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Methods of Analysis of Circuit Transient Performance

Abstract: This paper surveys some numerical methods applicable in obtaining the distribution of performance parameters associated with the transient behavior of switching circuits. The methods considered are: (1) *Monte Carlo*, in which sample circuits are simulated on the IBM 704 and their performance computed and tabulated. (2) *Parameter sensitivity methods*, including propagation of error, in which the first and second moments of the output distribution are estimated from parameter sensitivities; and a method using the total differential of the performance parameter to estimate the deviation of circuit behavior from its component deviations. (3) *A surface fitting method*, in which there is developed a formula for delay time in terms of circuit component values. Each method is used to determine the delay time of a simple system. Their merits and drawbacks are compared and discussed, and estimates are given of the IBM 704 machine time necessary for implementation on a ten-transistor switching circuit.

1. Introduction

When the reliability of an electronic circuit design is evaluated, component parameters are considered as random variables rather than fixed quantities. No matter how tight manufacturing control may be, there are initial variations from one component to another, and these together with subsequent component drift give rise, in circuits built from these components, to variations in performance from one circuit to another. Variables which characterize circuit performance, such as voltage at a node, or rise time, are called *performance parameters*.[1] The reliable design problem we are concerned with here is simply this: Given the component parameter statistical distributions, find the performance parameter distributions.

In engineering practice, the limits of a performance parameter distribution (beyond which cases exist with zero probability) may often be found by evaluating performance with various combinations of extreme adverse values of the circuit parameters. We may require that in a well-designed circuit these limits be within the region of satisfactory circuit performance. This is the "worst case" design philosophy. As noted in Ref. 2, this design method may be unduly costly and pessimistic. (The substantial discrepancy between worst case and actual performance is one attraction of asynchronous systems.) A design philosophy exploiting more of the output performance parameter distribution than its limits, would be more realistic.

Computationally, the problem of estimating a performance distribution may be of two distinct types, arising from the nature of the performance parameter being investigated. It may be a steady state level which can be found from the solution of a system of algebraic equations. In this case there are well-known techniques for finding the required distribution.[2] In particular, the Monte Carlo method has proved quite satisfactory in practice. But there are also performance parameters associated with the transient behavior of the circuit – for example, delay time – which are found from the solution of differential (rather than algebraic) equations. These performance parameters are important in the design of computer switching circuits. For a given circuit, therefore, the machine designer is interested not so much in the ideal circuit behavior, but in the distribution of performance for actual circuits with components deviating, within specifications, from their design values.

In this paper we are concerned with estimating transient distributions, that is, distributions of performance parameters associated with the transient behavior of a circuit. Analytically, we may suppose that the following system of differential equations is given:

$$\dot{x}_i = f_i(t; x_1, \ldots, x_n; a_1, \ldots, a_m)\, i = 1, \ldots, n \qquad (1)$$

and a function of the solution

$$v(t) = g(x_1(t), \ldots, x_n(t)) \,. \qquad (2)$$

We assume solution existence and uniqueness for given initial conditions. The system (1) may be network equations, where the x_i are unknown voltages or currents. The a's are random variables with known distributions corresponding to those of the component parameters. The function $v = g(t)$ defines the performance parameter. The problem is to estimate the distribution of v.

Note that v in (2) is a time-parameterized family of random variables — that is, a stochastic process. Although much is known about many important stochastic processes,[3] those generated by differential equations as indicated above appear to have been little investigated. Some recent work by F. S. Scalora[17] contributes in this area.

Note further that although we arrived at the above analytic formulation of the transient distribution problem in connection with the design of digital computers, the problem is an old one for analog computers. According to Dow[4] and others, when a differential analyzer is set up to solve a given equation, certain errors are introduced into the solution because the components of the computer are in error, or imperfect. In using analog computers we have the problem of estimating solution errors from component errors. The same problem is placed in the context of the reliability of switching circuits in a digital computer by a change in terminology; replace "error" by "deviation from the nominal." It is important to recognize this equivalence, for there is considerable work (especially analytic work, as in Ref. 4) on this problem in its analog computer context, which ought not be ignored by those primarily interested in the reliability of switching circuits. In particular, the works of Miller and Murray[5] and Dostupov and Pugachev[6] encompass both the digital and analog computer problems, and can be studied from either point of view. For example, Dostupov and Pugachev show that under certain conditions on the partial derivatives of the f_i, the time-parameterized family of distributions $p(t; x_1, \ldots, x_n; a_1, \ldots, a_m)$ satisfies the partial differential equation

$$\frac{\partial p}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} [f_i p] = 0 \,.$$

In this paper we outline some numeric approaches to the transient distribution problem. Beyond simply noting that the results are plausible, we make no attempt at hardware verification, nor at error bounds estimation. Our purpose is experimental: to try a few techniques, and so highlight some of their relative merits and weaknesses. We are concerned with those methods of transient distribution estimation that are especially suitable for use with a large digital computer, and which lend themselves to mechanization and incorporation in an automatic design procedure.

The remainder of this paper describes three transient-distribution estimation methods, with reference to an example circuit. The next section describes this circuit in detail, and the following sections describe estimation by the following methods.

Monte Carlo: In this we simulate and sample a large number of circuits by computing their behavior from network equations. This approach is used successfully on steady-state performance parameters, but because of the time required for numerical integration it is hardly practical for transient performance parameters at present.

Parameter sensitivities: Here we estimate response variance from component variances. This too is easily done for steady-state performance, but with transients the method entails deprecated numerical techniques.

Surface fitting: In this method we construct an expression for the performance parameter in terms of component values, recasting the problem as a steady-state one. The expression can be found from measured data by multiple regression, though our data are computed from circuit equations.

In the descriptions of these methods we will state certain assumptions regarding statistical independence of variables, and linearity of functions. To the extent that the assumptions for a method are invalid when applied to some circuit, the method is inapplicable to that circuit. However, the purpose of this paper is not a reliability analysis of the example circuit, but a discussion of methods of analysis on arbitrary circuits. For this reason we state, but do not justify, our assumptions before applying each method to the same example.

2. Example circuit

• *A. Circuit topology*

The numerical methods we have studied have been applied to a comparatively simple circuit, the transistor equivalent circuit with minimal external circuitry shown in Fig. 1 (Ref. 10). The differential equations for this circuit are:

$$C_c V'_{21} = -V_{21}(G_c + G_{cc}) + V_{e1}G_{cc} + \alpha I_h + G_{cc}(V_c - V_e)$$

$$C_e V'_{e1} = -V_{21}G_c - V_{e1}G_{bb} - (1-\alpha)I_h + G_{bb}(V_e - V_b) - C_c V'_{21}$$

where

$V_{21} = V_2 - V_1$, voltage difference from node 2 to node 1

$V_{e1} = V_e - V_1$, voltage difference from node E to node 1

and G's are reciprocals of corresponding R's.

$$I_h = I_{es}(e^{\lambda V_{e1}} - 1) \qquad (4)$$

$$C_e = \frac{C_{eo}}{(V_o - V_{e1})^N} + \frac{I_h \lambda}{2\pi F_{cb}} \qquad (5)$$

$$V_e = \frac{1}{G_a + G_{bb} + G_{cc}} [V_{e1}(G_{cc} + G_{bb}) - V_{21}G_{cc} + V_c G_{cc} + V_b G_{bb} + V_a G_a] \qquad (6)$$

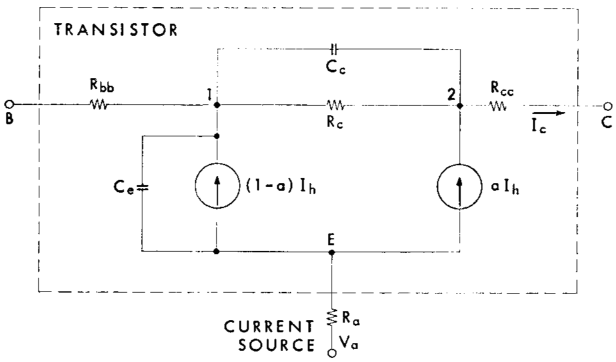


Figure 1 **Transistor equivalent circuit with minimal external circuitry.**

$$\alpha = \frac{\alpha'}{1+\alpha'} \qquad \alpha' = \frac{1}{b} - \frac{1}{AI_h + b}$$

$$b = \frac{1}{44.5}\,[0.028\,A + 0.075] \tag{7}$$

$$I_c = (V_{21} + V_e - V_{e1} - V_c)G_{cc}\,. \tag{8}$$

- *B. Parameter distributions*

The distributions on the circuit parameters are assumed to be uniform with the minimum and maximum values, as in the following tabulation. The uniformity assumption is incidental and does not restrict the applicability of the methods described. We also assume statistical independence of these variables. Where several parameters are associated with the same physical unit, such as a transistor, this assumption may not be justified, in which case the methods would have to be modified. In the Monte Carlo method we would need a more sophisticated and elaborate sampling scheme; in parameter sensitivity and surface fitting methods our formulas would have to include cross influence terms.

Parameter	*Units*	*Minimum*	*Maximum*
R_{bb}	ohms	25	75
R_c	ohms	10^5	10^6
C_c	$\mu\mu$fd	3	7
I_{es}	μamp	0.0367	1.28
C_{eo}	$\mu\mu$fd (volts)^N	12.5	18.36
N	-------	0.4	0.5
F_{cb}	mc/sec	70	200
A	-------	0.5	45

The following constants were not distributed:

$\lambda = (0.026)^{-1}$	-----	
$R_{cc} = 20$	ohms	
$V_o = 0.5$	volts	
$V_b = 0$ (grounded base)	volts	
$V_c = -4.5$	volts	
$R_a = 10{,}000$	ohms	

$$V_a = \begin{cases} \dfrac{50}{15\times 10^{-9}}\,t & \text{for} \quad 0 \leqq t \leqq 15\ \text{m}\mu\text{sec} \\ 50 & \text{for} \quad 15\ \text{m}\mu\text{sec} \leqq t \end{cases}$$

- *C. Input*

The input to the circuit is the current entering at node E, which is supplied by a 50-volt ramp acting through the 10K resistance R_a.

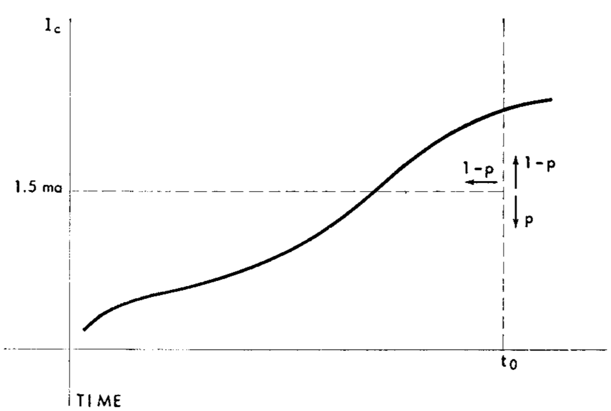
Note that everything in the differential equations, except the variables V_{e1} and V_{21}, is now in terms of distributed parameters, constants, and the input V_a. Equation (7) gives the parameter α' in terms of I_h and a distributed parameter A. A choice of A fixes the α' vs. I_h characteristic.

- *D. Performance parameter*

The performance parameter which we investigate will be *delay time*. This is defined, in this case, as the time it takes, after start of the input ramp of V_a, for the collector current I_c to reach 1.5 ma.

Since a transient performance parameter v is associated with the transient behavior of the system, it seemed natural to express v as a function of the solution of the differential equations governing the system, as in Eq. (2). A simple performance parameter to deal with would be the solution itself — for example, the collector current $I_c(t)$. Then $v(t) = I_c(t)$, and the statistical distribution of v at time t_o is the distribution of I_c at this time. But delay time as defined above is not the value of the solution at some t; it is the time t for some value (1.5 ma) of the solution. We are interested in the t distribution for a fixed I_c, rather than the I_c distribution for a fixed t. Under appropriate assumptions these distributions are related, and can be found, one from the other.

Figure 2 **Time distribution of I_c.**



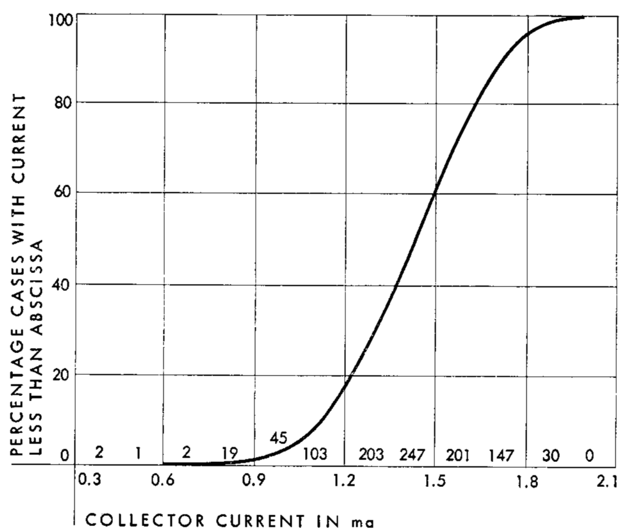
Suppose $I_c = I_c(t)$ is monotone for a sufficient interval of t, and suppose the distribution of I_c at time t_o is known. Then the fraction p of differential equation solutions such that $I_c \leq 1.5$ ma at time t_o is also known. The remaining fraction of differential equation solutions, $1-p$, are all greater than 1.5 ma at time t_o, as shown in Fig. 2. Since $I_c(t)$ is monotone, $1-p$ must also be the fraction of all solutions which have attained 1.5 ma before time t_o. Thus the distribution of I_c at t_o gives one point on the delay time distribution, for $I_c = 1.5$ ma.

3. Monte Carlo

In this method the circuit parameters are random variables with given distributions, and we solve the system of differential equations once for each set of randomly chosen parameters. In this way we simulate the existence and behavior of as many transistors as we wish. It is a simple matter to tabulate the delay times of the different samples, and this gives the desired distribution. An extensive discussion of many facets of this technique is contained in Ref. 11. The application of Monte Carlo to reliability problems is discussed in Ref. 2, while Ref. 12 describes some IBM 704 programs which have been used on *steady state* reliability problems. In order to apply the methods to the evaluation of a transient parameter we need to solve differential equations. If these equations can be solved analytically the problem reduces to a steady-state one, as in Ref. 16. Most often, however, it is necessary to apply numeric methods. There are many IBM 704 programs for this purpose, available through SHARE. We used PK NIDE.[7]

Since we wish to solve and resolve differential equations over and over again in this sampling procedure, the feasibility of this scheme depends on the speed at which integration proceeds. This is limited by the integration interval, which controls error and stability. We found that even the simple test circuit shown in Fig. 1 required about one and one-half hours for 1000 samples.

Figure 3 **Distribution of I_c at 7.45 mμsecs by Monte Carlo Method, nominal I_c=1.5 ma.**



For more complicated circuits involving several transistors, not only would the rate of solution be slower, but the transient would last longer, requiring that the solution be carried out further and taking a still longer time.

An advantage of this method over the next two to be described is that the delay time distribution may be obtained directly. In order to do this, however, frequent testing to determine when a solution has reached 1.5 ma would be necessary, and this would slow down the program. For this reason we chose instead to find the distribution of collector currents at a fixed time (print interval), and to obtain a rough estimate of our delay time distribution from this. The means of obtaining this estimate is described in connection with Fig. 2, using the I_c distribution data found at the 8th print. Of course, this gives just one point of the delay time distribution. Other points may be found in the same way. But Fig. 4 illustrates another way of getting the delay time distribution – by projecting population contours parallel to the nominal solution. Intuitively the assumption that population contours are approximately parallel is not as severe as the assumption that all individual solutions are parallel to each other. Solutions may cross each other without changing a population contour.

The distribution of 1000 cases of collector current at time 7.45 mμsec after start of the input ramp, as found by Monte Carlo, is shown in Fig. 3. Collector current is nominally 1.5 ma at this time. To obtain these results a random number generator[14] was used to pick random values for the distributed parameters, within the ranges specified in Section 2B. These random values, together with the fixed parameter values, were inserted in differential equation (3), which was then solved on the IBM 704 for I_c at time 7.45 mμsec. The I_c found was then tabulated with respect to 20 boundary values, spaced 0.15 ma apart, ten above the nominal value, and ten below. The entire process was then repeated again and again for a total of 1000 times, starting with newly generated random values each time. The resulting number of cases in each interval is shown at the bottom of the appropriate column in Fig. 3. These numbers are proportional to the height of histogram bars for the I_c distribution. The smooth curve shown was obtained from the cumulative totals. Figure 4 shows the start of a nominal solution. Assuming the shape of the I_c distribution is fairly constant in the time range from the 7th to the 10th print intervals, we may project the I_c distribution as shown, obtaining the delay time distribution shown in Fig. 5, as described previously.

4. Parameter sensitivity

We now describe two methods based on estimates of the sensitivity $\partial v/\partial a_i$ of the performance parameter v to the component parameter a_i. Implicit for the validity of these methods is the assumption that $v = g(t; a_1, \ldots, a_m)$ is approximately linear over the range of variation of the a_i. If the ranges of variation are sufficiently restricted the

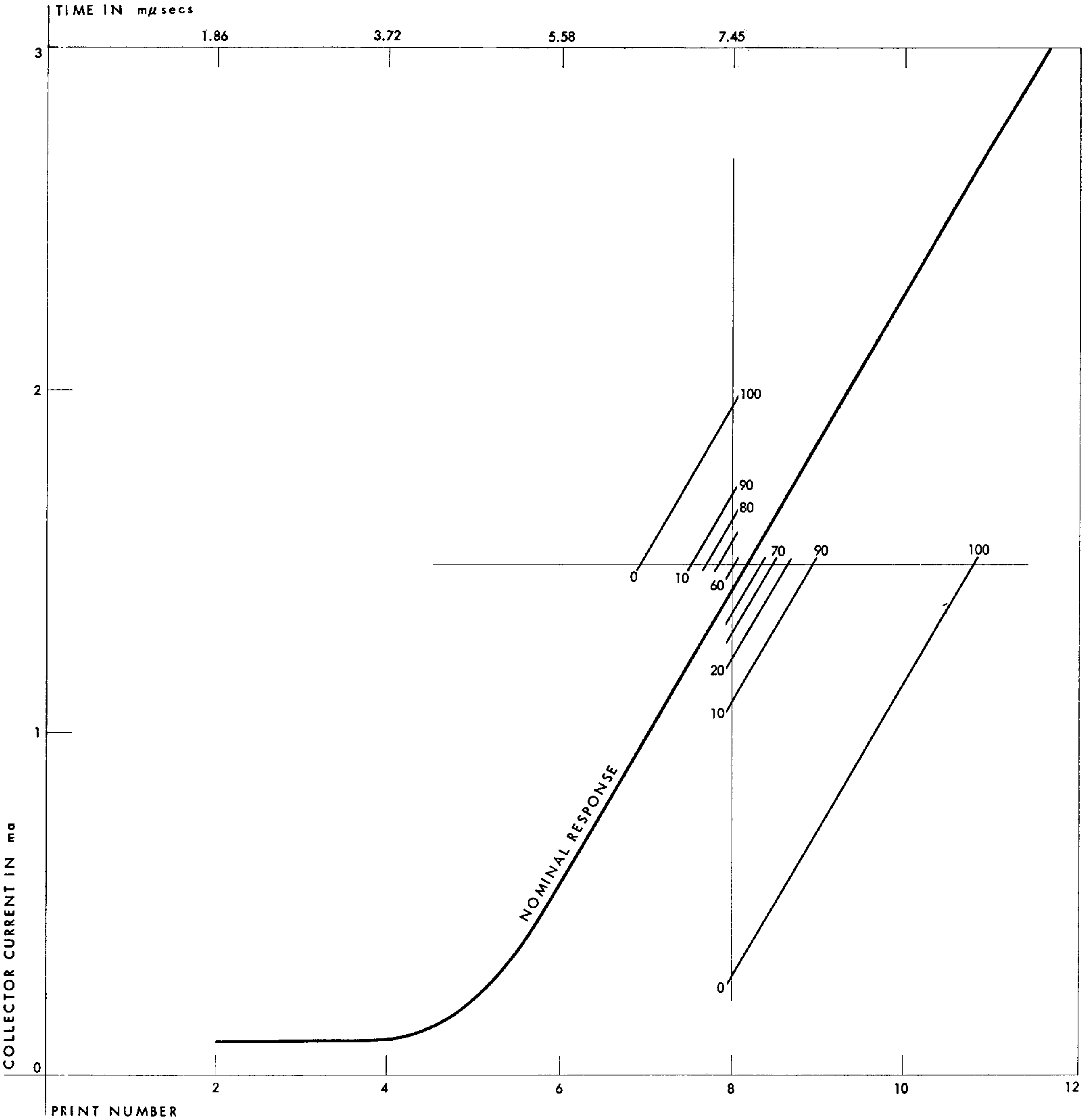


Figure 4 **Estimation of time distribution from I_c distribution.**

assumption can be justified, of course, but serious errors may be introduced if we have no control over the variations. The surface fitting method discussed later partially obviates this difficulty.

• *Sensitivity estimation*

The computational heart of these methods is determination of the partial derivatives $\partial v/\partial a_i$. If the solution of (1) is

$$x_i = x_i(t; a_1, \ldots, a_m)$$

we have

$$v = g(t; a_1, \ldots, a_m) .$$

The analytic expression for v in terms of the parameter and time is generally not known, but the *value* of v as a function of t can be found using a numerical integration routine. This makes possible the estimate of the partials by

$$\frac{\partial v}{\partial a_i} \approx \frac{g(a_i+\Delta a_i)-g(a_i)}{\Delta a_i} = \frac{\Delta v}{\Delta a_i} .$$

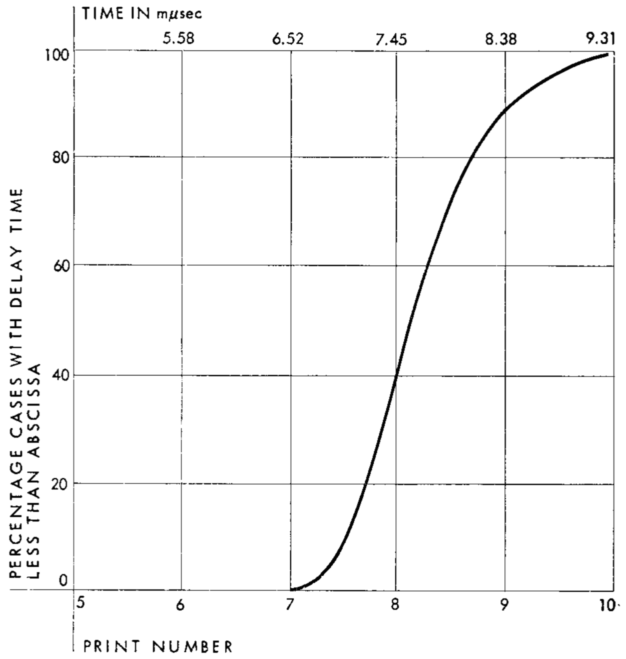


Figure 5 **Distribution of delay time, Monte Carlo method.**

This formula must be used with caution. Hildebrand[8] says "... numerical differentiation should be avoided whenever possible, ..." Δa_i must be small in order that $\Delta v/\Delta a_i$ be the slope of a tangent to the curve $v=g(a_i)$ rather than the slope of a line passing through the two distinct points $(a_i, v(a_i))$ and $(a_i+\Delta a_i, v(a_i+\Delta a_i))$. On the other hand, the smaller Δa_i the more significant the round-off error in Δa_i and the more significant the truncation and round-off errors in $v(a_i+\Delta a_i)-v(a_i)$. To find the slope of the tangent with a relatively large Δa_i second and higher forward differences may be used to estimate the derivative:

$$\frac{\partial v}{\partial a}=\frac{1}{\Delta a}\left(\Delta v-\frac{1}{2}\Delta^2 v+\frac{1}{3}\Delta^3 v\ldots\right),$$

where $\Delta v=v(a+\Delta a)-v(a)$. This and other formulas for numeric differentiation of polynomials may be found in Ref. 8.

Choosing the size of the parameter perturbations (the Δa_i's) is a difficult feature of this method. As noted above the perturbations must be neither too large nor too small. The optimum perturbation, moreover, will differ from parameter to parameter, from circuit to circuit, and even in the same circuit from one part of a solution to another part of the same solution. Although it may be possible to determine a good perturbation size from an error analysis of the differential equations, this is usually quite difficult, and in practice we can only experiment with various values.

Another way to estimate these partials is by means of sensitivity equations.[5] From Eq. (1) we have by differentiation with respect to a_j

$$\frac{\partial \dot{x}_i}{\partial a_j}=\frac{\partial f_i}{\partial a_j}+\sum_{k=1}^{n}\frac{\partial f_i}{\partial x_k}\frac{\partial x_k}{\partial a_j}\quad \begin{array}{l} i=1,\ldots,n \\ j=1,\ldots,m \end{array}. \tag{10}$$

For each j Eq. (10) is a system of linear differential equations in $\partial x_i/\partial a_j$, from which, with (2), we may find $\partial v/\partial a_j$.

In this method of sensitivity computation we bypass the problem of perturbation size, and the problem of difference order. If the partial derivatives are estimated by k differences, a circuit with n parameters requires $kn+1$ solutions, while the sensitivity equations method requires $m+1$ solutions. But in the latter case we must set up and solve a new system of differential equations, like (10), for each parameter sensitivity.

The sensitivity method has parameter contribution factors as a by-product. The contribution factor of the component a_i is defined as

$$c.f.(a_i)=\frac{\left(\dfrac{\partial v}{\partial a_i}\sigma_i\right)^2}{\displaystyle\sum_{j=1}^{m}\left(\frac{\partial v}{\partial a_j}\sigma_j\right)^2},$$

where σ_j is the standard deviation of a_j. Its significance is that it gives the relative contribution of the spread of a_i to the spread of the output. It tells how critical or how important each parameter is to the output. This may be useful information for design purposes.

Figure 6 shows the transistor response in relation to parameter contribution factors. Specifically, the contribution factor of R_{bb} was found from

$$c.f.(R_{bb})=$$

$$\frac{\left[\dfrac{\partial I_c}{\partial R_{bb}}\sigma(R_{bb})\right]^2}{\left[\dfrac{\partial I_c}{\partial R_{bb}}\sigma(R_{bb})\right]^2+\left[\dfrac{\partial I_c}{\partial C_c}\sigma(C_c)\right]^2+\ldots+\left[\dfrac{\partial I_c}{\partial A}\sigma(A)\right]^2}.$$

The partial derivatives were found from third differences; that is, to find $\partial I_c/\partial R_{bb}$ we used

$$\frac{\partial I_c}{\partial R_{bb}}\approx\frac{1}{\Delta R_{bb}}\left(\Delta I_c-\frac{1}{2}\Delta^2 I_c+\frac{1}{3}\Delta^3 I_c\right)$$

$$\approx\frac{1}{6\Delta R_{bb}}\left[2I_c(R_{bb}+3\Delta R_{bb})-9I_c(R_{bb}+2\Delta R_{bb})\right.$$

$$\left.+18I_c(R_{bb}+\Delta R_{bb})-11I_c(R_{bb})\right],$$

where the terms $I_c(R_{bb}+n\Delta R_{bb})$, $n=0,1,2,3$ were found by solving the differential equation system (3) first with R_{bb} and all other parameters at their nominal value, then again with R_{bb} replaced by $R_{bb}+\Delta R_{bb}$, then with $R_{bb}+2\Delta R_{bb}$, and finally with $R_{bb}+3\Delta R_{bb}$.

According to these results the most important parameters affecting collector current at the start of the transient (up to the 4th print interval) are R_{bb} and C_c.

From the 4th to the 9th print interval the collector current response is most sensitive to C_{eo}; from the 9th to steady state (the 20th print interval) F_{cb} is most important, and finally in steady state α is most important.

We now describe methods of distribution estimation using parameter sensitivities.

• *Propagation of error*

In this method we assume a normal output distribution with mean equal to the output nominal value, and standard deviation

$$\sigma(v)=\left\{\left[\frac{\partial g}{\partial a_1}\sigma(a_1)\right]^2+\ldots+\left[\frac{\partial g}{\partial a_m}\sigma(a_m)\right]^2\right\}^{\frac{1}{2}},$$

where $g(t;a_1,\ldots,a_m)$ is the performance parameter as a function of the circuit parameters a_i, and time t, and where $\sigma(a_i)$ is the standard deviation of a_i.

A difficulty with the propagation-of-error method is that the normal distribution curve has no sharp cut-off beyond which cases appear with zero probability. On the contrary, since the normal function is positive for any finite value of its argument, the probability of outputs in any region from $-\infty$ to $+\infty$ is positive.

Suppose the output of a circuit is the current in some branch and, from physical considerations, that this current must have a certain direction which corresponds to positive values of the current. Nevertheless, normality of the distribution of outputs implies a positive probability of negative outputs. This probability may be negligibly small, but there is a world of difference between "small" and "nonexistent," and there may be cases where the physics of a problem is embarrassing to the propagation of error technique.

Knowing the nominal $\bar{I}_c$ and its standard deviation $\sigma(I_c)$ as a function of time, the cumulative distribution of I_c is

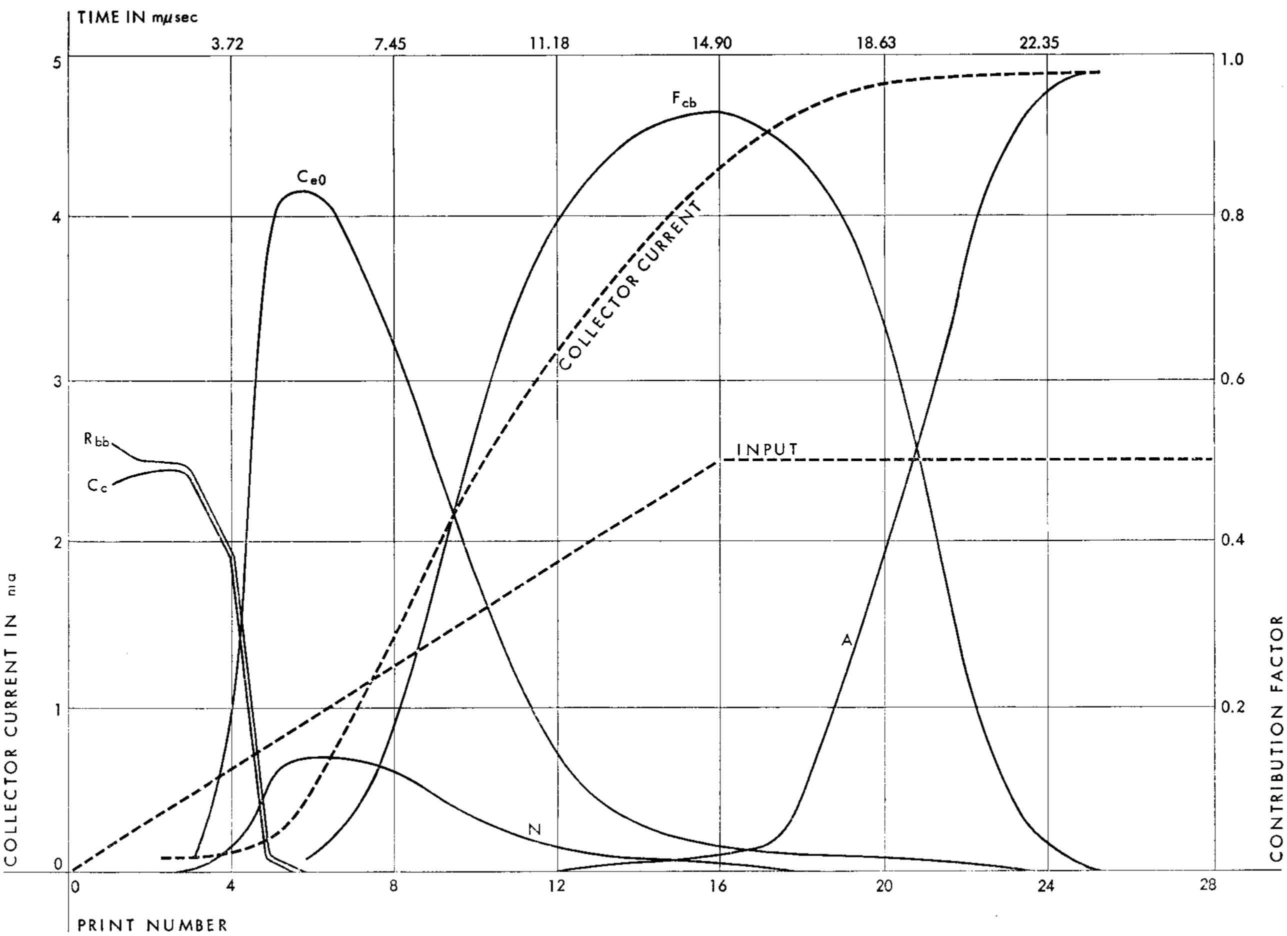
$$\frac{1}{\sigma(I_c)\sqrt{2\pi}}\int_{-\infty}^{I_c}\exp\left\{-\frac{(x-\bar{I}_c)^2}{2(\sigma(I_c))^2}\right\}dx\,.$$

The six points shown in Fig. 7 were found from the six I_c distributions at the times shown, by the conversion method described in connection with Fig. 2.

• *Total differential method*

This method is a combination of the Monte Carlo and propagation of error methods. The theory behind it is

Figure 6 **Sensitivity of collector current to parameters during transient response.**



this: If $v=g(a_1,\ldots,a_m;t)$ is the output as a function of parameters and time, then the deviation of v at time $t+dt$ from its nominal value at time t is

$$dv = \frac{\partial g}{\partial a_1} da_1 + \ldots + \frac{\partial g}{\partial a_m} + \frac{\partial g}{\partial t} dt\,, \tag{11}$$

where the partials are evaluated at time t and nominal component values, and where the differential $da_i=a_i-\bar{a}_i$ is the deviation of the parameter from its nominal value. For the deviation at time t the $(\partial g/\partial t)dt$ term is zero and omitted.

After the sensitivities in (11) are established, the distribution of dv is determined, using Monte Carlo, from the distributions of the da_i. The distribution of v is the distribution of dv displaced by the nominal value $\bar{v}$, since $v=\bar{v}+dv$.

This method has one obvious superiority over the previous method: nothing need be assumed about the shape of the output distribution. We need not assume a normal distribution, nor admit the possibility of outputs ranging from $-\infty$ to $+\infty$. Indeed, with the total differential method

$$\bar{v} \pm \sum_{i=1}^{m} \left|\frac{\partial g}{\partial a_i}\right| (\max|da_i|)$$

are the extreme possible outputs.

Monte Carlo was applied to (11) to obtain estimates of the collector current distribution at five points in time. These gave, using the method of Fig. 2, five points of the statistical distribution of delay time curve shown in Fig. 8.

5. Surface fitting

In this method we seek the function $t=g^{-1}(a_1,\ldots,a_m)$ which is satisfied, or is a close fit to, a set of data

$$\{(t_k, a_{1k},\ldots,a_{mk})\,|\,k=1,\ldots,s\}\,.$$

That is, we seek the equation for a multiple regression surface.[9] It is necessary to assume or determine the dependence or independence of variables, and the nature (linear, exponential, ...) of the dependence of t on the a_i.

Here we assume the deviation of a solution from its nominal is related to the deviations of the parameters by an expression of the form

$$v-\bar{v}=\sum_{i=1}^{m} g_i(a_i)-g_i(\bar{a}_i)\,,$$

where each g_i is a function of a_i and no other distributed parameter. Clearly, when each parameter has its mean value, the right member is zero and the solution is also nominal, $v=\bar{v}$.

To determine g_i, we set $a_j=\bar{a}_j$ for $j\neq i$ and find by experiment – solving the system with various values of a_i – the graph of v vs. a_i. This is the same as the graph of g_i displaced by the constant $v-g_i(a_i)$. If this happens to be the graph of a straight line, then

$$g_i(a_i)-g_i(\bar{a}_i)=ma_i+b-(m\bar{a}_i+b)=m\ (a_i-\bar{a}_i)\,,$$

and $\partial g_i/\partial a_i=m$ and this method is equivalent to the total differential method. But we have the advantage now that the g_i need not be assumed constant. It should be noted,

Figure 7 **Distribution of delay time, propagation-of-error method.**

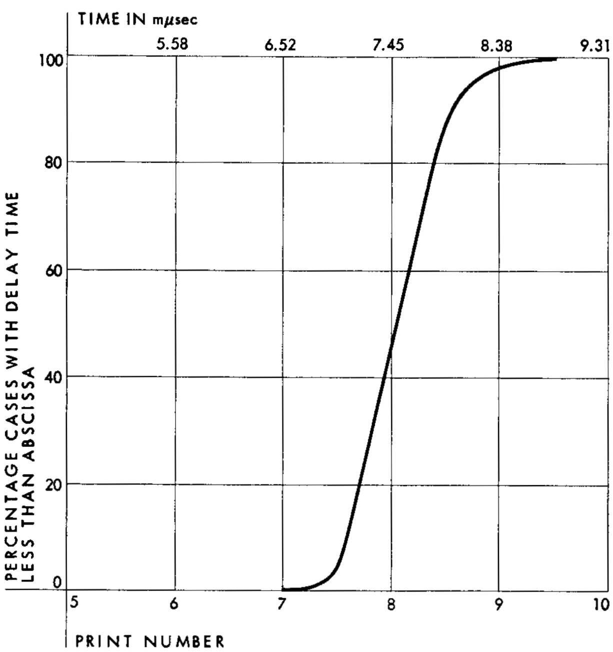
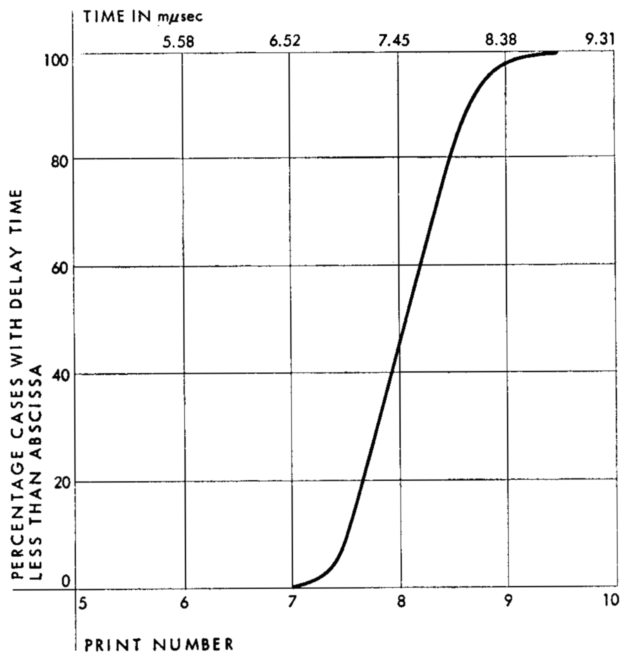


Figure 8 **Distribution of delay time, total differential method.**



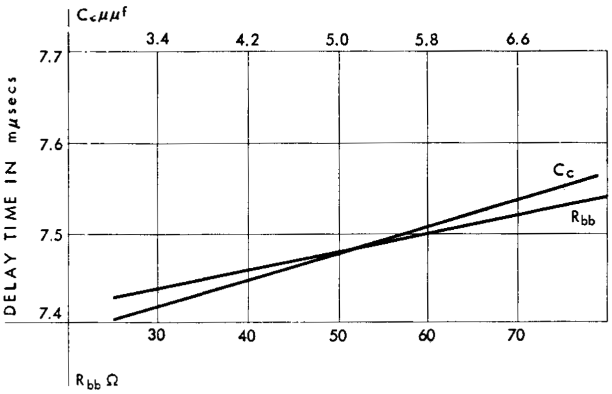


Figure 9a **Delay time vs R_{bb} and C_c.**

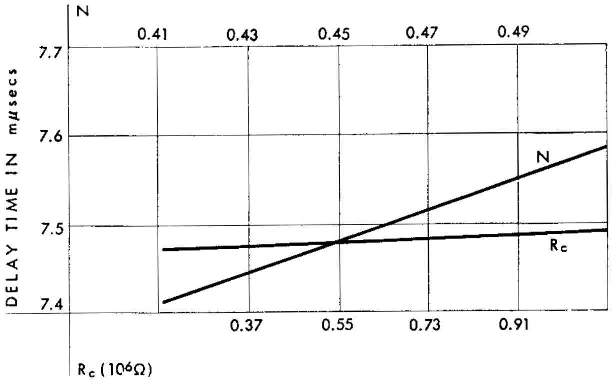


Figure 9b **Delay time vs R_c and N.**

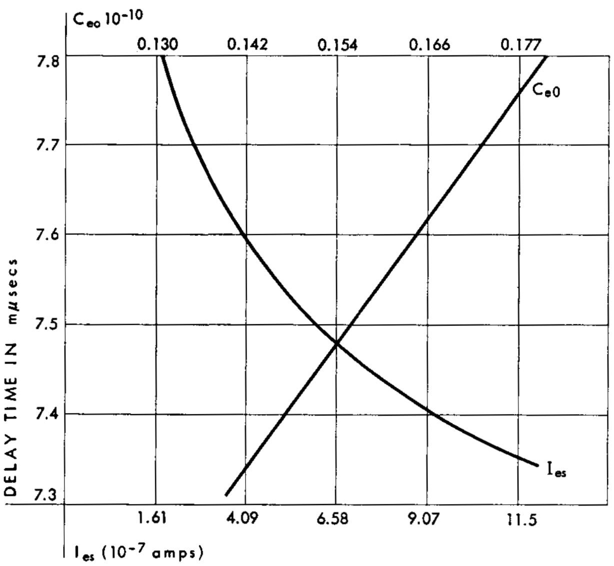


Figure 9c **Delay time vs I_{es} and C_{eo}.**

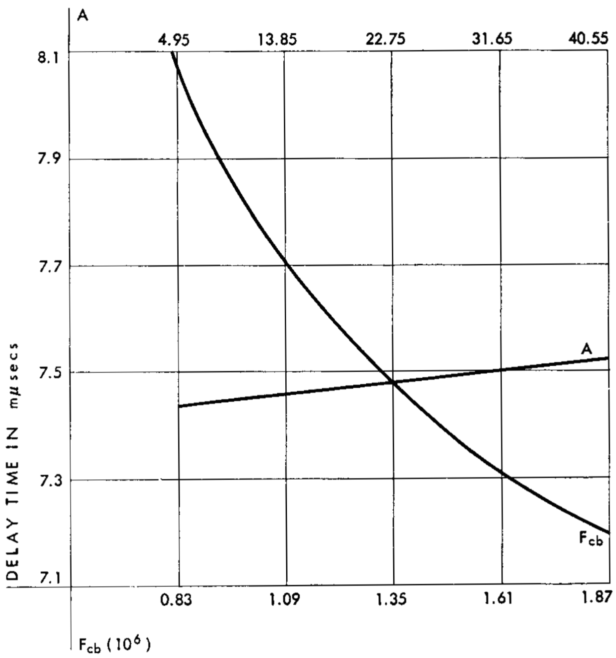


Figure 9d **Delay time vs F_{cb} and A.**

however, that ingenuity may be required in picking the analytic form of g_i and fitting it to the data.

Another advantage of this method is that we can use it to obtain delay time distributions directly, rather than indirectly from output distributions. The functions g_i may characterize the relation of any performance parameter to a_i, provided this parameter be physically determined by a_i, so that we may obtain a graph of the relationship. This is an advantage over the propagation-of-error and total-differential methods, for in those we can find directly only the distribution of those performance parameters for which we can write an analytic expression in terms of the dependent variables of the differential equations. In our case we could find the distribution of collector current because I_c can be expressed in terms of V_{e1} and V_{21}, formula (8). But any such formula for delay time is difficult to find, though it has been done in special cases.[16]

The dependence of delay time on each of the statistically distributed parameters is shown in the Figs. 9a to 9d. From this data we obtain the following expression for delay time.

$$
\begin{aligned}
DT-\overline{DT} = {} & 0.435\times 10^{-2}(R_{bb}-\overline{R}_{bb}) \\
& + 0.204\times 10^{-7}(R_c-\overline{R}_c) \\
& + 0.01447(C_c-\overline{C}_c) \\
& + 0.1176(C_{eo}-\overline{C}_{eo}) \\
& + 1.6(N-\overline{N}) \\
& + 0.234\times 10^{-2}(A-\bar{A}) \\
& + 0.464\left(\frac{1}{I_{es}+0.408}-\frac{1}{\overline{I}_{es}+0.408}\right) \\
& + 195\left(\frac{1}{F_{cb}+24.6}-\frac{1}{\overline{F}_{cb}+24.6}\right).
\end{aligned}
$$

Delay time DT is in mμsec if the parameter units are as indicated in Section 2B.

Note that the first six parameters are linearly related to delay time, but the relations of time with I_{es} and F_{cb} are hyperbolic. The distribution of delay time obtained using Monte Carlo on this formula is shown in Fig. 10. The histogram bar heights are indicated in the appropriate regions along the base of the Figure. Two thousand samples were taken.

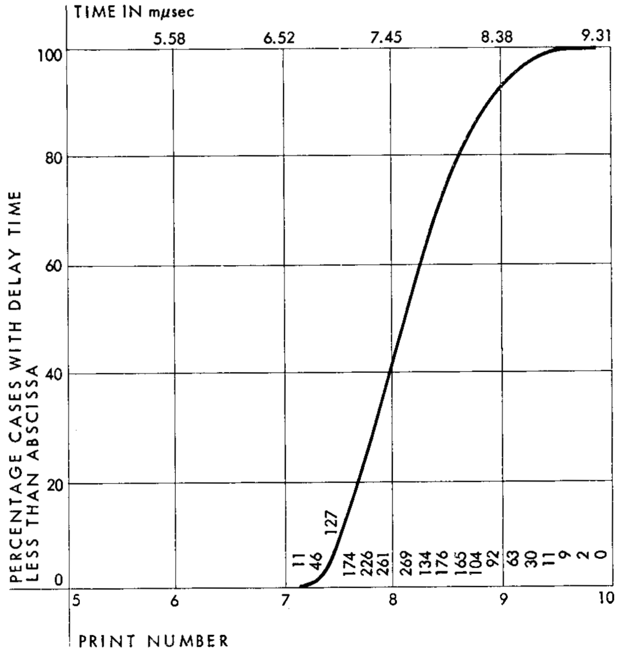


Figure 10 **Distribution of delay time, curve fitting method.**

6. Comparison of results and methods

A comparison of the delay time distributions found by various methods is shown in the following table:

	Best case delay (mμsec)	*Worst case delay* (mμsec)	*95% Circuits faster than* (mμsec)
Actual	6.15	9.35	–
Monte Carlo	6.5	9.4	8.7
Propagation of Error	$(-\infty)$	$(+\infty)$	8.16
Total Differential	6.43	8.66	8.16
Surface Fitting	6.71	9.03	8.40

The *actual* best case and worst case delays were obtained by solving the differential equations with best and worst extreme parameter values. The results by Monte Carlo, propagation of error and total differential are off by the approximations necessary to convert the I_c distribution to a time distribution. In addition, confidence in Monte Carlo is limited by the sample size, 1000. The parameter sensitivity methods are off by the linearity assumption, and the regression method by the assumed form of the expression for delay time. In all methods we assumed stochastic independence of the component parameters. Except for Monte Carlo, the apparent best and worst cases by the different methods are not as extreme as the actual best or worst cases using extreme parameters. This is true also of the Monte Carlo method if we compare I_c's rather than delay time. (The existence of a Monte Carlo extreme greater than the actual worst case arises in the conversion shown in Fig. 4.) The improbability of extremes was one of our reasons for getting away from "worst case" design.

The parameter sensitivity method required the least amount of IBM 704 machine time when applied to the simple transistor equivalent circuit. On larger circuits, however, this method is no longer too attractive timewise.

In our transistor equivalent circuit we have eight parameters. Using second differences to estimate the partials with respect to each parameter requires $2\times 8+1=17$ solutions of the differential equations. In a ten-transistor network, $2\times 80+1=161$ solutions would be required. Using Monte Carlo, this number of solutions could give a workable estimate of the delay time distribution, without the necessity of computing partials, standard deviations, and conversion from collector current to delay time distribution.

The surface fitting method is worse, timewise, since three points are hardly enough to estimate the relation of parameter to delay time. More likely, five points would be the minimum, and for an 80-parameter network this would require $4\times 80+1=321$ solutions.

At present it takes about a half-hour of machine time to obtain the complete solution of a ten-transistor switching block, using PE TAP.[15] If we could be satisfied with 100 samples the Monte Carlo method would require the least amount of machine time, 50 hours.

Neglecting time considerations, Monte Carlo is still the best method we have studied. The program for this method is the simplest to write and debug. It is easier to apply than the other methods and can give the desired result without intermediate data processing.

Summary

Several numerical techniques for estimation of transient distributions have been described and illustrated on a simple transistor circuit. Monte Carlo is simplest, but uses too much machine time on small circuits, compared to other methods. Parameter sensitivity methods may be faster in circuits with few parameters, but require more complicated data processing and considerable operator judgment to determine perturbations. Successful use of the regression analysis method hinges on a good correlation program to determine the performance parameter expression. This method may use empirical data.

Methods especially suitable for implementation on a high-speed digital computer are usually thought of as numerical methods. It should be noted, however, that these computers also lend themselves to symbol manipulatory functions, and that perhaps the best solution of the transient distribution problem is found here, or in symbol manipulation combined with numerical analysis. For, given an analytic solution of the differential equation (1), the transient distribution problem is reduced to the steady-state type, which may be handled easily. This was the motivation behind some recent work[13] on computer analytic methods for solving differential equations.

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