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The Dynamics of a Subharmonic Oscillator with Linear Dissipation

Abstract: A mathematical analysis of the dynamic behavior of subharmonic oscillators (parametrons) is made assuming a nonlinear reactance but a linear dissipation or resistance. Simple equations of motion for the subharmonic and pump amplitudes are derived in the quasistatic, or high-Q, approximation. Numerical solutions are obtained for two cases. The first shows the subharmonic amplitude changing from a small value to its steady state value when a constant pump signal is applied. The second shows decay when the pump signal is removed.

Introduction

Several authors have given mathematical treatments of the operation of subharmonic oscillators.[1] The case discussed here is that of a device with a nonlinear reactance, such as a capacitor whose potential depends quadratically on its charge but which has a linear dissipation, that is, a constant resistance. Two types of devices are treated: those having two modes, one tuned to the pump frequency and one to the subharmonic frequency; and those single-mode devices tuned to the subharmonic frequency. In both cases the devices are assumed to be tuned to perfect resonance. For each case we first define the chosen set of parameters and variables, write the equations of motion, manipulate them into a simple form with dimensionless parameters, discuss the qualitative properties of their solutions, and exhibit some particular numerical solutions. The solutions we discuss are *turning-on solutions* showing the subharmonic amplitude changing from an initially very small value to its steady state value when a constant pump signal is applied, and *turning-off solutions* showing the decay of the subharmonic amplitude when the pump signal is removed. These are solutions which may be appropriate to the computer applications of these devices discussed by Goto.[2]

The two-mode device

We begin the discussion of the two-mode devices by defining the nonlinear capacitance parameter, T, by the equation

$$V=q/c+Tq^2\,, \tag{1}$$

where V is the potential and q the charge of the nonlinear capacitor. The equations of motion of the system will be simplest when written in terms of the normal mode variables of the system considered, for this purpose, to be without nonlinear or dissipative terms. We take these variables, q_1 and q_2, to have the units of charge and to correspond to linear modes with frequencies ω and 2ω respectively, but do not, here, make any assumption as to their actual time dependence in the presence of the nonlinear capacitor. The charge q of the nonlinear capacitor may then be written as a linear combination of these variables:

$$q(t)=A_1q_1(t)+A_2q_2(t)\,, \tag{2}$$

where the A's are pure numbers. The energy of the system including the nonlinear capacitor, but still without dissipation, may be written as

$$E=\text{constant}\times[\tfrac{1}{2}(\dot{q}_1{}^2+\omega^2q_1{}^2)+\tfrac{1}{2}(\dot{q}_2{}^2+4\omega^2q_2{}^2)+\tfrac{1}{3}T(A_1q_1+A_2q_2)^3]\,, \tag{3}$$

where the dot above a quantity indicates a time derivative. The first two terms are the usual expressions for the energy of a linear vibrating system in terms of the normal mode variables, q_1 and q_2. The last term is obtained from Eq. (1) by integration. Equation (3) may be used as a Hamiltonian of a mechanical system to yield the equations of motion. To these equations we must add terms to account for the small linear dissipation and the driving force:

$$\ddot{q}_1+\omega^2q_1+A_1(A_1q_1+A_2q_1)^2=-(\omega/Q_1)\dot{q}_1+e_1\sin 2\omega t$$
$$\ddot{q}_2+4\omega^2q_2+A_2(A_1q_1+A_2q_2)^2=-(2\omega/Q_2)\,\dot{q}_2+e_2\sin 2\omega t\,, \tag{4}$$

where the left-hand side of the equations are the energyconserving terms and the right-hand side are the dissipation and driving-force terms, respectively.

The parameters Q_1, Q_2 are the ordinary quality factors for the respective modes; the parameters e_1, e_2 are proportional to the level of the pump signal and the efficiency with which it is coupled to the two modes. For a circuit these terms describe the effect of resistance and of a voltage generator of frequency, ω/π.

We now transform from the real charge variables, $q_{1,2}$, to complex amplitude variables, $a_{1,2}$, by the relations:

$$
\begin{aligned}
q_1 &= a_1 \exp(i\omega t) + a_1^* \exp(-i\omega t) \\
q_2 &= a_2 \exp(2i\omega t) + a_2^* \exp(-2i\omega t) \, .
\end{aligned}
\tag{5}
$$

If we were interested only in the steady state of these devices, the amplitude variables, $a_{1,2}$, would be complex numbers independent of time and would be proportional to the ordinary complex currents of alternating current circuit theory. Here we will make the assumption that the fractional change in the amplitudes is small in a period; i.e., that

$$
\dot{a}_{1,2} \ll \omega a_{1,2} \, . \tag{6}
$$

In this case, since we assume $Q_{1,2} \gg 1$, only terms of Eq. (4) of the corresponding frequencies will contribute to $a_{1,2}$. This follows by the argument that although potentials of all multiples of ω are generated by the nonlinear capacitor, these potentials generate very small currents because the linear part of the circuit is not resonant for them. Thus although these currents do combine in the capacitor to give contributions to a_1 and a_2 these contributions are small if $Q_{1,2} \gg 1$. Because of this we may obtain equations of motion for the amplitude variables, $a_{1,2}$, by substituting equations (5) into the equations of motion (4), equating coefficients of $e^{i\omega t}$ and $e^{2i\omega t}$ respectively and then dropping terms which are relatively small by virtue of the inequality (6). The result is

$$
\begin{aligned}
\dot{a}_1 &= -(A_1{}^2 A_2 T / i\omega) a_1^* a_2 - (\omega/2Q_1) a_1 \\
\dot{a}_2 &= -(A_1{}^2 A_2 T / 4i\omega) a_1{}^2 - (\omega/Q_2) a_2 - (e_2/8\omega) \, .
\end{aligned}
\tag{7}
$$

It should be noted that the nonlinearity parameters enter only in the combination $A_1{}^2 A_2 T$. The parameter e_1 has disappeared because the pump signal is out of resonance with the subharmonic mode. These equations may be further simplified by a change of scale and phase of the amplitude variables:

$$
\begin{aligned}
b_1 &= (\sqrt{Q_1 Q_2}\, A_1{}^2 A_2 T / 2\sqrt{i}\, \omega^2) a_1 \\
b_2 &= (2 Q_1 A_1{}^2 A_2 T / \omega^2) a_2 \\
u &= (Q_1 Q_2 A_1{}^2 A_2 T / 4\omega^2) V_2
\end{aligned}
\tag{8}
$$

yielding
$$
\begin{aligned}
\dot{b}_1 &= (\omega/2Q_1)(b_1^* b_2 - b_1) \\
\dot{b}_2 &= (\omega/Q_2)(u - b_1{}^2 - b_2) \, .
\end{aligned}
\tag{9}
$$

The time-dependent behavior of these oscillators is thus determined by three parameters, one of which could be removed by scaling the time and another, namely u, which is proportional to the pump signal. The change of complex phase of the variables is significant in that it has yielded equations whose coefficients are real. This suggests that only the real parts of b_1 and b_2 need to be considered. A complete demonstration of this requires that the solutions with the imaginary parts of b_1 and b_2 equal to zero be stable against small disturbances or noise. This is easily shown to be the case by assuming the real parts of b_1 and b_2 in equations (9) are fixed and showing that the characteristic frequencies of the resulting linear differential equations for the imaginary parts of b_1 and b_2 have negative real parts. In the following it is therefore assumed that b_1 and b_2 are real quantities.

The steady state solutions of equations (9), which may be obtained by setting the time derivatives to zero and solving for b_1 and b_2, are the trivial one, $(b_1 = 0, b_2 = u)$ and the oscillatory steady state, $(b_1 = \pm\sqrt{u-1}, b_2 = 1)$. The latter set of values is a solution only for $u > 1$, and therefore $u = 1$ is the threshold level for the pump signal. The two possible signs of b_1 represent the two stable phases of the subharmonic which are used to represent a binary ONE or ZERO in the computer applications. The equations (9) can be transformed into a single second-order equation by first performing the substitution $x = \ln b_1$, which gives

$$
\begin{aligned}
\dot{x} &= (\omega/2Q_1)(b_2 - 1) \\
\dot{b}_2 &= (\omega/Q_2)[u - \exp(2x) - b_2] \, .
\end{aligned}
\tag{10}
$$

If the time derivative of the first equation above is taken, a total of three equations may be obtained, from which b_1 and $\dot{b}_1$ may be eliminated to give

$$
\begin{aligned}
&\ddot{x} + (\omega/Q_2)\dot{x} \\
&\quad + (\omega^2/2Q_1Q_2)[\exp(2x) - (u-1)] = 0 \, .
\end{aligned}
\tag{11}
$$

This differential equation may be compared to that for the position, x, of a particle with a mass, m, subject to a linear damping proportional to the velocity, $\dot{x}$, and subject to a force given by the third term of the equation:

$$
(m/2)\ddot{x} + (\omega/Q)\dot{x} - F(x) = 0 \, .
$$

This force may be integrated with respect to x to give the potential energy

$$
V(x) = (\omega^2/2Q_1Q_2)[\tfrac{1}{2}\exp(2x) - (u-1)x] \, , \tag{12}
$$

which is a simple curve, concave upwards. This analogy is valuable since it gives one a feeling for the qualitative properties of the solutions of Eq. (9). The obvious question to ask is whether the particle is over- or underdamped — that is, does it settle down to its equilibrium value monotonically or does it undergo damped oscillations? The question is answered by expanding the third term of Eq. (11) about its zero value and finding the criterion that the resulting linear system be overdamped. This procedure gives

$$
u - 1 < (Q_1/4Q_2) \, .
$$

It thus becomes apparent that, although the rate of expo-

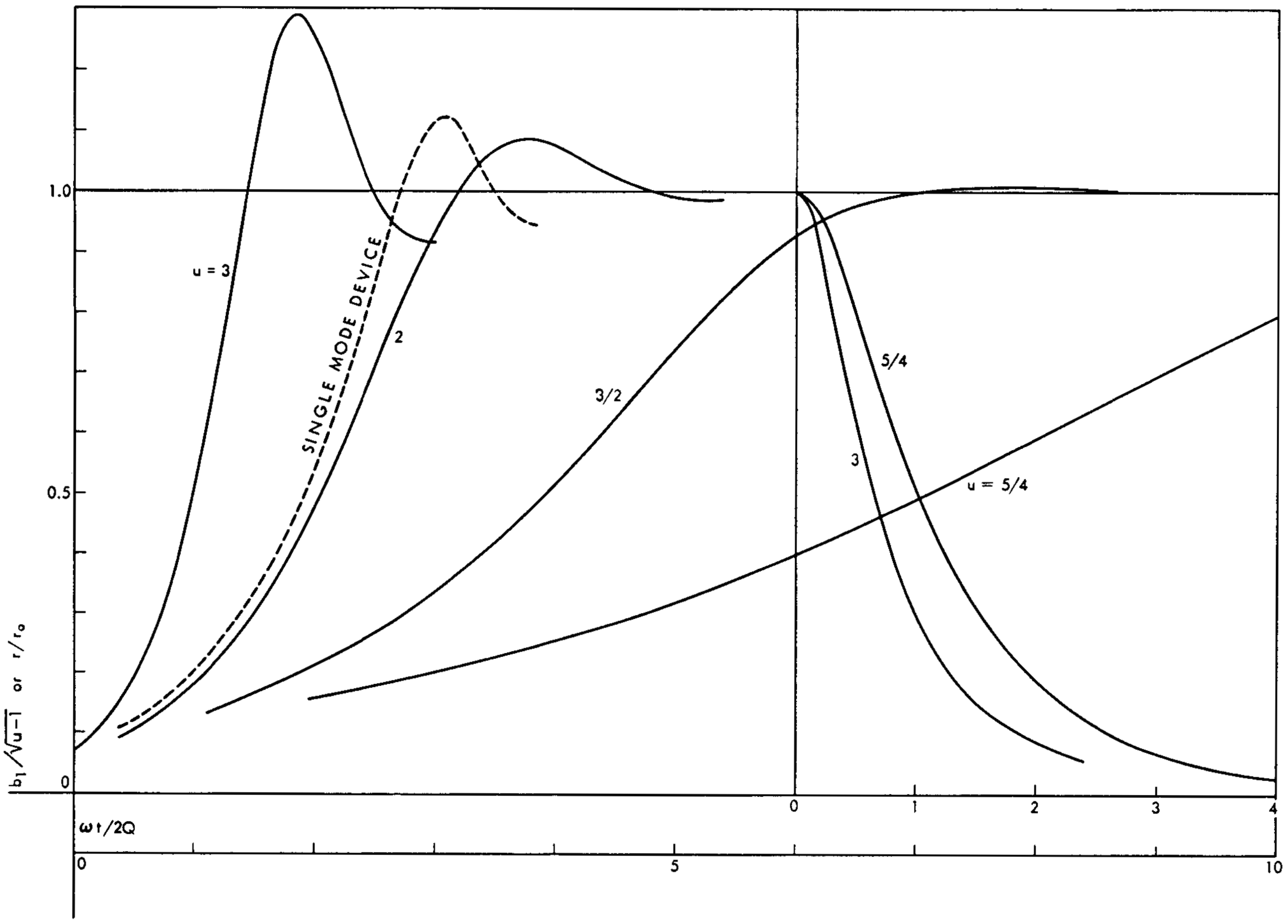

Figure 1 **Numerical solutions for the two mathematical models of subharmonic oscillators. Subharmonic amplitudes vs time.**

nential rise of the subharmonic amplitude is proportional to that excess pump signal level, $(u-1)$, the use of too large a pump signal will cause the subharmonic amplitude to execute many large oscillations about its steady value before settling down. This suggests that there is an optimum pump signal level which gives a fast rate of rise without causing oscillations of the amplitude which are large enough to cause trouble.

Figure 1 shows in ascending curves the "turning on" solutions for a two-mode subharmonic oscillator with $Q_1/Q_2=1$ with the initial condition $b_2=u$ and b_1 positive but very small. These initial conditions are equivalent to assuming that subharmonic amplitude b_1 has such an extremely small initial value that the pump amplitude b_2 reaches the value u before b_1 becomes sizable. Since b_1 increases as $e^{(b_2-1)t}$ this requires an unrealistically small initial value of b_1; the rate of increase of b_1 will be somewhat less than the solutions of Figure 1 if b_1 is initially larger. There are solutions for four values of the pump signal, u, the lowest one corresponding to critical damping. The optimum value of u for the ratio of Q's lies well into the underdamped region since the oscillations for $u=3/2$ and $u=2$ are sufficiently small not to cause errors. The turning off solutions, that is, the change of the subharmonic amplitude from its steady value to zero when the pump signal is turned off, are shown by the descending curves. Here the dependence on pump signal level is much less marked. These solutions were obtained by numerical integration of the equations (11).

The single-mode device

The case of a subharmonic oscillator which is not tuned to resonance with the pump signal will be discussed with most attention paid to the differences from the double-tuned case. We assume the system has but one mode with an equation of motion for the corresponding normal coordinate, q:

$$\ddot{q}+\omega^2 q=-(\omega/Q)\dot{q}+Tq^2+e\sin 2\omega t \qquad (13)$$

and then make the substitution[3]

$$q=a_1\exp(i\omega t)+a_2\exp(2i\omega t) + \text{complex conjugate}\,. \qquad (14)$$

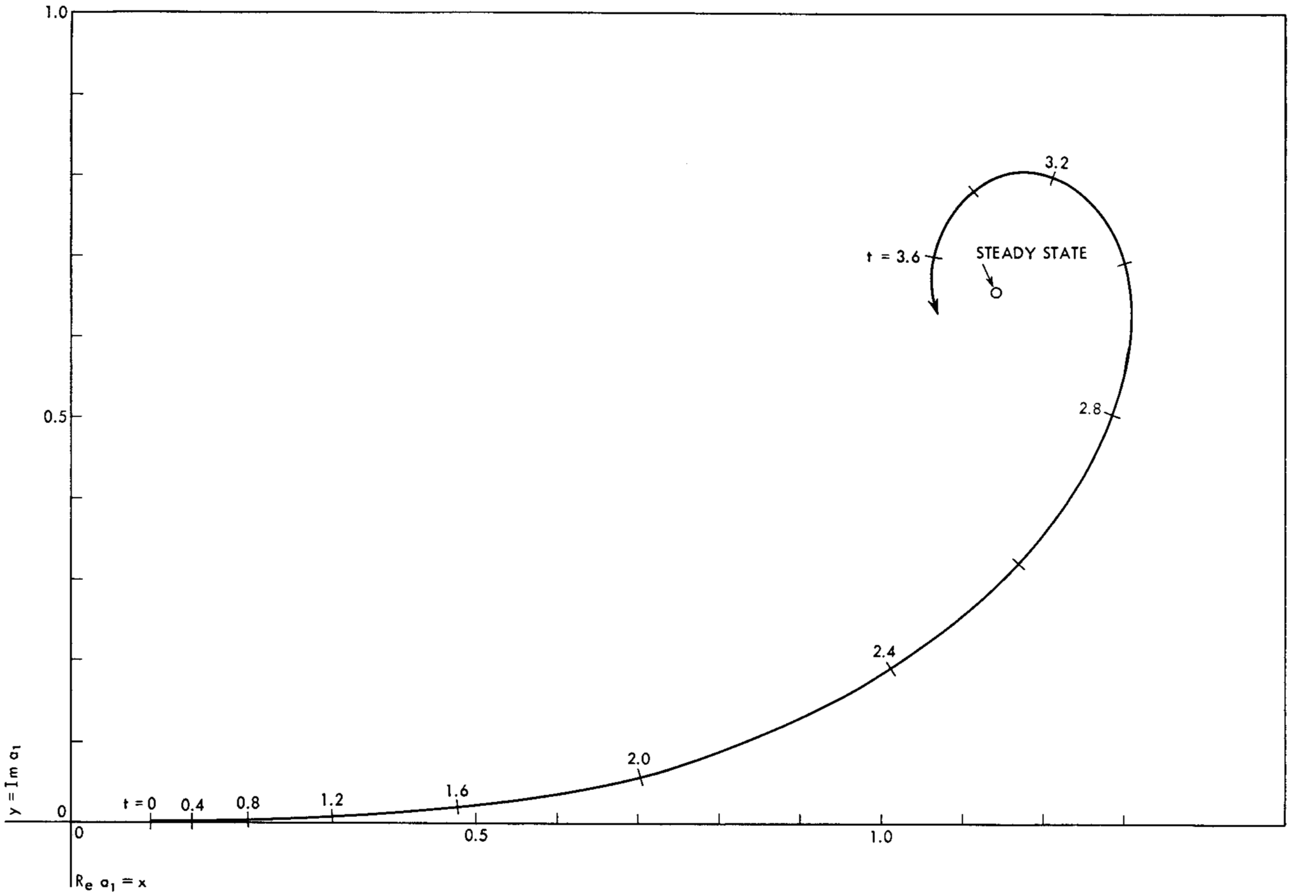

Figure 2 **Solutions for the single-mode model showing complex phase of the subharmonic amplitude.**

Here the justification for neglecting the higher frequencies requires some calculation. If higher frequency terms are included in Eq. (14), they contribute terms to the differential equation for a_1 of order

$$\frac{1}{Q} \times \frac{\text{Pump signal level}}{\text{Threshold pump level}}$$

compared to the terms computed below. As in the single mode case, we neglect terms which can be shown to be small by invoking the slow-variation assumption of the inequality (6). The result of making the substitution (14) into equation (13) and equating coefficients of $\exp(i\omega t)$ and $\exp(2i\omega t)$ is:

$$\begin{aligned} 2i\omega\dot{a}_1 &= -(i\omega^2/Q)a_1 + 2Ta_1{}^*a_2 \\ -3\omega^2 a_2 &= Ta_1{}^2 + (e/2i)\,. \end{aligned} \qquad (15)$$

The second of these equations gives the amplitude, a_2, in terms of a_1 and this may be substituted into the first equation to give

$$\begin{aligned} \dot{a}_1 = -(\omega/2Q)a_1 - (T^2/3i\omega^3)(a_1{}^*a_1)a_1 \\ + (TV/6\omega^3)a_1{}^*\,. \end{aligned} \qquad (16)$$

This time the real and imaginary parts of the amplitude are necessary to describe the behavior of the device. Letting $a_1 = x + iy$,

$$\begin{aligned} \dot{x} &= (v-\tau)x - nr^2y \\ \dot{y} &= nr^2x - (v+\tau)y \end{aligned} \qquad (17)$$

where $r^2 = x^2 + y^2 = a_1{}^*a_1$

$v = TV/6\omega^3$

$\tau = \omega/2Q$

$n = T^2/3\omega^3\,.$

The nontrivial steady state solution of equations (17) is easily obtained by setting the determinant,

$$\begin{vmatrix} (v-\tau) & -nr^2 \\ nr^2 & -(v+\tau) \end{vmatrix},$$

of the right-hand side of the equations equal to zero and solving for r^2:

$$r_0{}^2 = \sqrt{v^2 - \tau^2}\,/n\,, \qquad (18a)$$

and by substituting this result back into one of the equations to obtain

$$y_0/x_0=\sqrt{(v-\tau)/(v+\tau)}\ . \tag{18b}$$

Thus we have the threshold condition, $v=\tau$ and the phase change with increasing pump signal level from zero degrees ($y=0$; a_1, real) at threshold to 45 degrees ($x=y$) in the limit of high pump level. Examination of equations (17) shows that for small a_1 the solution begins as

$$a_1=(\text{real constant})\times\exp(v-\tau)t\,, \tag{19}$$

showing that the magnitude of the amplitude behaves much like the first case, but that phase change from zero to the steady phase takes place during the turning on process. The initial exponential character of this solution with the implied rise time was given by Goto.[2] To investigate the nature of the solutions in the neighborhood of the steady state we substitute

$$\begin{aligned} x&=x_0+\varepsilon\exp(\lambda t)\\ y&=y_0+\delta\exp(\lambda t) \end{aligned} \tag{20}$$

into the equations (17), keep only term linear in ε and δ, and find λ so that a nontrivial solution exists. This gives

$$\lambda=-\tau\pm\sqrt{3\tau^2-2u^2}\ . \tag{21}$$

Again the subharmonic amplitude will execute damped oscillations about its steady value if pump level is high. The equations (17) have an analytic solution for $u=0$. Thus when the pump signal is turned off, the subharmonic amplitude will have this time dependence:

$$\begin{aligned} |a_1|&=r=r_0\exp(-\tau t)\\ \theta&=arg(a_1)=\arctan(y/x)\\ &=\theta_0+(nr_0^2/2\tau)[1-\exp(-2\tau t)]\ . \end{aligned} \tag{22}$$

The latter equation shows a continuous phase change taking place as the oscillation dies out. The dashed line of Figure 1 shows a "turning on" solution for $\tau/v=2$, $n/\tau=1$ normalized to the same steady value as the two mode solutions. This is qualitatively a great deal like the solutions of the single mode case. Figure 2 shows the complex phase of the subharmonic for various times.

Conclusions

The "turning on" and "turning off" solutions of two mathematical models of subharmonic oscillators with linear dissipation have been discussed in the low-loss or quasistatic approximation. The initial exponential rise of the subharmonic amplitude, Eq. (19), is a result commonly found in the literature; Goto's Eq. (15) is an example. The contribution of this paper, aside from a somewhat different mathematical approach, is in providing relatively complete and quantitative solutions with special attention to the phenomenon of the oscillation of the subharmonic amplitude about its steady value at high pump levels. The results of these two analyses show that this phenomenon is similar for the one- and two-mode models.

Acknowledgment

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References and footnotes

1. S. Bloom and K. K. N. Chang, *RCA Review*, **18**, 578 (1957).
   J. Hildebrand and W. R. Bean, *RCA Review*, **20**, 229 (1959).
   A. H. Nethercot, Jr., *IBM Journal*, **4**, 402 (1960).
   Hildebrand and Bean discuss devices whose stable state is established by the forward current of a diode; that is, by a very nonlinear dissipation. Nethercot treats the switching of subharmonic oscillators by an over-riding input signal, using an energy balance method of N. M. Kroll and I. Palocz, *IBM Journal*, **3**, 345 (1959). This method might also be used in our case.
2. For an illustration of the actual use of these devices, there termed *parametrons,* in computer circuitry see E. Goto, *Proc. IRE*, **47**, 1304 (1959). His circuits seem to be at least qualitatively similar to our single-mode case.
3. It has been pointed out to the author that a more general assumption would include a quasiconstant term in this expression and upon equating constant terms below yield a third equation to the set (15). This would have the effect of adding a term to Eq. (16); changing the value of the parameter n to $2T/3\omega^2$; changing the sign of terms containing n in equations (17); and, finally, reversing the sign of the phase, Eq. (18b). The subsequent solution and discussion would remain the same. Whether or not such a term exists depends upon the particular system treated, but as Eq. (13) stands its inclusion is necessary to give a correct solution.

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