Edward F. Moore*

Minimal Complete Relay Decoding Networks

Abstract: The standard relay-contact-tree network has been used extensively for many years. If n is the number of relays involved, it has always been assumed that the $2(2^n-1)$ contacts used in the standard tree network is the smallest possible number of contacts with which such a network could be made. This paper proves that this is true, provided no sneak paths are allowed. This is in contrast to the result obtained by Lupanov, who showed that when n is five or more it is possible to save contacts below the usual number by permitting sneak paths.

This paper proves further theorems about any network which satisfies the same specifications as an n-relay tree without sneak circuits, and which is built with the minimal number of contacts. In particular, these theorems characterize such a network well enough that it can be shown to be one of the standard forms of relay tree network.

Introduction

A relay is a device, somewhat like an ordinary electric light switch, having contacts which are used to connect and disconnect within electrical networks. There is a certain class of these networks, called *relay trees*, which is frequently used and practically important. Each relay tree has a network configuration which is topologically a tree in the graph-theoretic sense. A relay tree for n relays is used to perform a certain job of connecting to any one of 2^n specified terminals. A relay tree is made up of relay contacts only, and uses exactly $2(2^n - 1)$ of these. These trees have long been known [13], [14], and various papers have been written [4], [9], [13] solving various theoretical problems about relay trees. However, one problem which remained open until recently was the question as to whether these circuits used the smallest possible number of contacts. It had been assumed or conjectured by most persons working on the subject that a standard relay tree for n variables used the minimal number of contacts, that is, was the most economical network for performing the specified job. A recent paper by Lupanov [8], however, gives a relay network for five relays, using only 60 contacts instead of the customary 62. Lupanov's network, shown in Fig. 6, is not topologically tree-like, hence it seems appropriate to use the term "complete relay decoding network" to denote any solution to the problem, and to reserve the term "relay tree" for the kind of network previously known. Lupanov's paper indicates that the network he gives can be generalized to larger values of n, and that in the limit for large n the generalized network uses a number of contacts which asymptotically approaches half the number of contacts of a standard relay tree. However, there is a certain stricter version (excluding sneak paths) of the statement of the network requirements which his network fails to satisfy, although the standard tree does satisfy this stricter statement of the requirements. In certain applications, his networks could be used instead of trees, but in other applications Lupanov's networks would not be admissible because of the sneak paths. The present paper shows that if sneak paths are excluded, then $2(2^n - 1)$ is the smallest number of contacts with which a complete relay decoding network can be built. That is, in this stricter version of the problem, the relay tree is the most economical solution possible.

This paper also demonstrates a number of characteristic properties of those complete relay decoding networks which do not have sneak paths

*Bell Telephone Laboratories, Incorporated, Murray Hill, New Jersey.

and do have the smallest possible number of contacts. In particular, it is shown that these must actually be relay trees of the kind which is already standard.

Contact networks

In this paper the terminology for graphs will be used which is customary in electrical network theory, and by this terminology a *graph* will be said to have *nodes* (which other terminologies call points or vertices) and *branches* (which other terminologies call lines or edges).

A *relay* is a device having two admissible *states*, which will be called 0 and 1. At any given time a relay will be said to be in one or the other of these two states. State 1 will be said to be the *operated* state, and the relay will be said to be operated if it is in state 1. Similarly, state 0 will be said to be the *released* state, and the relay will be said to be released if it is in state 0.

Given a set of n relays, the variables which indicate which state each relay is in will be taken in this paper to be the last n small letters of the Latin alphabet. The *states* of such a set of relays are the ordered n-tuples of states of the n relays. Hence a set of n relays has 2^n states. For instance, the statement that the set $\{w, x, y, z\}$ of relays is in the state 0100 merely asserts that relays w, y, and z are released, and relay x is operated.

Associated with each relay are electrical circuit elements called *contacts.* Each contact is a two-terminal device which connects or disconnects (depending on the state of the relay) electrically the two nodes which are its *terminals.* There are two kinds of contacts, indicated in Figs. 1 and 2. A *front contact* of a relay z is a contact which *closes* (that is, makes a connection or a short circuit between the two terminals) when the relay z is operated, and *opens* (that is, disconnects the two terminals from each other), when the relay z is released. A front contact is denoted by the variable associated with the relay, without any prime, as in Fig. 1. A *back contact* of a relay z is a contact which closes when z is released, and opens when z is operated. A back contact is denoted by adding a prime to the variable associated with the relay, as in Fig. 2.

A *contact network* is a graph of which each branch is taken to be a relay contact. Figs. 3, 4, 5, and 6 are examples of contact networks, drawn with the symbols for the relay contacts drawn along each branch of each graph. The *state* of a contact network is the state of the set of all relays whose contacts occur in that network.

Given two nodes F and G of a contact network, a *path* between F and G is a sequence $F_0, F_1, \cdots F_D$ of distinct nodes and a sequence $Q_1, Q_2, \cdots Q_D$ of contacts, such that $F_0 = F$, $F_D = G$, and such that Q_i is a contact joining node F_{i-1} to node F_i, whenever i satisfies $1 \leq i \leq D$. If all the conditions of the previous sentence are satisfied, the path will be said to have *length* D, and F and G will be said to be the *endpoints* of the path. A path will be said to be *electrically connected in state* W, if when the relays of the network are in state W, each of the contacts of the path is closed. Node F will be said to be *electrically connected to* G *in state* W if there is a path between F and G which is electrically connected in state W. A path will be said to be *sometimes connected* if there is a state W of its network such that the path is electrically connected in state W. Given any two nodes F and G, the *distance* from F to G is the smallest length of any path between F and G which is sometimes connected. A contact network will be called a *topological tree* if for any two nodes F and G, there is exactly one path between them.

Relay decoding networks

A contact network is said to be a *relay decoding network* if it contains one special node (called the *root*) and other special nodes (called *leaves*) such that each leaf is electrically connected to the root in exactly one of the states of the network.

An *external node* of a relay decoding network is a node which is either the root or a leaf. This terminology is justified by the fact that the external nodes are the only ones referred to in the definition of a relay decoding network. In the applications which are made of such a network, it is customary to make it become a part of a larger electrical network by attaching other circuit elements to the external nodes but not to any nodes other than the external nodes.

The definition of a relay decoding network does not prohibit such a network from having other unused states in which the root is not electrically connected to any leaf. However, a relay decoding network will be said to be a *complete relay decoding*

Figure 1 **Front contact on relay z.**

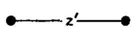

Figure 2 **Back contact on relay z.**

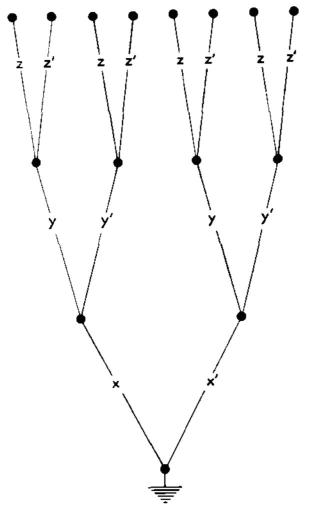

Figure 3 **Ordinary relay tree for 3 relays.**

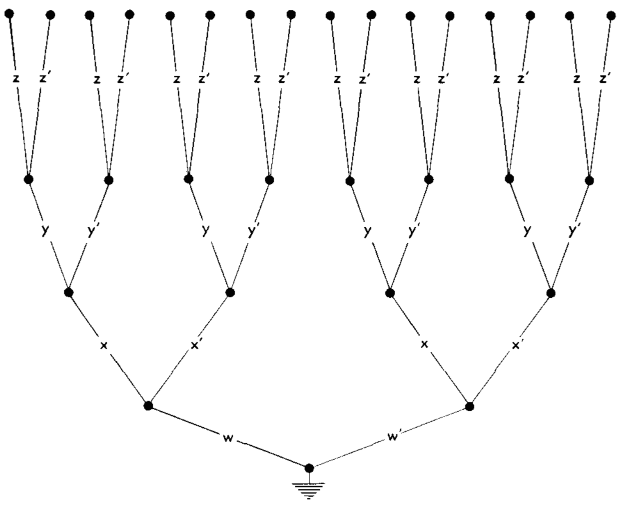

Figure 4 **Ordinary relay tree for 4 relays.**

network if there is one leaf for every state of the network. Since in a network of n relays the number of states is 2^n, a complete relay decoding network has exactly 2^n leaves.

For each n, there is a well-known complete relay decoding network called a *relay tree.* Fig. 3 shows an ordinary (or nonfolded) relay tree having 3 relays, and Fig. 4 shows an ordinary relay tree for 4 relays.

The terms "leaf", "root", "tree", and "decoding" are intended to be mnemonic. All of the trees shown in this paper (Figs. 3, 4, and 5) are drawn with the root at the bottom attached to the electrical ground symbol, and with the leaves at the top of the trees. Each state of the circuit is represented by a sequence of binary digits, which can be considered to be a code word, and this word may be said to be the coded name of the corresponding leaf. The operation of putting the network into a state W then decodes the code word W by connecting the root of the tree to the leaf named W.

The ordinary relay tree for n relays has 2 contacts from one relay, 4 from another, 8 from another, until 2^n contacts occur on the n^{th} relay. The total number of contacts used in the network thus amounts to

$$\sum_{i=1}^{n} 2^i = 2(2^n - 1).$$

Another well-known contact network, called a *folded tree*, is obtained from the ordinary relay tree by an operation called folding, whenever $n \geq 3$. A folded tree which has $n = 4$ is shown in Fig. 5. A folded tree has the same total number of contacts as an ordinary relay tree, but the number of contacts on the individual relays are not constrained to be successive powers of 2, as they are in an ordinary relay tree. The exact extent to which the numbers of required contacts can be redistributed among the various relays is indicated in [5], [6], [14] and investigated in [4] and [13].

Sneak paths

In general, the statement that a contact network has a sneak path is used somewhat loosely to mean that there is a path through the network which electrically connects certain nodes to each other, which the designer of the network either did not intend to electrically connect at all, or did not intend to electrically connect when the network was in certain states in which this path is electrically connected. Hence determining whether a network has sneak paths cannot be done by merely inspecting the network. It must also relate to the intentions of the designer. This subjective part of the determination of sneak paths can be evaded in this paper, since the only networks considered are relay decoding networks. It will be assumed that the designer of the network intentionally puts into it only those paths which are necessary to satisfy the definition of a relay decoding network.

A relay decoding network will be said to have a *sneak path* if there exist two external nodes F and G and a state W of the network, such that F is electrically connected to G in state W, but the definition of the relay decoding network does not require F to be electrically connected to G in state W.

Since the definition of a relay decoding network specifies exactly when the root is connected to each leaf, a sneak path of the network electrically connects two leaves in a state in which neither one of these leaves is electrically connected to the root.

Lupanov [8] constructed a complete relay decoding network for 5 relays which uses only 60 contacts instead of the 62 contacts which the corresponding tree would use. This network permits sneak paths, and it is shown in Fig. 6. By way of comparison, no relay tree permits sneak paths.

Minimality

In this section theorems will be proved which show that a complete relay decoding network cannot have fewer contacts than a relay tree for the same number of relays, unless sneak paths are permitted. Further theorems will give various characterizations of complete relay decoding networks having this minimal number of contacts and having no sneak paths.

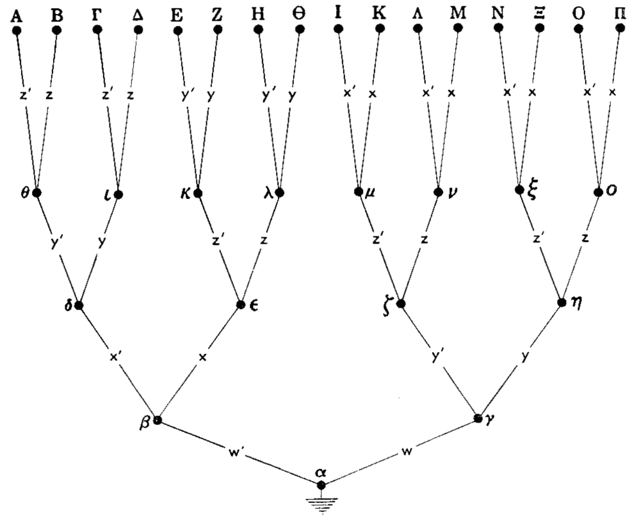

Figure 5 **Folding tree having 4 relays.**

• *Theorem 1*

In a contact network containing n relays, if there is a sometimes connected path of length K between two nodes, then the number R of states in which this path is electrically connected satisfies $R \geq 2^{n-K}$. *Furthermore,* $R = 2^{n-K}$ *if, and only if, all of the contacts along this path belong to different relays.*

Proof: When $K = 0$ the theorem is vacuously true. To complete the proof by induction on K, assume that it holds for all paths of length $K - 1$. Consider a path of length K between two nodes F and G, and let H be the node occurring just after F along this path. Then there is a contact joining node F to node H, and there is a shorter path of length $K - 1$ which is closed in a number R of states satisfying $R \geq 2^{n-(K-1)}$.

Case I. The contact between F and H belongs to a relay x different from any of the relays whose contacts are along the shorter path. Then the shorter path is electrically connected independent of the state of x. That is, for each state in which the shorter path is electrically connected and $x = 1$, there is another state in which the shorter path is electrically connected and $x = 0$. Thus the set of R states in which the shorter path is electrically connected is divided into two subsets having $R/2$ members each. The path of length K is electrically connected in the states of one or the other of these subsets, depending on whether the contact on x is a front contact or a back contact. Hence the number R' of states in which the path of length K is electri-

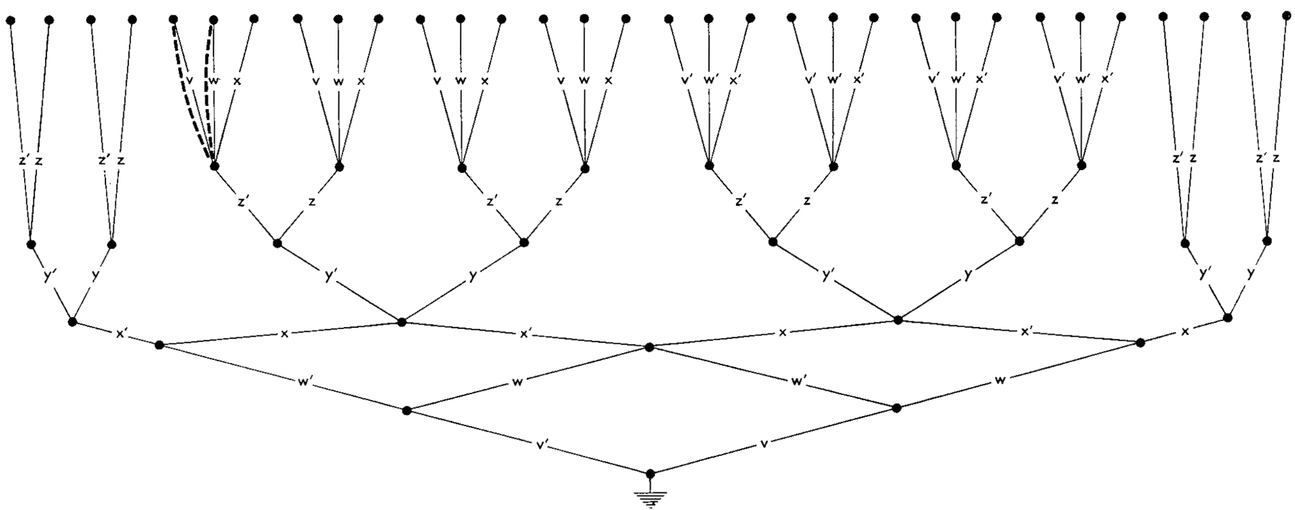

Figure 6 **Lupanov's network, with a sneak path indicated by dotted lines.**

Table 1

The matrix C corresponding to the network of Fig. 5.

The quadruples of binary digits are the states of the network, and the lower-case Greek letters are the nodes which are not leaves.

	α	β	γ	δ	ϵ	ζ	η	θ	ι	κ	λ	μ	ν	ξ	o
0000	Α	Α	Ι	Α	Ε	Ι	Ν	Α	Γ	Ε	Η	Ι	Λ	Ν	Ο
0001	Β	Β	Λ	Β	Η	Λ	Ο	Β	Δ	Ε	Η	Ι	Λ	Ν	Ο
0010	Γ	Γ	Ν	Γ	Ζ	Ι	Ν	Α	Γ	Ζ	Θ	Ι	Λ	Ν	Ο
0011	Δ	Δ	Ο	Δ	Θ	Λ	Ο	Β	Δ	Ζ	Θ	Ι	Λ	Ν	Ο
0100	Ε	Ε	Κ	Α	Ε	Κ	Ξ	Α	Γ	Ε	Η	Κ	Μ	Ξ	Π
0101	Η	Η	Μ	Β	Η	Μ	Π	Β	Δ	Ε	Η	Κ	Μ	Ξ	Π
0110	Ζ	Ζ	Ξ	Γ	Ζ	Κ	Ξ	Α	Γ	Ζ	Θ	Κ	Μ	Ξ	Π
0111	Θ	Θ	Π	Δ	Θ	Μ	Π	Β	Δ	Ζ	Θ	Κ	Μ	Ξ	Π
1000	Ι	Α	Ι	Α	Ε	Ι	Ν	Α	Γ	Ε	Η	Ι	Λ	Ν	Ο
1001	Λ	Β	Λ	Β	Η	Λ	Ο	Β	Δ	Ε	Η	Ι	Λ	Ν	Ο
1010	Ν	Γ	Ν	Γ	Ζ	Ι	Ν	Α	Γ	Ζ	Θ	Ι	Λ	Ν	Ο
1011	Ο	Δ	Ο	Δ	Θ	Λ	Ο	Β	Δ	Ζ	Θ	Ι	Λ	Ν	Ο
1100	Κ	Ε	Κ	Α	Ε	Κ	Ξ	Α	Γ	Ε	Η	Κ	Μ	Ξ	Π
1101	Μ	Η	Μ	Β	Η	Μ	Π	Β	Δ	Ε	Η	Κ	Μ	Ξ	Π
1110	Ξ	Ζ	Ξ	Γ	Ζ	Κ	Ξ	Α	Γ	Ζ	Θ	Κ	Μ	Ξ	Π
1111	Π	Θ	Π	Δ	Θ	Μ	Π	Β	Δ	Ζ	Θ	Κ	Μ	Ξ	Π

cally connected satisfies $R' = R/2 \geq 2^{n-K}$. If equality holds in the longer path, then it must also hold in the shorter path, and so all of the contacts are different from each other. Similarly, if all of the contacts are different from each other, equality holds in the shorter and hence in the longer path.

Case II. The contact between F and H belongs to a relay x which is the same relay to which some contact on the shorter path belongs. Then if one of these two contacts is a front contact and the other is a back contact, we have one of them closed when the other is open, which violates the assumption that the path is sometimes connected. Hence we must have these two contacts be either both front or both back contacts. In either case the contact between F and H is closed whenever the other contact is closed, hence the path of length K is electrically connected in all of those states in which the shorter path is electrically connected. Hence the number of states in which the path of length K is closed is $R \geq 2^{n-K+1} > 2^{n-K}$. Since $R = 2^{n-K}$ cannot occur in Case II, this completes the proof of both parts of the theorem.

• *Theorem 2*

If a complete relay decoding network for n relays has no sneak paths, the number of contacts in the network is at least $2(2^n - 1)$. *Furthermore, if the number of contacts equals* $2(2^n - 1)$, *then the network is a topological tree.*

Proof: Given any relay decoding network for n relays which has no sneak paths, consider the matrix C which has one row for each state of the network and one column for each node which is not a leaf. Let each matrix entry c_{jk} be the name of the leaf which is electrically connected to node k when the circuit is in state j, if there is such a leaf, and let it be blank otherwise. The absence of sneak paths is used in constructing this matrix, since this condition insures that the matrix C has at most one leaf whose name is supposed to occupy each position of the matrix.

To illustrate the ideas involved in this proof, Table 1 is the matrix C which has been constructed for the folded tree network of Fig. 5. The leaves of Fig. 5 were labelled with capital Greek letters, and the nodes of Fig. 5 other than the leaves were labelled with lower-case Greek letters. In addition to the matrix itself, the label (i.e., the state) corresponding to each row and the label (i.e., the lower-case Greek letter for the non-leaf node) corresponding to each column have been indicated. It should be noted that every entry of the matrix of this example is the name of a leaf, rather than being blank. The remainder of the proof will be for the general network, without referring specifically to this particular example.

For each leaf L, let $d(L)$ be the distance along a shortest, sometimes connected path from L to the root. For each L and each integer i satisfying $i \leq d(L)$, let $f(L, i)$ be the node at distance i along a sometimes connected path of length $d(L)$ between L and the root. Then for each i, the node $f(L, i)$ is connected to L in at least 2^{n-i} states of the circuit, by Theorem 1.

But $f(L, d(L))$ is electrically connected to L in exactly one state of the circuit. Hence $d(L)$ is at least n, and the name L must have at least

$$\sum_{i=1}^{n} 2^{n-i} = 2^n - 1 \qquad (1)$$

occurrences in the matrix C.

But since there are 2^n different leaves, and the name of each of these leaves must have at least $2^n - 1$ occurrences in the matrix C, the total number of non-blank entries in C must be at least $2^n(2^n - 1)$. The number of rows of the matrix C equals the number 2^n of states of the network. Hence, to have this many entries, the matrix must have at least $2^n - 1$ columns, so we may conclude that there are at least $2^n - 1$ nodes which are not leaves. Since there are 2^n leaves, there must be a total of at least $2^{n+1} - 1$ members of the set V of all leaves and all those nodes for which there is some leaf to which they are sometimes connected. Each leaf is sometimes connected to the root, hence the subnetwork consisting of all branches which have both their nodes in V must be connected in the topological sense. Then by Theorems 14 and 15 in [7], the number of branches of this subnetwork is at least $2^{n+1} - 2$, and if it has exactly this many branches, this subnetwork must be a topological tree. If the entire network has exactly this many branches, then the subnetwork must be identical with the entire network, which must therefore also be a topological tree. Since each branch of a relay contact network is considered to be a relay contact, the network must have at least $2(2^n - 1)$ contacts, and if equality occurs, the network must be a topological tree. This completes the proof.

Although it has been shown above that the network must be a topological tree, the theorems given later in this paper must be proved before the network will have been shown to be a relay tree. This is an appropriate point at which to mention the different uses to which the word "tree" has been put. When writing about graph theory and about the theory of these electrical networks which do not contain relay contacts, most authors use the word "tree" without any preceding adjective to denote what is called a topological tree in this paper. When writing about relay contact networks, most authors use the word "tree" without any preceding adjective to denote what is called a relay tree in this paper. Since this paper deals with both subjects, it is impossible to be consistent with both terminologies. The two longer terms are used in this paper to avoid confusion, but at the same time to be fairly close to the standard terminology from both fields.

Theorem 3

If a complete relay decoding network for n relays has no sneak paths and uses exactly $2(2^n - 1)$ *contacts, then the distance from the root to each leaf is exactly n, and in each state of the network, each node of the network is connected to exactly one leaf.*

Proof: For a complete relay decoding network which has n relays and uses exactly $2(2^n - 1)$ contacts, the matrix C of the proof of Theorem 2 must have exactly $2^n(2^n - 1)$ entries, none of which are blank. Hence by the definition of the matrix C, in each state of the network each node of the network is electrically connected to exactly one leaf.

But in order for the matrix to have only this many entries, we must have each name of a leaf occurring only $2^n - 1$ times. Since if the distance from any leaf L to the root were different from n, the sum corresponding to (1) would have more terms and the name of L would occur more than $2^n - 1$ times.

Theorem 4

If a complete relay decoding network for n relays has no sneak paths and uses exactly $2(2^n - 1)$ *contacts, then every sometimes connected path between nodes of the network is a part of the path from the root to some leaf.*

Proof: Consider any nodes F and G of the network, such that F is sometimes connected to G. Let W be a state of the network such that F is electrically connected to G in state W. But since none of the entries of the matrix C can be blank, we must have some leaf L which is connected to G in state W. But then L is obviously also connected to F in state W. Then it suffices to prove that both F and G lie on the path from the root to L.

To prove this, we need only show that if any node F of a network is electrically connected to leaf L in state W, then F lies on the path from the root to L. If we assume otherwise we will arrive at a contradiction. By (1), the name L must have at least $2^n - 1$ occurrences in those columns of C which correspond to the nodes which occur on the path from the root to L. But the name L must have at least one occurrence in the column corresponding to F, and hence the name L must have more than the required number of occurrences, which gives a contradiction.

Theorem 5

If a complete relay decoding network for n relays has no sneak paths and uses exactly $2(2^n - 1)$ *contacts, then every sometimes connected path between nodes of the network passes through a set of nodes whose* *distances from the root are all different.*

Proof: Suppose that there were a sometimes connected path which passes through two distinct nodes G and H, and that the distance from the root to G is the same as the distance from the root to H. Without loss of generality we may consider G and H to be the endpoints of the path. But by Theorem 4 nodes G and H lie on the unique path from the root to some leaf L. But since the network is a topological tree, the unique path from the root to G must be contained in this path, and the unique path from the root to H is also contained in this same path, and since the distances from the root to G and to H are equal to each other, we must have $G = H$, which contradicts the assumption that G and H were distinct.

Theorem 6

If a complete relay decoding network for n relays has no sneak paths and uses exactly $2(2^n - 1)$ *contacts, then for each node F of the network, if F is not a leaf, there is a relay such that front and back contacts of that relay go from F to other nodes whose distances from the root are greater than the distance of F.*

The above theorem, together with the preceding ones, characterizes these networks by making it possible to show that each of the networks is a folded tree in the sense defined in [4] (page 14). As applied to the examples of Figs. 3, 4, and 5, each node which is not a leaf must have exactly two contacts between it and the nodes drawn above it in the diagrams, and these two contacts must be a front contact and a back contact on the same relay.

Proof: By Theorem 3, in each state of the network F is electrically connected to exactly one leaf. Hence in each state of the network there must be a closed contact which is the first contact along the path connecting F to this leaf. But by Theorems 4 and 5 the other end of this contact must be farther from the root than F. Consider the set J of all contacts joining F to some node farther from the root. Then if J has only one member, we could put the network into a state in which this contact was open, contradicting Theorem 3. But if J has three or more members, or if it has two members which are contacts of different relays, we could find a set of two contacts in J which are closed in the same state of the network, contradicting Theorem 5. Hence J must consist of exactly two contacts on the same relay. If these two contacts were both front contacts or both back contacts, there would again be a state of the network in which both of them would be closed. Hence one must be a front contact and the other must be a back contact.

Other papers on decoding networks

Besides those publications cited in this paper, others are listed in the references. It should be mentioned that [1], [2], [3], and [12] deal with the minimizing of electronic rather than relay decoding networks. Marcus [9] gives a technique for constructing economical incomplete folded relay trees, when only some specified subset of the 2^n leaves are required. Realization by means of relay contact networks of certain generalizations of relay decoding networks are given in [5] and [11]. An abstract giving Theorem 2 of this paper was published as [10], before Theorems 3, 4, 5, and 6 of this paper were proved.

References

[1] H. H. Aiken and Staff of the Computation Laboratory, "Synthesis of electronic computing and control circuits," *The Annals of the Computation Laboratory of Harvard University*, Harvard University Press, Cambridge. **27**, 68-76, 1951.

[2] D. R. Brown and N. Rochester, "Rectifier networks for multiposition switching," *Proceedings of the Institute of Radio Engineers*, **37**, 139-147 (1949).

[3] A. W. Burks, R. McNaughton, C. H. Pollmar, D. W. Warren and J. B. Wright, "Complete decoding nets: general theory and minimality," *Journal of the Society for Industrial and Applied Mathematics*, **2**, No. 4, 201-243 (1954).

[4] A. W. Burks, R. McNaughton, C. H. Pollmar, D. W. Warren and J. B. Wright, "The folded tree," *Journal of the Franklin Institute*, **260**, No. 1, 9-24 (1955), and No. 2, 115-126 (1955).

[5] S. H. Caldwell, *Switching circuits and logical design*, John Wiley Sons, Inc., New York, 1958, pp. 211-226.

[6] W. Keister, A. E. Ritchie and S. H. Washburn, *The design of switching circuits*, D. Van Nostrand Co., Inc., New York, 1951, 306-326.

[7] D. König, *Theorie der endlichen und unendlichen Graphen*, Chelsea Publishing Company, New York, 1950, pp. 53, 54.

[8] O. B. Lupanov, "O sinteze kontaktnykh skhem," *Doklady Akademii Nauk SSSR*, **119**, No. 1, 23-26 (March 1958). Translation into English published as "On the synthesis of contact networks," *Automation Express*, **1**, No. 3, 7-8 (1958).

[9] M. P. Marcus, "Minimization of the partially-developed transfer tree," *Transactions of the IRE Professional Group on Electronic Computers*, **EC-6**, No. 2, 92-95 (1957).

[10] E. F. Moore, "Minimality of complete relay decoding networks," *Notices of the American Mathematical Society*, **6**, No. 6, 622 (November 1959).

[11] E. F. Moore, "Relay selecting circuit," U. S. Patent 2,864,008, December 9, 1958.

[12] Z. Pawlak, "Decoding nets and the theory of graphs," *Journal of the Society for Industrial and Applied Mathematics*, **7**, No. 1, 1-5 (March 1959).

[13] C. E. Shannon, "The synthesis of two-terminal switching circuits," *The Bell System Technical Journal*, **28**, No. 1, 59-98 (1949).

[14] S. H. Washburn, "Relay trees and symmetric circuits," *Transactions of the American Institute of Electrical Engineers*, **68**, Pt. 1, 582-586 (1949). Reprinted in *Bell Telephone System Monograph* B-1688 (1949).

Received March 2, 1960