K. U. von Hagenow* and H. Koppe†

On the Influence of Free Path on the Meissner Effect

Abstract: The influence of impurities on the behaviour of a superconductor is computed by introducing a scattering potential and averaging over all positions of the scattering centers. This procedure only takes into account scattering and free path effects and neglects changes in the elastic constants and electron density due to actual impurities. Using perturbation theory, it is shown that the free energy and the static Meissner effect are not influenced by scattering.

1. Introduction

Mattis and Bardeen[1] have recently given a theory of the response of a superconductor to electromagnetic waves, where the free path due to scattering by randomly distributed impurities is introduced by a simple approximation, which roughly corresponds to the "optical model" in nuclear physics. This approximation consists in replacing expressions

$$\sum_{E=\text{const.}} \psi_k(r)\psi_k{}^*(r') \left(=\frac{\sin|k|R}{|k|R} \text{ for free electrons}\right) \qquad (1)$$

wherever they occur by

$$\rho_E(\mathbf{r}, \mathbf{r}') = \frac{\sin kR}{kR} \exp[-R/2l]\,, \qquad (2)$$

where $R = |\mathbf{r} - \mathbf{r}'|$ and l is the free path. Their result is

$$\mathbf{j}(\mathbf{r}, t) = \sum \frac{e^2 N(0) V_0}{2\pi^2 \hbar c} \int \frac{\mathbf{R}(\mathbf{R}\cdot\mathbf{A}_\omega)}{R^4} \times I(\omega, R, T) \exp[-R/l] d\mathbf{r}' \exp[-i\omega t]\,. \qquad (3)$$

For $\omega = 0$, the function $I(\omega, R, T)$ becomes identical with the function $I(R, T)$ introduced previously in the theory of the Meissner effect, which as a function of R behaves approximately like the Pippard kernel $\exp[-R/\xi_0]$. Since only for $l = \infty$ is (3) identical with the formula for the Meissner effect as given by Bardeen, Cooper and Schrieffer, (3) would indicate an influence of free path on the static Meissner effect. Since the latter is a property of thermal equilibrium, such an influence should not exist.

For this reason, we calculated the influence of a scattering potential on the Meissner effect by straightforward thermodynamical perturbation theory. This should be permissible on account of (3), at least as long as l is much larger than the coherence length ξ_0, since in this case j can be expanded in a series of descending powers of l, which is equivalent to a perturbation expansion with respect to the scattering potential.

2. Influence of free path on free energy

We introduce the scattering centers into the Hamiltonian as a term of the form

$$V = \frac{1}{\Omega} \sum_{\mathbf{r}_j} \sum_{\mathbf{k},\mathbf{k}'} v(\mathbf{k} - \mathbf{k}') \exp[i(\mathbf{k} - \mathbf{k}')\cdot\mathbf{r}_j] \times \{C^*(\mathbf{k}\uparrow)C(\mathbf{k}'\uparrow) + C^*(-\mathbf{k}'\downarrow)C(-\mathbf{k}\downarrow)\} \qquad (4)$$

and put

$$\mathscr{H} = \mathscr{H}_0 + V, \qquad (5)$$

where $\mathscr{H}_0$ is the reduced Hamiltonian of BCS. Since the terms in (4) with $\mathbf{k} = \mathbf{k}'$ are of the form $v(0)[n_{k\uparrow} + n_{k\downarrow}]$ they can be absorbed into the Fermi-energy ζ and omitted in V. We obtain then, up to second-order terms in V, $F = F_0 + u^{(2)}$ with

$$u^{(2)} = \frac{1}{2\Omega^2} \sum_{\substack{\mathbf{k},\mathbf{k}'\\ j,j'}} |v(\mathbf{k} - \mathbf{k}')|^2 \times \exp[i(\mathbf{k} - \mathbf{k}')\cdot(\mathbf{r}_j - \mathbf{r}_{j'})] L(\varepsilon, \varepsilon') \qquad (6)$$

* Max Planck Institute for Physics and Astrophysics, Munich.
† University of Munich.

$$L(\varepsilon, \varepsilon') = \frac{1}{\varepsilon - \varepsilon'}\left\{\frac{\varepsilon}{E} \tanh \frac{E}{2k_B T} - \frac{\varepsilon'}{E'} \tanh \frac{E'}{2k_B T}\right\}.$$

Since only the difference between the free energies of the superconducting and the normal state is of interest, we need only to calculate $u^{(2)}(\varepsilon_T) - u^{(2)}(0)$. After averaging over the r_j, and some minor simplifications (which are possible on account of the fact that contributions arise only from states in the immediate neighborhood of the Fermi surface), it can be shown that this difference vanishes under the condition that $|v(\mathbf{k} - \mathbf{k}')|^2$ is bounded. Therefore, scattering centers do not influence the free energy of a superconductor up to second order in the scattering potential v, whereas correction terms of the order $1/l$ should occur in second-order perturbation.

In view of the well-established fact that impurities do influence the transition temperature it is necessary to point out that the formal addition of a potential like (4) to the Hamiltonian is not quite the same thing as the actual addition of impurities to a superconductor. The potential (4) adequately introduces scattering and free-path effects. Beyond this impurities can act as electron donors or acceptors, thereby changing the electron density, and can influence the elastic properties, and thereby the electron-phonon interaction; and the transition temperature depends sensitively on both.

3. Influence of scattering on the static Meissner effect

The static Meissner effect is essentially obtained by the evaluation of expressions of the form

$$\begin{aligned} M_{\mathbf{q},\mathbf{q}'} = \int_0^\tau d\tau'\, Tr\{\mathbf{j}_\mathbf{q} \exp[-(\tau - \tau')(\mathscr{H}_0 + V)] \\ \times \mathbf{j}_{-\mathbf{q}'} \exp[-\tau'(\mathscr{H}_0 + V)]\} \\ \times [Tr\{\exp[-\tau(\mathscr{H}_0 + V)]\}]^{-1}, \end{aligned} \quad (7)$$

where $\tau = 1/kT$ and $\mathbf{j}_\mathbf{q}$ is the Fourier component of the current operator. If one develops the operators $\exp[-\tau(\mathscr{H}_0 + V)]$ in the usual way, one obtains, with $w_\tau = \exp[-\tau\mathscr{H}_0]$, expressions of the type

$$\begin{aligned} Tr\{(\mathbf{l}_1, \mathbf{l}'_1)_j w_{\tau_1} (\mathbf{k}_1, \mathbf{k}'_1)_V w_{\tau_2} \\ \cdots (\mathbf{l}_2, \mathbf{l}'_2)_j w_{\tau_r} (\mathbf{k}_r, \mathbf{k}'_r)_V \cdots \}. \end{aligned} \quad (8)$$

$(\mathbf{l}_1, \mathbf{l}'_1)_j$ indicates a term of the form $C^*(\mathbf{l}\uparrow)C(\mathbf{l}'\uparrow)$ or $C^*(-\mathbf{l}'\downarrow)C(-\mathbf{l}\downarrow)$ occurring in $\mathbf{j}_\mathbf{q}$, and $(\mathbf{k}_1, \mathbf{k}'_1)_\mathbf{q}$ an analogous term occurring in V. Traces of this kind vanish, unless the primed momenta are some permutation of the unprimed ones. Averaging over the positions of the scattering centers introduces the further condition, that the primed k's are already a permutation of the unprimed k's. Therefore, we must have $l_1 = l'_2$ and $l_2 = l'_1$ and only the $M_{\mathbf{q},\mathbf{q}}$ are different from zero. If the l are different from all the k, (8) can be factorized into a product of a trace containing only the $\mathbf{j}$ and a trace containing only the V. Since terms where the l are not all different from the k give only contributions of order $1/\Omega$ in subsequent integrations, we obtain

$$\begin{aligned} &\frac{Tr\{\mathbf{j}_\mathbf{q} \exp[-(\tau-\tau')(\mathscr{H}_0+V)]\mathbf{j}_{-\mathbf{q}} \exp[-\tau'(\mathscr{H}_0+V)]\}}{Tr \exp[-\tau\mathscr{H}_0]} \\ &= \frac{Tr\{\mathbf{j}_\mathbf{q} \exp[-(\tau-\tau')\mathscr{H}_0]\mathbf{j}_{-\mathbf{q}} \exp[-\tau'\mathscr{H}_0]\}}{Tr \exp[-\tau\mathscr{H}_0]} \\ &\quad \times \frac{Tr \exp[-\tau(\mathscr{H}_0+V)]}{Tr \exp[-\tau\mathscr{H}_0]} (1 + O(1/\Omega)). \end{aligned} \quad (9)$$

Inserting this result into (7) we obtain finally

$$\begin{aligned} M_{\mathbf{q},\mathbf{q}'} = \int_0^\tau \frac{Tr\{\mathbf{j}_\mathbf{q} \exp[-(\tau-\tau')\mathscr{H}_0] \times \mathbf{j}_{-\mathbf{q}} \exp[-\tau'\mathscr{H}_0]\}}{Tr \exp[-\tau\mathscr{H}_0]} d\tau' \\ \times (1 + O(1/\Omega)), \end{aligned} \quad (10)$$

which indicates that scattering has no influence on the static Meissner effect.

It is interesting to note that the derivation of (9) depends vitally on the fact that the momenta in the current operators do not mix with the momenta contained in the V's, which was a consequence of the random distribution of the scattering centers. It would be possible to calculate the influence of a single scatterer on the current distribution. In this case, the $M_{\mathbf{q},\mathbf{q}'}$ would not vanish for $\mathbf{q} \neq \mathbf{q}'$ and would give a contribution, since for them a factorization of traces would no longer be possible.

Reference

1. D. C. Mattis and J. Bardeen, *Phys. Rev.* **111,** 412 (1958).

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