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Inductive Proof of the Simplex Method

Abstract: Instead of the customary proof of the existence of an optimal basis in the simplex method based on perturbation of the constant terms, this paper gives a new proof based on induction. From a pedagogical point of view it permits an earlier and more elementary proof of the fundamental duality theorem via the simplex method. Specifically we shall show that there exists a finite chain of feasible basis changes, which results in either an optimal feasible solution or in an infinite class of feasible solutions, such that the objective form tends to minus infinity.

Instead of the customary proof of the existence of an optimal basic set of variables in the simplex method based on perturbation of the constant terms we shall give a new proof based on induction [1]. From a pedagogical point of view it permits an early elementary proof of the fundamental duality theorem via the simplex method, which is favored by some [2].

The general linear programming problem is to find $x_j \geq 0$ and Min z satisfying

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\ c_1x_1 + c_2x_2 + \cdots + c_nx_n = z. \end{array} \qquad (1)$$

Our objective is to prove the following theorem: *If a basic feasible solution to* (1) *exists, then there exists a finite number of changes in the feasible basic sets resulting in either an optimal basic feasible solution or in an infinite class of feasible solutions for which z has no lower bound.*

Let the canonical form for (1) with respect to the assumed initial set of basic variables, say x_1, $x_2, \cdots, x_m$, be

$$\begin{array}{l} x_1 \qquad\qquad + \bar{a}_{1m+1}\, x_{m+1} + \cdots + \bar{a}_{1s}x_s + \cdots + \bar{a}_{1n}x_n = \bar{b}_1 \\ \quad \ddots \; x_r \qquad + \bar{a}_{rm+1}\, x_{m+1} + \cdots + \bar{a}_{2s}x_s + \cdots + \bar{a}_{rn}x_n = \bar{b}_r \\ \qquad\quad \ddots \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \qquad\qquad x_m + \bar{a}_{m,m+1}x_{m+1} + \cdots + \bar{a}_{ms}x_s + \cdots + \bar{a}_{mn}x_n = \bar{b}_m \\ \qquad\qquad\qquad \bar{c}_{m+1}\, x_{m+1} + \cdots + \bar{c}_s\, x_s + \cdots + \bar{c}_n\, x_n = z - \bar{z}_0, \end{array} \qquad (2)$$

where $\bar{z}_0$ is a constant and $\bar{a}_{ij}$ are the new values of the coefficients obtained by eliminating $x_1, \cdots, x_m$ from all but one of the equations and the $\bar{b}_i \geqq 0$ for $i = 1, 2, \cdots, m$. The basic feasible solution is obtained by assigning to the nonbasic variables the values zero and solving for the values of the basic variables, including z.

The simplex method may be outlined as follows. Each iteration begins with a canonical form with respect to some set of basic variables. The associated basic solution is also *feasible*, i.e., the constants $\bar{b}_i$ (as modified) remain nonnegative. The procedure terminates when a canonical form is achieved for which either the $\bar{c}_j \geq 0$ for all j (in which case the basic feasible solution is optimal), or else, in some column with $\bar{c}_s < 0$, the coefficients are all nonpositive $\bar{a}_{is} \leq 0$, (in which case a class of feasible solutions exists for which $z \to -\infty$). In all other cases a "pivot" term is selected in a column, s, and row, r, such that $\bar{c}_s =$ Min c_j is < 0 and $\bar{b}_r/\bar{a}_{rs} =$ Min $(\bar{b}_i/\bar{a}_{is})$ for $\bar{a}_{rs}$, $\bar{a}_{is}$ positive. The variable x_s becomes a new basic variable replacing one in the basic set—namely, by using the equation with the pivot term to eliminate x_s from the other equations.

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When the coefficient of the pivot term is adjusted to be unity, the modified system is in canonical form, and a new basic feasible solution is available in which the value of $z = \bar{z}_0$ is decreased by a positive amount if $\bar{b}_r > 0$. In the nondegenerate case, we have all $\bar{b}_i$'s positive. If this remains true from iteration to iteration, then a termination must be reached in a finite number of steps, because (1) each canonical form is uniquely determined by choice of the m basic variables; (2) the decrease in value of $\bar{z}_0$ implies that all the basic sets are strictly different; and (3) the number of basic sets is finite; indeed, not greater than $\binom{n}{m}$.

In the degenerate case it is possible that $b_r = 0$; this results in $\bar{z}_0$ having the same value before and after pivoting. It has been shown (by examples due to Hoffman and Beale) that the procedure can repeat a basic set and hence cycle indefinitely without termination. This phenomenon occurs (as can be inferred from what follows) when there is ambiguity in the choice of pivot term by the above rules. A proper choice among them will always get around the difficulty. To show this we make the following inductive assumption:

We assume for $1, 2, \cdots, m - 1$ *equations that only a finite number of feasible basic set changes are required to obtain a canonical form such that the* z *equation has all nonnegative coefficients* ($\bar{c}_j \geq 0$), *or some column* s *has* $\bar{c}_s < 0$ *and all nonpositive coefficients* ($\bar{a}_{is} \leq 0$).

We first show the truth of the inductive assumption for $m = 1$. If the initial basic solution is nondegenerate, $\bar{b}_i > 0$, then we note that each subsequent one is also. It follows that the finiteness proof of the simplex method outlined above is valid, so that a final canonical form will be obtained that satisfies our inductive assumption. The degenerate case $\bar{b}_1 = 0$, is established by invoking the following convenient lemma:

If the inductive assumption holds for m, *where not all* $\bar{b}_i$ *are initially zero, then it holds when all* $\bar{b}_i$ *are zero.*

Change one or more $\bar{b}_i = 0$ to $\bar{b}'_i = 1$ (or any other positive value) and then, by hypothesis, a sequence of basic set changes exists such that the final one has the requisite properties. If exactly the same sequence of pivot choices are used for the totally degenerate problem, each basic solution remains feasible (namely zero). Since the desired property of the final canonical form depends only on the choice of basic variables and not on the right-hand side, the lemma is demonstrated.

To establish the inductive step, suppose our inductive assumption holds for $1, 2, \cdots, m - 1$, and that $\bar{b}_i \neq 0$ for at least one i in the m equation system (2). If we are not at the point of termination, then the iterative process is applied until on some iteration, a further decrease in the value of $\bar{z}_0$ is not possible, because of degeneracy. By rearrangement of equations, let $\bar{b}_1 = \bar{b}_2 = \cdots = \bar{b}_r = 0$ and $\bar{b}_i \neq 0$ for $i = r + 1, \cdots, m$. Note for *any* iteration that $r < m$ holds because it is not possible to have *total* degeneracy on a subsequent cycle if, as assumed, at least one of the $\bar{b}_i \neq 0$ initially. According to our inductive assumption there exists a finite series of basic set changes using pivots from the first r equations that results in a subsystem satisfying all $\bar{c}_j \geq 0$ or, for some s, all $\bar{a}_{is} \leq 0$, $1 \leq i \leq r$ and $\bar{c}_s < 0$.

Since the constant terms for the first r *equations are all zero, their values will all remain zero throughout the sequence of pivot term choices for the subsystem; this means we can apply the same sequence of choices for the entire system of* m *equations without replacing* $x_{r+1}, \cdots, x_m$ *as basic variables or changing their values in the basic solutions.*

If the final basis for the subsystem has all $\bar{c}_j \geq 0$, then the same property holds for the system as a whole. On the other hand, suppose the final basis of the subsystem has for some s, $\bar{c}_s < 0$ and $\bar{a}_{is} \leq 0$ for $i = 1, 2, \cdots, r$; in this case we *either* have $\bar{a}_{is} \leq 0$ for $i = r + 1, \cdots, m$ (in which case the inductive property holds for m) *or* the variable x_s can be introduced into the basic set for the system as a whole, producing a *positive decrease* in $\bar{z}_0$ since $\bar{b}_i > 0$ for $i = r + 1, \cdots, m$. We have seen earlier that this value of z can decrease only a finite number of times. Hence the iterative process must terminate, but the only way it *can* is when the inductive property holds for the m equation system.

This completes the proof for m equations, except for the *completely* degenerate case where $\bar{b}_i = 0$ for *all* $i = 1, 2, \cdots, m$. The latter proof, however, now follows directly from the lemma. Q. E. D.

References

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[2] R. D. Luce and H. Raiffa, *Games and Decisions*, John Wiley and Sons, Inc., New York, 1957.

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