H. J. Kump
T. G. Greene

# Magnetization of Uniaxial Cylindrical Thin Films

**Abstract: An analysis is given of the magnetization of a cylindrical thin film exhibiting a uniaxial anisotropy in the circumferential direction. The magnetization and demagnetizing fields are derived for the cylinder where the magnetization is not uniform. The derivation is accomplished by dividing up the cylinder into a large number of uniformly magnetized, coaxial cylindrical regions in superposition and by integrating their individual contributions. Some applications of the derivation are shown for several specified field geometries. The technique of superposition may be applied to other film geometries.**

## Introduction

In order to properly understand the performance of a magnetic device it is often necessary to be able to describe in detail the magnetization and demagnetizing fields in the magnetic medium. We certainly cannot hope to describe all fields and variations in the magnetization; however, certain fields important to the operation of the device may be derived. One of these is the field resulting from the *shape anisotropy*, that is, the demagnetizing field.

We will consider here a uniaxial[1] cylindrical thin film, i.e., a cylindrical sheet in which a uniaxial anisotropy exists in the circumferential direction. The "hard" direction of magnetization then will lie parallel to the cylinder axis. The film, in general, is switched so that the entire film is aligned in one direction. Actually, this is true only from the macroscopic viewpoint since the microscopic magnetization will vary appreciably with variations in the film structure.[2] In that this is in a circumferential direction, *no poles will exist* (other than stray poles*), since the magnetization closes on itself. To reverse the magnetization of the film, the magnetization is driven into the hard direction, thereby forming poles at the ends of the cylinder. These poles result in a demagnetizing field which, in turn, reduces the magnetization in that direction. It is these demagnetizing fields in which we are primarily interested and which we shall derive.

While the accurate solution of demagnetizing fields of ellipsoidal geometries may be found in the literature, the accurate solutions of other geometries are virtually unknown. It is often customary in the solution of nonuniform magnetization problems, involving shape anisotropy, to assume the magnetic medium to be uniformly magnetized and to argue that the result in the central region will be reasonably accurate. The edges are expected to be very inaccurate. Some authors[3] reduce the edge inaccuracies by treating them separately. However, where the length-to-thickness ratio of the magnetized material is small, serious errors may result in the central region as well.

Since we desire an accurate derivation of the magnetization and the demagnetizing fields everywhere, a solution must be found in which the accuracy is independent of the position. In our method of solution, the magnetized cylinder is replaced by a large number of uniformly magnetized coaxial cylindrical sheets of different lengths in superposition. The contributions from all the cylindrical sheets are integrated to give the demagnetizing fields, and consequently the magnetization, everywhere over the magnetized cylinder. The accuracy of the solution is seen to be dependent only on the number of coaxial cylinders chosen and the rate of change of the magnetization through the point, and is independent of the position of the point on the cylinder in reference to an edge.

As a demonstration of the usefulness of the derivation several practical cylindrical geometries and driving fields are treated in the section entitled "Applications."

## Theory

We shall assume, initially, that the cylinder is uniformly magnetized in the hard direction and the resulting poles concentrated at the edges of the thin films at either end of the cylinder (Fig. 1). The elemental pole may be given by:

$$dm = \lambda(R)RdRd\theta, \tag{1}$$

*We ignore any contribution from the exchange coupling, which may be shown to be negligible for practical memory applications.

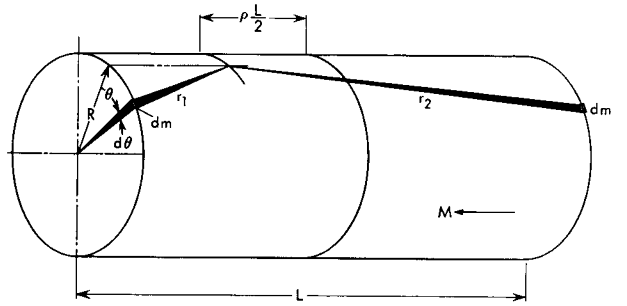


*Figure 1* **Uniformly magnetized cylinder.**

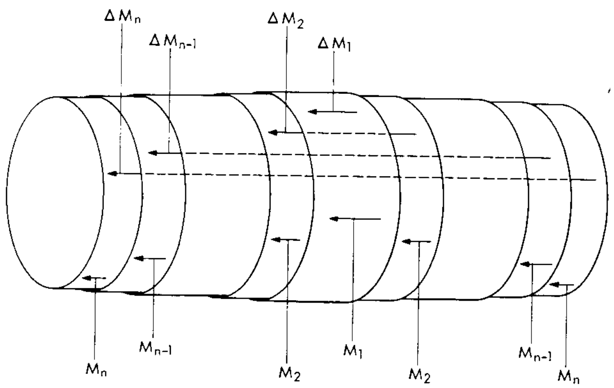


*Figure 2* **Concept of superposed cylinders adopted for the analysis.**

where $\lambda(R)$ is the magnetic charge surface density and $R$ and $\theta$ are as shown in Fig. 1.

Assuming the film to be infinitesimally thin, the field associated with the elemental pole at some distance $r$ from the pole is

$$d\mathbf{H} = \frac{m_0}{2\pi r^2}\, d\theta\, \mathbf{r}_0 \,, \tag{2}$$

where $m_0$ is the total pole strength of the uniformly magnetized cylinder and $\mathbf{r}_0$ is the unit vector in the $r$ direction. Taking only the component of the field parallel to the cylinder axis and adding the contributions over $2\pi$ for both ends of the cylinder, we have

$$H_\delta(\rho) = \frac{m_0 L}{4\pi} \int_0^{2\pi} \left( \frac{1-\rho}{r_1^3} + \frac{1+\rho}{r_2^3} \right) d\theta, \tag{3}$$

where

$$r_1^2 = (1-\rho)^2(L/2)^2 + 2R^2(1 - \cos\theta)$$

$$r_2^2 = (1+\rho)^2(L/2)^2 + 2R^2(1 - \cos\theta)$$

and $\rho$ is the decimal part of the half-length of the cylinder, $L/2$. The value $\rho$ denotes the position of the cylinder at which $H_\delta$ is determined. The pole strength, $m_0$, at the ends of the cylinder may be expressed as

$$m_0 = 2\pi t R \Delta M, \tag{4}$$

where $t$ is the film thickness and $\Delta M$ is (for the single cylinder) the difference in magnetization between the magnetic cylinder and air. Combining (4) with (3) and carrying out the integration over $\theta$, we have

$$H_\delta = \frac{t\Delta M}{R} \left[ \frac{k_1^2 E(k_1)}{(1-k_1^2)^{1/2}} + \frac{k_2^2 E(k_2)}{(1-k_2^2)^{1/2}} \right],$$

where

$$k_1^2 = \frac{4R^2}{\left[\frac{L}{2}(1-\rho)\right]^2 + 4R^2}$$

and

$$k_2^2 = \frac{4R^2}{\left[\frac{L}{2}(1+\rho)\right]^2 + 4R^2}, \tag{5}$$

and $E(k)$ is the complete elliptic integral of the first kind. (The foregoing result was obtained in a different manner by T. H. O'Dell.[3])

Now if we replace the single cylindrical shell of Fig. 1 by $N$ coaxial cylindrical shells in superposition (Fig. 2), each of a different length $(n/N)/L$ and with different uniform magnetization, we may add their contributions vectorially to obtain the total demagnetizing field:

$$H_D(\rho) = \sum_{n=1}^{N} H_\delta(n), \tag{6}$$

where

$$H_\delta(n) = \frac{t\Delta M_n}{R} \left[ \pm \frac{k_1^2 E(k_1)}{(1-k_1^2)^{1/2}} + \frac{k_2^2 E(k_2)}{(1-k_2^2)^{1/2}} \right]$$

(the choice of sign in $H_\delta(n)$ is that of the term $(n/N - \rho)$ in $k_1$), where

$$k_1^2 = \frac{4R^2}{\left[\frac{L}{2}\left(\frac{n}{N} - \rho\right)\right]^2 + 4R^2},$$

$$k_2^2 = \frac{4R^2}{\left[\frac{L}{2}\left(\frac{n}{N} + \rho\right)\right]^2 + 4R^2}$$

and

$$\Delta M_n = M_n - M_{n+1}.$$

The value for $M_n$ may be found from basic magnetics, since $4\pi M \approx \mu H$. The permeability, $\mu$, for a uniaxial film to saturation, in the hard direction, is approximately $M_s/H_k$ and unity beyond saturation.[1] The magnetizing field, $H$, is given as the sum of the applied field, $H_a$, and the demagnetizing field, $H_D$. We have

$$M_n = \begin{cases} \dfrac{M_s}{H_K}\,[H_a(\rho) + H_D(\rho)] & \text{for } H_a + H_D \le H_K \\ M_s & \text{for } H_a + H_D > H_K \end{cases} \tag{7}$$

We may, if we wish, express (6) succinctly in the form of an integral:

$$H_D(\rho) = \int_0^1 K(\rho, x)\,\frac{\partial}{\partial x}\,[M(x)]\,dx \tag{8}$$

where

$$K(\rho, x) = \frac{t}{R}\left[\pm\frac{k_1^2 E(k_1)}{(1-k_1^2)^{1/2}} + \frac{k_2^2 E(k_2)}{(1-k_2^2)^{1/2}}\right]$$

and

$$x \equiv \frac{n}{N},$$

where $x$ is now the dummy variable for $\rho$ and represents the pole position. It is immediately seen from (6) and (8) that as the pole position, $x$, nears the observer position, $\rho$, a singularity exists in the expressions. However, it is evident (in the true physical picture) that such a singularity does not occur, since the sum of the components of the demagnetization factor cannot exceed $4\pi$.

The singularity is removed by invoking the Cauchy Principal Value Theorem:

$$H_D(\rho) = P\int_0^1 K(\rho, x)\,\frac{\partial M}{\partial x}\,dx,$$

where $P$ is the Cauchy principal value. If we consider the region within some arbitrary radius of the observer, the principal value of the contribution from that region is seen, from Eq. (4), to depend on the second derivative change of the magnetization through the region, $P \propto \partial^2 Mx/\partial x^2$ since a contribution to $H_D$, from the region, can occur only if the pole strength on one side of the singularity differs from that on the other side. The second derivative, in general, will be quite small and the contribution to the demagnetization may be ignored.

The singularity is avoided in our solution by adopting an observer position between the concentric ring charges and ignoring any contribution to the field that may have resulted by the material in that region.

## Applications

Equation (6) was applied to typical problems involving uniaxial cylindrical shells. Only the results of the computer solution to (6) will be discussed here. A discussion concerning the method of computation is found in the Appendix.

The driving fields applied to the cylinders are either uniform or are those arising from a current through one or more line-current drives in the form of wire loops. The equation for uniform drive is adjusted so as to saturate only the center of the cylinder:

$$H_a = H_D + H_K. \tag{9}$$

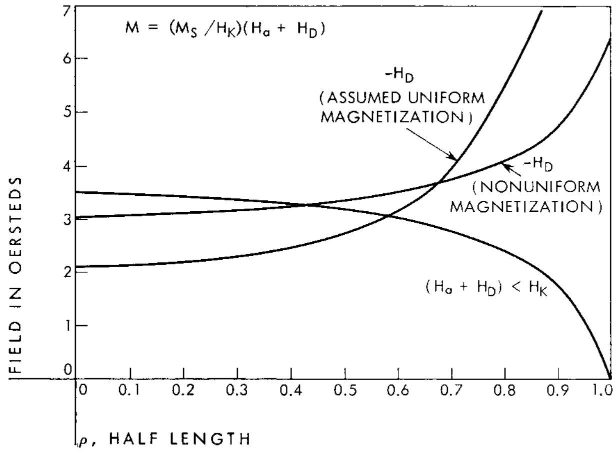


*Figure 3* **Uniform drive field.** *A cylindrical uniaxial thin film is driven by a uniform drive field so as to saturate only the central region of the cylinder (Eq. 9). The demagnetizing fields are shown as calculated by assuming the film to be uniformly magnetized and by the method of superposed cylindrical sheets. The resulting magnetization,* $H_a + H_D$*, is also shown. The parameters are* $L = 50$ *mils,* $D = 20$ *mils,* $t = 0.02$ *mils,* $M_s = 640$ *gauss/*$4\pi$, $H_k = 3.5$ *oe,* $H_a = H_D$ $(\rho = 0) + H_k$.

$H_D$ is at the point $\rho = 0$.

The equation for the wire loop drive is[4]

$$H_a = \frac{I}{2\pi}\,\frac{1}{\left[(a+R)^2 + \left(\rho\,\frac{L}{2}\right)^2\right]^{1/2}} \cdot\left[K + \frac{a^2 - R^2 - \left(\rho\,\frac{L}{2}\right)^2}{(a-R)^2 + \left(\rho\,\frac{L}{2}\right)^2}\,E\right], \tag{10}$$

where $K$ and $E$ are elliptic integrals of the first and second kinds, respectively, and

$$k^2 = \frac{4aR}{(a+R)^2 + \left(\rho\,\frac{L}{2}\right)^2}.$$

Also $I$ is the driving current and $a$ the distance from the surface of the cylinder to the wire loop.

A comparison is made in Fig. 3 between the demagnetizing fields calculated from (6) and the equation for a uniformly magnetized model, in order to show the large error that may exist over the entire cylinder due to the simple model. The calculated demagnetizing field from (6) is found to approach an asymptotic value rapidly with 50 superposed cylinders and is given here for a model composed of 70 superposed cylinders. It is, of course, not surprising that a very large error is found at the ends of

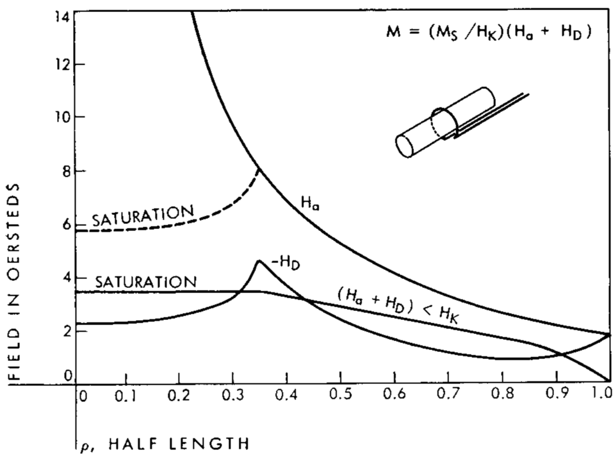


*Figure 4* **One-wire loop drive.** *A cylindrical uniaxial thin film is driven by a single wire loop located mid way between the ends and the demagnetizing field and magnetization calculated by the method of superposed cylindrical sheets. The parameters are $L = 100$ mils, $D = 50$ mils, $t = 0.024$ mils, $a = 5$ mils, $M_s = 800$ gauss/$4\pi$, $H_k = 3.5$ oe, $I = 1.2$ amp and $b = 0$.*

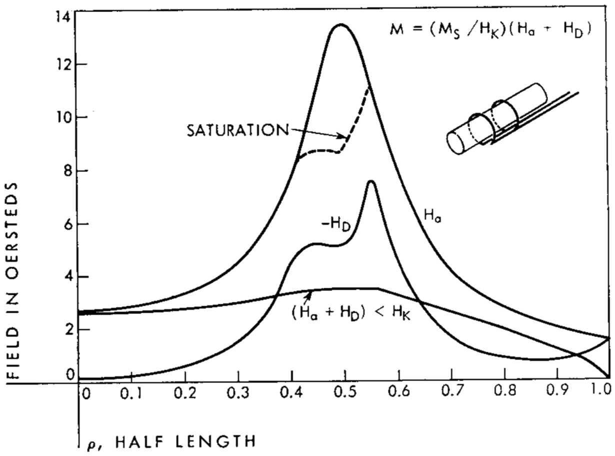


*Figure 5* **Two-wire loop drive.** *A cylindrical uniaxial thin film is driven by two wire loops located midway on either side of center and the demagnetizing field and magnetization calculated by the method of superposed cylindrical sheets. Parameters are the same as for Fig. 4, except $I = 0.3$ amp and $b = \pm 0.25$ L inch.*

the cylinder. The 60% error found at the center of the cylinder may, however, be unexpected, and is due to the distribution of the poles toward the center from both ends. That the poles are distributed is easily seen from the magnetization curve $(H_a + H_D)$, calculated from the demagnetizing field derived from (6). That the magnetization is zero at the very ends of the cylinder may not be factual and represents only the change in magnetization between the 70th cylinder and the region of nonmagnetic material beyond.

It is apparent from (7) and (8) that the shape of the demagnetizing field is as much a function of the geometry of the applied field as of the geometry of the magnetic cylinder. For that matter, if the cylinder is very long, the shape of the demagnetizing field is a function of the applied field geometry only. This is demonstrated in Figs. 4 through 6, where the applied field is derived from one or more wire loops expressed by Eq. (10). A line-current drive in the form of a single-wire loop positioned at the center of the cylinder is shown in Fig. 4. The driving field at the ends of the cylinder is not sufficient to cause saturation. As a consequence, the peak of the demagnetizing field occurs towards the center, where the magnetization deviates from saturation. In Fig. 3, where the driving field was uniform, the peak occurred at the ends of the cylinder, even though the cylinder was saturated only at the center. The dotted curve in the Figures indicates the applied field needed to saturate, in that region, for the magnetization configuration shown. All energy used in creating the fields beyond this point is, of course, of no avail.

A two-wire drive is shown in Fig. 5, where the wires are placed at the midpoint on either side of the center. The driving current is intentionally low in order to demonstrate the usefulness of Eq. (6). The "unpredictable" nature of demagnetizing fields is now readily seen, while the configuration of the resulting magnetization curve is as expected.

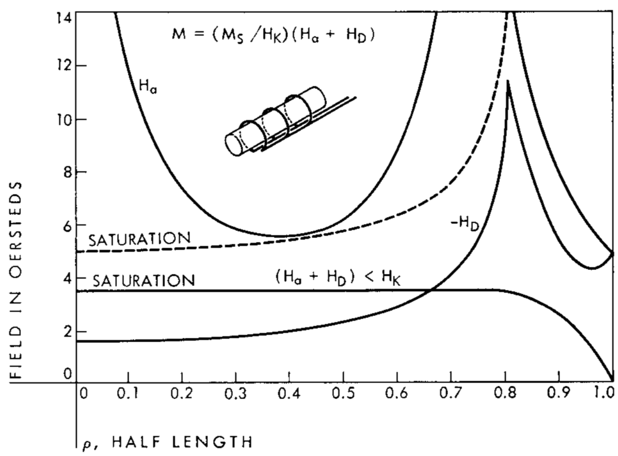


*Figure 6* **Three-wire loop drive.** *A cylindrical uniaxial thin film is driven by a single wire loop at the center and two wire loops midway on either side (all with equal currents) and the demagnetizing field and magnetization calculated by the method of superposed cylindical sheets. Parameters are the same as for Fig. 4 except $I = 0.4$ amp and $b = \pm 0.75$ L, 0.*

Finally, a three-wire loop drive is shown in Fig. 6, where the magnetization now approaches that of uniform magnetization, and the demagnetizing fields approach that of the simple model of Eq. (5).

## Summary

The demagnetization of a nonuniformly magnetized uniaxial cylindrical sheet can be found everywhere on the cylinder by the use of Eq. (6). It is of interest to note that the accuracy of the solution is not dependent on the distance from the ends of the cylinder, but rather on the spacing between the superposed cylinders of the model and the slope of the magnetization. It was also shown that the configurations of the demagnetization over the length of the cylinder was as strongly influenced by the configuration of the applied field as the geometry of the cylinder (exclusive of the thickness). The foregoing method may be applied to other film geometries.

## Appendix. Solution of the nonlinear integral equation

The integral equation for the demagnetization (Eq. 8) is nonlinear in form, and may be complicated still further where the applied field, $H_a$, is allowed to vary with the demagnetization, $H_D$. This may be expressed in general form as

$$y(\rho) = \int_a^b K(\rho, x)\left[\frac{\partial F}{\partial x} + \frac{\partial y}{\partial x}\right] dx. \tag{11}$$

A solution to the nonlinear integral equation can be found by standard numerical methods. The "direct iterative" approach (where the solution of $y(\rho)$ is successively fed back into the integral) converges, but onto a loop where the spread in $y(\rho)$ is much too large to be considered a solution. Rather than reduce the loop of convergence, it was deemed advisable to try other methods of solution.

A solution by "matrix methods" is not possible where the set of equations is homogeneous, because the determinant of the set of equations does not vanish. A solution to the nonlinear integral equation can be found through the use of "direct search".[5] In direct search, a point (in this case the slope at the point) is varied and a comparison is made between the input curve and the generated curve. If the change in the point is such so as to reduce the area between the curves, the change is stored and the succeeding points are tested. Finally, all the changes are inserted into the function. A pattern has now been established and is repeated until the area between the input curve and generated curve is no longer reduced. A new pattern is now found and the procedure is repeated. In order to apply the method of direct search, the integral (11) is put into a set of simultaneous equations:

$$y(\rho) = \sum_{x=(a+\Delta x)}^{b} K\left(\rho, x - \frac{\Delta x}{2}\right) \times \left[\frac{Z(x) - Z(x - \Delta x)}{\Delta x} + \frac{y(x) - y(x - \Delta x)}{\Delta x}\right]\Delta x. \tag{12}$$

This set of simultaneous equations may be reduced to the equivalent set:

$$y(\rho) \approx y^*(\rho) = R(\rho) + \sum_{x=(a+\Delta x)}^{b} K\left(\rho, x - \frac{\Delta x}{2}\right) \times [Z(x) + y(x) - Z(x - \Delta x) - y(x - \Delta x)], \tag{13}$$

where $R(\rho)$ is the residue at any point. In each of the simultaneous equations, the kernel, $K$, is evaluated at the center of each segment (avoiding the singularities), whereas the boundary conditions of each segment determines the slope of the function at the center of each segment. In that the function $Z + y$ represents the magnetization of the cylinder, it must be bounded by $H_k$; that is, the magnetic material is permitted to saturate.

A solution to the set of equations is obtained when $y(\rho) = y^*(\rho)$ that is, when the residues $R(\rho)$ vanish. The residues are, therefore, used to determine the degree of fit between the assumed curve and the generated curve. The residues are summed as their squares, and the sum is reduced to zero:

$$S = \sum_{i=1}^{N} R_i^2$$

in the manner already described.

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