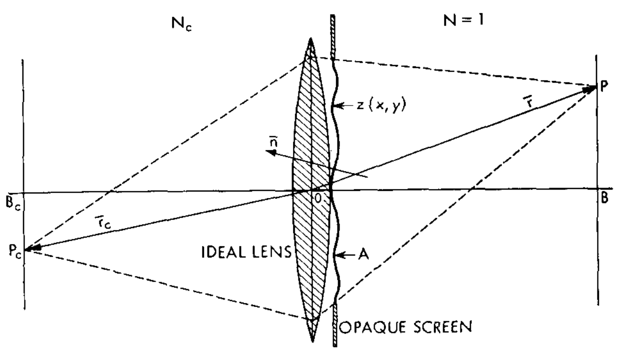
According to Fraunhofer's theory the amplitude distribution $U(P)$ at the field point P due to light of wave length λ emanating from the source at P_c is such that[2,3,4]

$$U(P) \sim \int_A \psi \exp \{ i2\pi[\phi z + (\mu_c - \mu)x + (\omega_c - \omega)y] \} \, dA, \qquad (1)$$

where integration is carried over the surface $z(x, y)$ within the region A and where the obliquity factor is $\psi = \frac{1}{2}[N_c \cos(n, r_c) - \cos(n, r)]$; the phase deviation is $2\pi\phi$ with $\phi = (N_c \cos \alpha_c - \cos \alpha)/\lambda$; and where $\mu_c = (N_c \sin \beta_c)/\lambda$, $\mu = (\sin \beta)/\lambda$, $\omega_c = (N_c \sin \gamma_c)/\lambda$, and $\omega = (\sin \gamma)/\lambda$ are spatial frequencies.[5] The quantities $\alpha, \alpha_c, \beta, \beta_c, \gamma, \gamma_c, r, r_c$ are as shown in Fig. 2, n is the outward normal to the diffracting surface, and N_c is the index of refraction of the diffracting object. (It should be noted that the above result is derived in Ref. 1 subject to the approximation $\psi \approx 1$).

If the source is assumed to be a long line parallel to the y axis and of uniform intensity, and if the object does not have any variations along this axis, i.e., if $z = z(x)$, then variations of amplitude with y can be neglected. The amplitude distribution is then such that

Figure 1 **Configuration of source, object and Fraunhofer domains.**



$U =$

$$K' \int_{-\infty}^{\infty} \psi(x) o(x) e^{i2\pi(\nu_c - \nu)x} \sqrt{1 + (dz/dx)^2}\, dx \qquad (2a)$$

where K' is a constant of proportionality which depends on the intensity of the source, the distance between the source, object and field domains, and the wave length of the light emitted by the source. The object function $o(x) = \exp[i2\pi\phi(\nu, \nu_c)z(x)]$ if x is in A and it is zero elsewhere, and

$$\nu_c = (N_c \sin \alpha_c)/\lambda \qquad \nu = (\sin \alpha)/\lambda. \qquad (2b)$$

The obliquity factor is a function of x, ν and ν_c while the phase deviation depends only on ν and ν_c. Clearly, the amplitude distribution $U(\nu)$ differs from $O(\nu)$, the Fourier transform of $o(x)$; the diffraction and object domains are thus not canonically conjugate. Furthermore, ϕ and ψ are not functions of the difference $\nu - \nu_c$ and thus the system is not space invariant, or isoplanatic. These two nonlinear effects are important if large values of α and α_c are of interest. To obtain a linear theory we assume that both ϕ and ψ are independent of ν and ν_c, an assumption which is valid only if α and α_c are small. Thus

$$\phi = (N_c - 1)/\lambda = \phi_0/2\pi \qquad (3a)$$

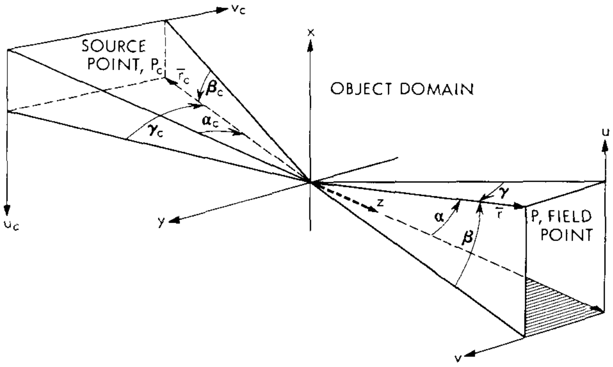
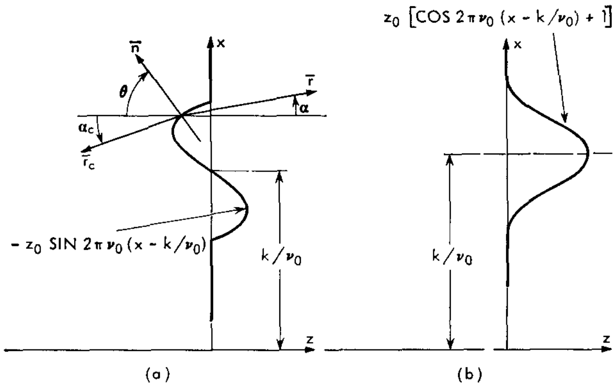


Figure 2 **Geometry associated with the definition of spatial frequency.**

Figure 3 **Groove geometries.**



$$\psi = (N_c + 1)\cos\theta = (N_c + 1)/\sqrt{1 + (dz/dx)^2} \qquad (3b)$$

so that

$$U = K \int_{-\infty}^{\infty} o(x) \exp[i2\pi(\nu_c - \nu)x]\, dx \qquad (4)$$

for all surface shapes $z(x)$. The quantity $K = K'(N_c + 1)$ is a constant. Thus under the foregoing assumptions we get the usual result that U is the Fourier transform of $o(x)$.

The results derived below on the basis of Eq. (4), and thus on the basis of the linear theory, are limited with respect to their realms of applicability to small angles of incidence and of diffraction. For a comparison of the diffraction patterns predicted by the linear and nonlinear theories for blazed diffraction gratings see Ref. 7.

The sinusoidal groove

To obtain the power spectrum of a sinusoidal groove (Fig. 3a), let

$$z_1(x) = -z_0 \sin 2\pi\nu_0[x - (k/\nu_0)], \qquad (5a)$$

where ν_0 is the fundamental frequency, and k is a constant defining the position of the object centerline at k/ν_0. Let the object function be

$$o(x) = \begin{cases} e^{i\phi_0 z_1} & |x - (k/\nu_0)| \leq 1/2\nu_0 \\ 0 & \text{elsewhere} \end{cases} \qquad (5b)$$

so that k/ν_0 is the position of the groove centerline. From Eq. (4) we get

$U_s(p) =$

$$\frac{K}{2\pi\nu_0} \exp(-i2\pi pk) \int_{-\pi}^{\pi} \exp[-i(\phi_0 z_0 \sin\zeta + p\zeta)]\, d\zeta, \qquad (6)$$

where $p = (\nu - \nu_c)/\nu_0$. Eq. (6) can be rewritten as

$U_s(p) =$

$$\frac{K}{\pi\nu_0} \exp(-i2\pi pk) \int_0^{\pi} \cos(\phi_0 z_0 \sin\zeta + p\zeta)\, d\zeta. \qquad (6a)$$

Using Sommerfeld's integral representation[6] for the Bessel function involving integration in the complex plane, we obtain

$$U_s(p) = \frac{K}{2\pi\nu_0} \exp(-i2\pi pk) \Big\{ 2\pi J_{-p}(\phi_0 z_0) - \int_{-\pi+i\infty}^{-\pi} \exp[-i(\phi_0 z_0 \sin\zeta + p\zeta)]\, d\zeta - \int_{\pi}^{\pi+i\infty} \exp[-i(\phi_0 z_0 \sin\zeta + p\zeta)]\, d\zeta \Big\}$$

or, with a bit of manipulation,

$$U_s(p) = \frac{K}{\nu_0} \exp(-i2\pi pk) \Big\{ J_{-p}(\phi_0 z_0) - \frac{1}{\pi} \sin\pi p \int_0^{\infty} \exp[-(\phi_0 z_0 \sinh\xi - p\xi)]\, d\xi \Big\}. \qquad (6b)$$

Expressions corresponding to Eqs. (6a) and (6b) for the groove shown in Fig. 3b with

$$z_2(x) = z_0\{\cos 2\pi\nu_0[x - (k/\nu_0)] + 1\} \quad (7a)$$

$$o(x) = \begin{cases} e^{i\phi_0 z_2} & |x - (k/\nu_0)| \le 1/2\nu_0 \\ 0 & \text{elsewhere} \end{cases} \quad (7b)$$

are respectively

$$U_c(p) = \frac{K}{\pi\nu_0} \exp\,[-i(2\pi pk - \phi_0 z_0)] \int_0^\pi [\cos\,(\phi_0 z_0 \cos\,\zeta) + i \sin\,(\phi_0 z_0 \cos\,\zeta)] \cos p\zeta\; d\zeta \quad (8a)$$

$$U_c(p) = \frac{K}{\nu_0} \exp\,[-i(2\pi pk - \phi_0 z_0)]\left[e^{-i\pi p/2} J_{-p}(\phi_0 z_0) - \frac{1}{\pi} \sin \pi p \int_0^\infty \exp\,(-i\phi_0 z_0 \cosh \xi + p\xi)\; d\xi\right]. \quad (8b)$$

We can readily see from Eq. 6a that if the asymmetric groove of Fig. 3a is centered on the optic axis ($k = 0$) then the amplitude distribution is real and asymmetric in the non-dimensional spatial frequency p; while from Eq. 8a we can see that if the symmetric groove of Fig. 3b is centered on the optic axis ($k = 0$) then the amplitude distribution is complex and symmetric. It is of interest to note that in the case of amplitude modulation with $o_1(x) = \frac{1}{2}(1 + \sin 2\pi\nu_0 x)$ if $|\; 2\nu_0 x \;| \le 1$ and $o_1(x) = 0$ elsewhere, an asymmetric case, the amplitude distribution in the Fraunhofer field is asymmetric in p and complex, (the real part is symmetric and the imaginary part is antisymmetric). In the symmetric case, $o_2(x) = \frac{1}{2}(1 + \cos 2\pi\nu_0 x)$ if $|\; 2\nu_0 x \;| \le 1$ and $o_2(x) = 0$ elsewhere, the amplitude distribution is real and symmetric in p.

Finite grating

Let the object consist of N adjacent sinusoidal grooves

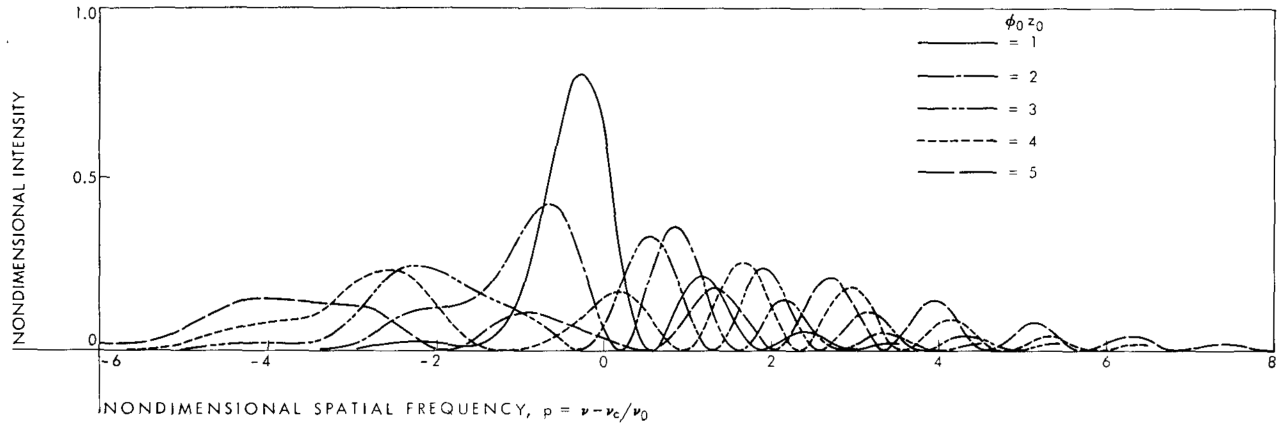
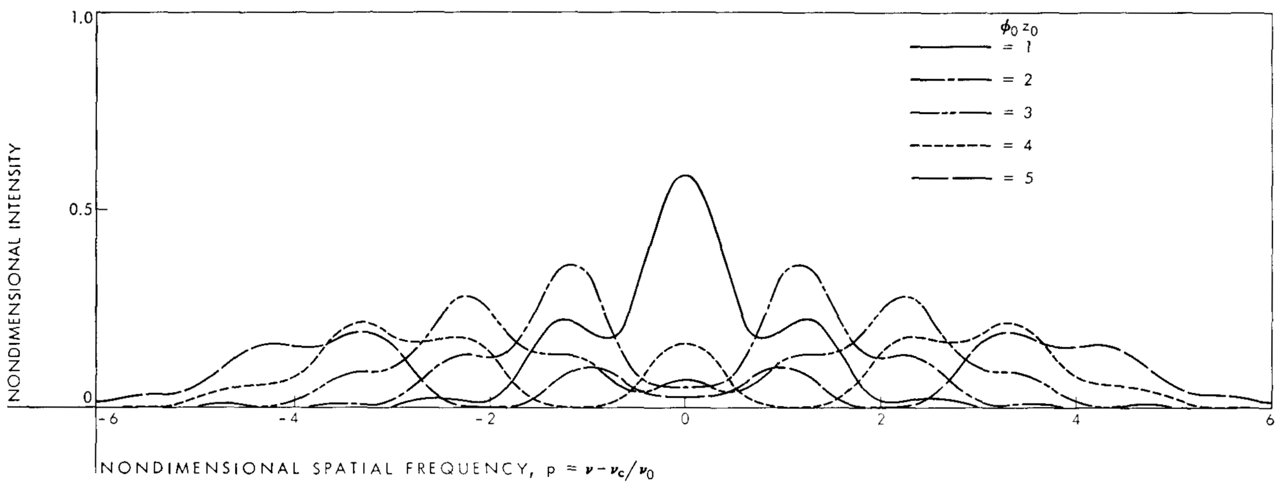


Figure 4 **Non-dimensional power spectrum for a sinusoidal groove: the antisymmetric case.**

Figure 5 **Non-dimensional power spectrum for a sinusoidal groove: the symmetric case.**



centered on $x = k_n/\nu_o$, where if N is odd $k_n = k + n$, $n = 0, \pm 1, \pm 2, \cdots \pm \frac{1}{2}(N-1)$, and where if n is even $k_n = k + \frac{1}{2} + n$, $n = 0, \pm 1, \cdots, \pm(\frac{1}{2}N - 1), -\frac{1}{2}N$, (in either case the object centerline is at $x = k/\nu_o$). The diffraction pattern is the sum of the patterns of the individual grooves, i.e.

$$U_N(p) = U(p)I(p), \tag{9}$$

where $U(p)$ is as given by Eqs. 6 or 8 for the gratings defined by Eqs. 5a or 7a respectively, and $I(p)$ is the interference function given by

$$I(p) = \frac{\sin N\pi p}{\sin \pi p}. \tag{9a}$$

It is simply the sum of the geometric series

$$\sum_n \exp\left[-i2\pi p(k_n - k)\right] \quad \text{over all} \quad n.$$

If N is infinite $I(p)$ becomes the comb function $\sum_{n=-\infty}^{\infty} \delta(p - n)$ where δ is the Dirac delta function. The function $I(p)$ is non-zero only when p is an integer. The amplitude distributions U_N consist of discrete lines whose magnitudes are proportional to $J_n(\phi_o z_o)$, and which occur at the spatial frequencies $\nu = \nu_c + n\nu_o$. Indeed, as would be expected, the intensity distributions $U_N U_N^*$ (where U^* is the complex conjugate of U) for the objects defined by z_1 and z_2 are identical if N is infinite, while the amplitude distributions differ only in phase.

When N is finite the diffraction spectrum is no longer discrete. Finite light intensities appear in regions which are centered on the integral values of p and whose widths are of the order $2/N$. If N is small (say of the order of three or four, a case which is of interest in color recording on thermoplastics) the diffraction spectra for all p become important.

Let us consider then the non-dimensional power spectrum of a single groove, namely

$$\mathfrak{I}(p, \phi_0 z_0) = UU^*\nu_0{}^2/KK^*. \tag{10}$$

In Fig. 4 $\mathfrak{I}_s$ (for the groove defined by z_1) is plotted for $\phi_o z_o = 1, 2, 3, 4, 5$ and in Fig. 5, $\mathfrak{I}_c$ (for the groove defined by z_2) is plotted for the same values of $\phi_o z_o$.

As would be expected $\mathfrak{I}_s(n, \phi_o z_o) = \mathfrak{I}_c(n, \phi_o z_o)$ for any integer n, i.e., at the positions where the integral orders of a grating would appear. Furthermore, the larger $\phi_0 z_0$ the more energy is thrown into the higher orders. This, too, should have been anticipated since the higher the index of the object and/or the deeper the groove (and thus the greater the surface slope) the greater the angle of diffraction. Furthermore, since p is inversely proportional to ν_o, the greater the density of recording the greater the angle of diffraction. Thus, at high densities we are limited to low values of $\phi_o z_o$ if we wish to make use of the major portion of the diffracted energy. For instance, if $\nu_o = 10^4$ lines/

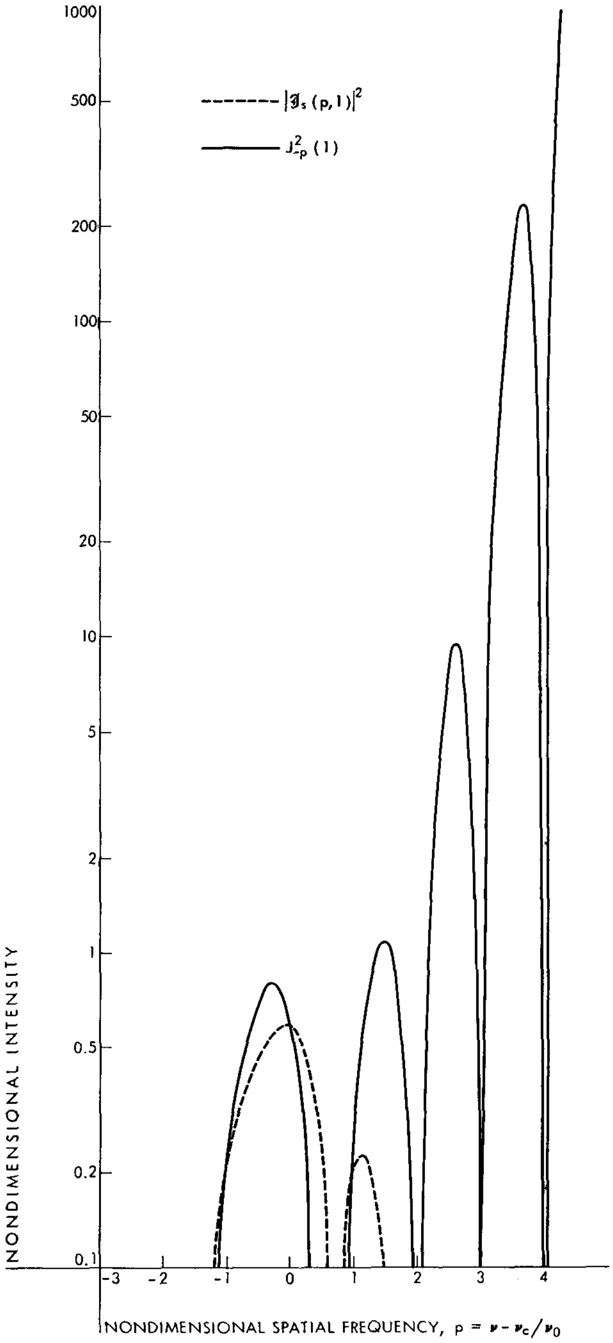


Figure 6 **Comparison of** $|\mathfrak{I}_s(p,1)|^2$ **and** $J^2_{-p}(1)$. *Note that* $|\mathfrak{I}_s(n,1)|^2 = J^2_{-n}(1)$ *for* $n = 0, \pm 1, \cdots$

inch, and if we are operating with visible light and are limited to $NA = 0.5$ then the system is inefficient (i.e., an undue amount of energy is diffracted out of the system) if $\phi_o z_o > 2.5$.

In Fig. 6 $\mathfrak{I}_s(p, \phi_o z_o)$ and $J^2_{-p}(\phi_o z_o)$ are plotted for $\phi_o z_o = 1$. It is obvious from Fig. 6 that although J^2_{-p} and $\mathfrak{I}_s$ are identical at the integral orders (at integral values of p) they are radically different elsewhere. Clearly, unless N is very large, the exact expressions for U must be used; the simple approximation $U(p) \approx J_{-p}$ is not valid.

Summary

The exact expressions for the diffraction pattern produced by a single sinusoidal groove and for finite gratings have been derived and compared with the expressions commonly used. We have shown that the usual approximations are valid only in the limit of gratings with an infinite number of grooves. Attention is given to the case of color thermoplastic recording, where the number of grooves is expected to be few.

References

1. F. Fischer and H. Thiemann, "Theoretical Considerations on a New Method of Large-Screen TV Projection," *Schweiz. Arch. angew. Wiss. Tech.* 7, Nos. 1, 2, 11, 12 (1941); **8**, Nos. 1, 5, 6, 7, 10 (1942).
2. R. V. Pole, "Spatial Phase Modulation and Remodulation" appearing in E. L. O'Neill, *Communications and Information Theory Aspects of Modern Optics*, General Electric Co., Electronics Laboratory, Syracuse, New York, August, 1962.
3. M. Born and E. Wolf, *Principles of Optics*, Pergamon Press, New York, 1959.
4. R. P. Madden and J. Strong, "Diffraction Gratings", Appendix P of J. Strong, *Concepts of Classical Optics*, W. H. Freeman and Co., San Francisco, 1958.
5. R. V. Pole, private communication.
6. E. Jahnke and F. Emde, *Tables of Functions*, Dover Publications, New York, 1945, p. 148.
7. E. S. Barrekette and R. L. Christensen, unpublished report.

Received November 16, 1962