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# Synthesis of Transfer Admittance Functions Using Active Components

**Abstract: A formal synthesis procedure is developed for active networks. The transfer admittance function is realized in parallel *RC* subnetworks, one of which contains a current-reversal negative impedance converter. This procedure offers several advantages over existing methods.**

## Introduction

A procedure for synthesizing a network, terminated in a one-ohm impedance, from any real, stable, rational transfer admittance function is formulated in this paper. The network is realized in terms of the $y$ parameters of two subnetworks and consists of one current-reversal negative impedance converter (NIC) and two ladder-type subnetworks.

This procedure offers several distinct advantages: 1) at most two ladder networks and an NIC are required; 2) the decomposition process allows identification of the $y$ parameters of the two ladder subnetworks in a relatively simple manner; and 3) the designer may select the form of the network structure to improve reliability, reduce sensitivity, and/or reduce component count.

The influence of the NIC can easily be extended to the source or load impedance connected to the network. This paper also suggests a method for compensating unwanted shunt admittance in the load as seen by the network.

Of the many synthesis procedures for active networks (containing NIC's) described in the literature, that developed by T. Yanagisawa[1] for voltage transfer functions stands out by virtue of its simplicity; it uses two paralleled subnetworks, one containing a current reversal NIC. However, the voltage transfer functions must always be realized by inverse $L$ subnetworks in conjunction with the current reversal NIC. The procedure described herein does not have this restriction. Further, it can be shown that Yanagisawa's method may be regarded as a special case of the procedure developed here.

## Partial fraction expansion theorem

The following theorem is of paramount importance in the development of this procedure.

• *Theorem*

Given a polynomial $P(s)$ with real coefficients and any polynomial $D(s)$ with real, negative, simple roots only and with the degree of $D(s)$ equal to or greater than one less than the degree of $P(s)$, the ratio $P(s)/D(s)$ may be expanded in admittance partial fractions such that

$$P(s)/D(s) = [P_1(s)/D_1(s)]-[P_2(s)/D_2(s)], \tag{1}$$

where both $P_1(s)/D_1(s)$ and $P_2(s)/D_2(s)$ are $RC$ driving-point admittance functions.

• *Proof*

Expand $P(s)/D(s)$ in admittance partial fractions. Since $P(s)$ has real coefficients and the roots of $D(s)$ are negative and real, each residue in the poles of $P(s)/D(s)$ will be real and positive or real and negative. Thus in admittance partial fraction form:

$$\frac{P(s)}{D(s)} = k^{(\infty)}\, s+k^{(0)}+\sum_{i=1}^{L}\frac{k_i s}{s+\sigma_i} - \left[k'^{(\infty)}\, s+k'^{(0)}+\sum_{i=L+1}^{N}\frac{k'_i s}{s+\sigma_i}\right]. \tag{2}$$

Equation (2) is recognized as the difference between two $RC$ driving-point admittance functions. In general $k^{(\infty)}\, k'^{(\infty)} = 0$, i.e., only one factor in the product can be non-zero. $k^{(0)}$ is always positive, provided the proper convention is used.[2]

## Synthesis of $Y_{12}$ functions

The terminated transfer admittance, $Y_{12}(s)$, of Fig. 1a may be written in terms of the $y$ parameters as

$$E_2/E_1 = I_2/E_1 = -Y_{12}(s) = -y_{12}/(1+y_{22}). \tag{3}$$

Two subnetworks $A$ and $B$, connected in parallel, form the network of Fig. 1a. The $y_{12}$ and $y_{22}$ parameters of Fig. 1a are related to the $y$ parameters of subnetworks $A$ and $B$ by

$$y_{22} = y_{22a}+y_{22b} \tag{4}$$

and

$$y_{12} = y_{12a}+y_{12b},$$

*Figure 1* **a) Basic form of the network. b) Internal structure of the network in (a). c) Internal structure of the *B* subnetwork.**

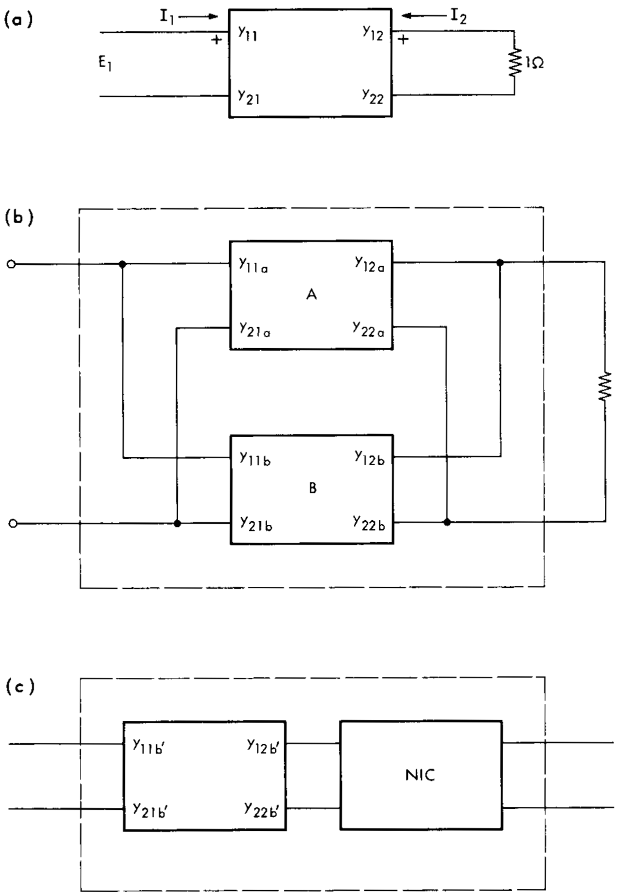
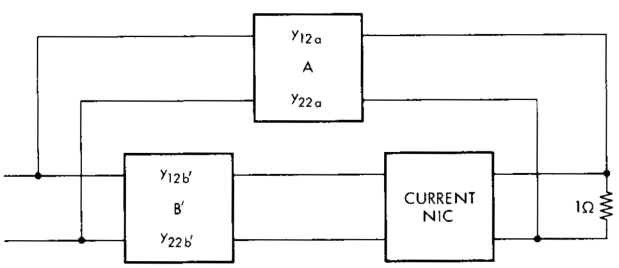



*Figure 2* **Complete network with *A* and *B'* subnetworks.**




where the lower case subscripts refer to the network parameters of the $A$, $B$, and $B'$ networks.

Subnetwork $B$ (Fig. 1c) is composed of two networks, a negative impedance converter (current reversal) and a network $B'$. The relationship between parameters of subnetwork $B$ and $B'$ are

$$y_{12b} = -y_{12b'}$$

$$y_{22b} = -y_{22b'}. \tag{5}$$

Under these conditions, the $Y_{12}(s)$ function may be written in terms of the $y$ parameters of networks $A$ and $B'$ as

$$-Y_{12}(s) = \frac{-y_{12a}+y_{12b'}}{1+y_{22a}-y_{22b'}}. \tag{6}$$

The complete network is shown in Fig. 2.

It will be useful to expand the $Y_{12}(s)$ function such that the numerator will appear as the difference between two $RC$ transfer admittances and the denominator will appear as the difference between two $RC$ driving point admittances as required by Eq. (6). Moreover, the synthesis procedure will be simplified considerably if the $RC$ transfer admittance parameters can bé identified with a particular network structure.

To guarantee that the above expansion and identification can always be performed, the following justification is offered. Given a real, stable, rational function in $s$,

$$\frac{s^r+A_rs^{r-1}+\cdots+A_{-1}}{s^t+b_ts^{t-1}+\cdots+b_{-1}} = \frac{P(s)}{Q(s)},$$

having no poles at infinity, choose a polynomial $D(s) = \prod_{i=1}^{N} (s+\sigma_i)$, such that all roots are real and negative and such that $P(\sigma_i)/Q(\sigma_i)>0$, for all $i$, and $N\geq t-1$. (That this step can always be taken is shown in the following paragraph.) Divide numerator and denominator of $P(s)/Q(s)$ by $D(s)$, obtaining

$$-Y_{12}(s) = \frac{P(s)/D(s)}{Q(s)/D(s)}. \tag{7}$$

Under the conditions stated above, if $Q(s)/D(s)$ and $P(s)$ are expanded in admittance partial fractions, it is readily seen that except for the pole at $s = \infty$, the residue in each particular finite pole of $P(s)/D(s)$ and $Q(s)/D(s)$ will be real and have the same sign. To clarify this point, note that the ratio of the two residues is $P(s)/Q(s)$, evaluated at the corresponding pole. The term $y_{12a}$ is identified as the sum of all terms in the admittance partial fraction expansion of $P(s)/D(s)$ having positive residues. The sum of the remaining terms is $-y_{12b}$. Thus, the residues of both $y_{12a}$ and $y_{12b}$ are positive. This in turn implies that the zeros of $y_{12a}$ and $y_{12b}$ will all be on the negative real axis. Networks $A$ and $B$ may now be realized without resorting

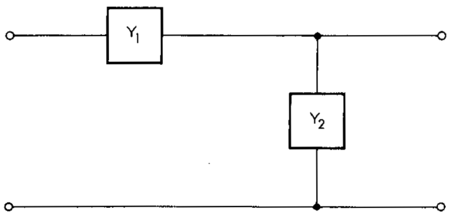



*Figure 3* **An *L* section.**
$-y_{12} = Y_1. \quad y_{22} = Y_1+Y_2 = (-y_{12})+Y_2.$

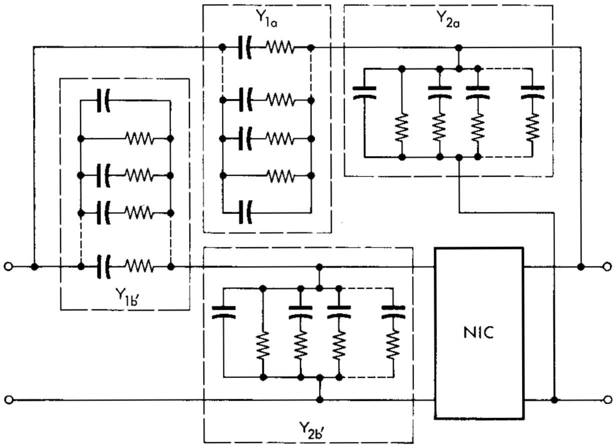



*Figure 4* **General primitive ladder realization.**

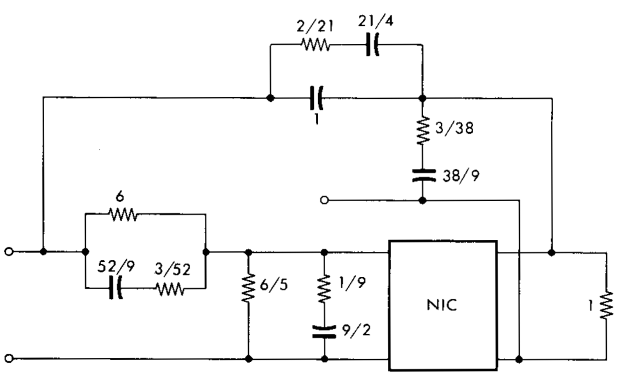



*Figure 5* **Primitive ladder realization.**

to complex zero producing sections. In fact, the subnetworks will be simple ladders. The result, after expansion and multiplication by $s$ will have the form

$$-Y_{12}(s) = \frac{k^{(0)}+k^{(\infty)}s+\sum_{i=1}^{L}\frac{k_i s}{s+\sigma_i} - \left[k'^{(\infty)}s+\sum_{i=L+1}^{N}\frac{k'_i s}{s+\sigma_i}\right]}{h^{(0)}+h^{(\infty)}s+\sum_{i=1}^{L}\frac{h_i s}{s+\sigma_i} - \left[h'^{(\infty)}s+\sum_{i=L+1}^{N}\frac{h'_i s}{s+\sigma_i}\right]}, \quad (8)$$

where $(k^{(\infty)})\,(k'^{(\infty)}) = (h^{(\infty)})\,(h'^{(\infty)}) = 0$.

By adding $(1-1)$ to the denominator of (8) and referring to (6), the following parameter identifications are obtained:

$$-y_{12a} = k^{(0)}+k^{(\infty)}s+\sum_{i=1}^{L}\frac{k_i s}{s+\sigma_i}$$

$$y_{22a} = h^{(0)}+h^{(\infty)}s+\sum_{i=1}^{L}\frac{h_i s}{s+\sigma_i}$$

$$-y_{12b'} = k'^{(\infty)}s+\sum_{i=L+1}^{N}\frac{k'_i s}{s+\sigma_i}$$

$$y_{22b'} = h'^{(\infty)}s+\sum_{i=L+1}^{N}\frac{h'_i s}{s+\sigma_i}+1.$$

From (9) it can be seen that $y_{12a}$, $y_{12b'}$ and $y_{22b'}$ have the form of *RC* driving point admittances.

In the case of non-minimum phase functions with an odd number of right-plane zeros, $y_{12b'}$ or $y_{12a}$ may contain a term $k^{(\infty)}s$ while $y_{22b'}$ or $y_{22a}$ does not. To satisfy the residue condition for this case add to $y_{22a}-y_{22b'}$ a term of the form, $k^*{}_1s-k^*{}_1s$, such that the residue condition is satisfied at the infinite pole. The addition of such a term to the $y_{22a}-y_{22b'}$ function will not adversely affect the synthesis of either network. This is true because the residue conditions will not be violated for the pole at infinity, even if $k^{(\infty)}{}_{12} = 0$ or $h^{(\infty)}{}_{12} = 0$ since $k^{(\infty)}{}_{11} \times k^{(\infty)}{}_{22} - 0 > 0$ and $h^{(\infty)}{}_{11}\, h^{(\infty)}{}_{22} - 0 > 0$.

### General realization of $Y_{12}(s)$ in a primitive ladder

To demonstrate that it is always possible to synthesize $Y_{12}(s)$ by means of a pair of ladder networks, consider the parameter identifications of (9), and assume that the appropriate terms have been added to $y_{22a}$ and $y_{22b}$ such that the residue condition is satisfied at every pole.

The simplest ladder network is a single *L* section. A general method for synthesizing the two networks, whose parameters are given by (9) with two *L* sections is now shown. This realization does not result in a minimum number of components, but it does demonstrate that subnetworks of (9) may always be realized as a pair of primitive ladders. The equations of an *L* section are given in Fig. 3.

Realization of $y_{12a}$ and $y_{22a}$ in an *L* section requires that each residue of $y_{22a}$ be no smaller than the corresponding residue of $y_{12a}$. If this condition is not satisfied, add a term of the form $k^*{}_1s/(s+\sigma_i)-k^*{}_1s/(s+\sigma_i)$ to $y_{22a}-y_{22b'}$ such that the residue in every pole of $y_{22a}$ is equal to or greater than the residue in the corresponding pole in $y_{12a}$. This introduces new poles in $y_{22b'}$. A similar operation is performed with $y_{12b'}$ and $y_{22b'}$. The $-Y_{12}(s)$ function takes the form

$$-\frac{y_{12a}-y_{12b'}}{1+y^*{}_{22a}-y_{22b'}{}^*}.$$

New parameter identifications are made by separating the denominator function into the appropriate form in the following way:

$$\begin{aligned} y_{12a} &= Y_{1a} & y_{12b'} &= Y_{1b'} \\ Y_{2a} &= y^*_{22a} - Y_{1a} & Y_{2b'} &= y^*_{22b'} - Y_{1b'} \end{aligned} \quad . \tag{10}$$

The final network is shown in Fig. 4.

It has been demonstrated that any real, stable, rational voltage transfer function in $s$ may be realized by two primitive ladder networks and one NIC of the current inversion type. However, the primitive ladder often results in more elements than necessary.

In general, it is more convenient and economical to realize the $y_{12a}$, $y_{22a}$, and $y_{12b'}$, $y_{22b'}$ functions by means of general ladder networks, and to use a zero shifting technique to obtain the real zeros of the $y_{12}$ parameters which are not coincident with the zeros of the $y_{22}$ parameters.

## Selection of $(-\sigma_i)$

If the roots of $D(s)$ are selected from the segments of the $-\sigma$ axis for which the function $P(s)/Q(s)$ is positive, then the sign of the residues will, in each finite pole, be the same for corresponding poles of $P(s)/sD(s)$ and $Q(s)/sD(s)$.

By proper use of the reference convention[2] for the voltage, roots for $D(s)$ can always be chosen from convenient segments of the $-\sigma$ axis such that $P(s)/Q(s)$ is positive for each chosen root. This not only guarantees that the desired expansion can be obtained but also permits some control over element values.

The location of the roots of $D(s) = \prod_{i=1}^{N} (s+\sigma_i)$ also influences pole-zero sensitivity[3] as well as the spread of element values. These three constraints on the location of the roots of $D(s)$ have a rather significant effect when dealing with practical networks.

## Example

In order to clarify the above procedure one example will be considered.

Given: $-Y_{12}(s) = \dfrac{(s-1)(s^2-s+1)}{(s+1)(s^2+s+1)}$ (all pass function).

Choose: $D(s) = (s+2)(s+3)$, since $-Y_{12}(s) > 0$ for $s < -1$.

Then:

$$-Y_{12}(s) = \frac{\dfrac{(s-1)(s^2-s+1)}{(s+2)(s+3)}}{\dfrac{(s+1)(s^2+s+1)}{(s+2)(s+3)}} = \frac{s+\dfrac{\frac{21}{2}s}{s+2}-\dfrac{1}{6}-\dfrac{\frac{52}{3}s}{s+3}}{\dfrac{1}{6}+s+\dfrac{\frac{3}{2}s}{s+2}-\dfrac{\frac{14}{3}s}{s+3}}.$$

To satisfy the residue condition in the $B$ network at $s = 0$ and to put the denominator in the proper form, add $(5/6-5/6)$ to the denominator of $Y_{12}(s)$:

$$-Y_{12}(s) = \frac{s+\dfrac{\frac{21}{2}s}{s+2}-\dfrac{1}{6}-\dfrac{\frac{52}{3}s}{s+3}}{1+s+\dfrac{\frac{3}{2}s}{s+2}-\dfrac{5}{6}-\dfrac{\frac{14}{3}s}{s+3}}.$$

After adding and subtracting the appropriate terms necessary to obtain a realization in canonical $L$ sections, the parameter identifications are:

$$-y_{12a} = s+\frac{\frac{21}{2}s}{s+2}, \qquad -y_{12b'} = \frac{1}{6}+\frac{\frac{52}{3}s}{s+3},$$

$$y_{22a} = s+\frac{\frac{3}{2}s}{s+2}+\frac{\frac{18}{2}s}{s+2}+\frac{\frac{38}{3}s}{s+3},$$

$$y_{22b'} = \frac{5}{6}+\frac{\frac{14}{3}s}{s+3}+\frac{\frac{18}{2}s}{s+2}+\frac{\frac{38}{3}s}{s+3}.$$

$Y_{1a}$, $Y_{2a}$, $Y_{1b'}$ and $Y_{2b'}$ are obtained as in (10) and Fig. 3. The final network is shown in Fig. 5.

## Load compensation

Under certain circumstances it will be useful to cancel the undesirable effect of some $RC$ load admittance, $Y_L$, associated with the network. For example, it might be required that the effective shunt capacitance measured at the input terminals to a vacuum tube circuit or a transmission line be removed; in this case the unwanted admittance can be removed by regarding it as part of the $A$ subnetwork and compensating its effect in the $B$ subnetwork.

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## References

1. T. Yanagisawa, "RC-Active Networks Using Current Inversion Type Negative Impedance Converters," *IRE Transactions on Circuit Theory*, **CT-4**, 140 (1957).
2. N. Balabanian and W. R. LePage, "What is a Minimum-Phase Network?" AIEE Basic Sciences Paper, Presented at AIEE Fall General Meeting, Chicago, Illinois, Oct. 3-7, 1955.
3. R. E. Thomas, "Active Network Synthesis by Lattice Reduction," Ph.D. Thesis, University of Illinois, 1959.

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